Talk E: S(AM) Applied A.G
Ford Collins, Ca 2013

1) Notation: \( X \subset \mathbb{P}(V^*) \), \( d = \dim(V^*) = n+1 \)

\[ X \text{ smooth}, \quad Xd = \mathbb{V}d(X) \subset \mathbb{P}(\text{Sym}^d(V)) \]

\[ \text{im } Xd \text{ is the image of } X \text{ under the } d\text{-th Veronese embedding} \]

\[ Pd = \mathbb{V}d(P^r) \subset \mathbb{P}(\text{Sym}^d(V)^r) \]

\[ \langle X \rangle = \text{linear span of } X \]

\[ \text{or } (X) = \bigcup_{P \in \text{Ex}} \langle P \rangle \]

2) Results discussed are from

I-K: Iarrobino-Kanev: Power Sums, Gorenstein Algebras & Determinantal Loci

B-G-L: Determinantal Equations
In Secant Varieties, and the
Eisenbud-Stillman Conjecture
(Buczyński-Gorinsteijn-lands by)

2a) Buczyński-Buczyński 8-D
Secant Varieties to high degree
Veronese reembeddings, Catalecticant
matrices I smoothable Gorenstein schemes

2a) I apologize for inexact incorporate
with but

3) Classical Fact (Mumford) if d>20
Then Xd C<x_d> is defined by
(low rank) quadrics

3a) Pf: Pf is defined for M
3a) \( \mathbb{P}d \subset \mathbb{P}(\text{Sym}^d(V)) \) is defined by

(chunk rank) quadrics & if \( d \geq 8 \),

\[ I(x) \text{ is a \textit{quadric in degree} } d, \text{ then} \]

\[ \forall D(I(x)) \text{ is generated by linear forms} \]

\[ \forall \mathbf{L} \subset \text{ideal} \]

\[ <x_d> \subset \mathbb{P}(\text{Sym}^d(V)) \]

\[ x_d \subset <x_d> \]

is then defined by the quadrics defined \( x_d \)

\[ \forall x_d = <x_d> \cap \mathbb{P}d \]

4) So in recent varieties questions:

(A) \( \sigma_d(\mathbb{P}d) \) defined by \textit{nice} eqns

(B) \( \sigma_d(x_d) = <x_d> \cap \mathbb{P}(\mathbb{P}d) \)

5) We only have \textit{set-theoretic results},

Don\'t know about scheme theoretically.

6) (B) was addressed in \( B-G-L \).
Theorem (B-G-L) \[ \mathfrak{g} \succeq \text{Gut}(k_{\mathfrak{g}}) + r - 1 \]

set theoretically \( \mathfrak{g}_2(k_{\mathfrak{g}}) = \langle \mathfrak{g}_{\mathfrak{g}} \rangle \Lambda \mathfrak{g}_2(k_{\mathfrak{g}}) \)

6A.) I expect this to be true as a scheme theoretic statement. \text{Gut}(k_{\mathfrak{g}}) is essentially the regularity of \( \mathfrak{g}_{\mathfrak{g}} \).

7.) It is more difficult (not well done)

8.) Subfact: \( (I-K, B-B) \) we don't know the equations of \( \mathfrak{g}_2(k_{\mathfrak{g}}) \).

9.) Notation dump

\[
\begin{align*}
\mathfrak{g}^* & \xrightarrow{\iota} \mathbf{P}(\text{Sym}^d(v)^*) & \mathbf{P}(\text{Sym}^c(v)) & \mathbf{P}(\text{Sym}^{d+c}(v))^* \\
\mathbf{P}^* & \xrightarrow{\iota} \mathbf{P}(\text{Sym}^d(v)^*) & \mathbf{P}(\text{Sym}^c(v)) & \mathbf{P}(\text{Sym}^{d+c}(v))^*
\end{align*}
\]

\[
S_c = \mathbf{P}(\text{Sym}^c(v)^*) \times \mathbf{P}(\text{Sym}^{d+c}(v))^*
\]
Theorem: If $P$ is the random forest in an edge $\text{Sym}^d(V) \circ \text{Sym}^r(V) \circ \text{Sym}^{d-r}(V)$.

Hence $\sigma_r(P) \subseteq \sigma_r(S)$

$S_r = \sigma_r(S) \cap \text{IP}(\text{Sym}^d(V))$

$\sigma_r(S_r) \cap \text{all } ...$

b) Second variables to $S_r$: embeddings are understood - their signs are the $(r \times r) \times (r \times r)$ minors of the catalecticant matrix

$F_{ij}$ - column $F_j$ x rows $w$

$r_i$ spanning new basis of $\text{Sym}^d(V)$ by columns of $\text{Sym}^{d-r}(V)$

The signs of $S_r$ are the 'new' signs of $\sigma_r(V)$.
1. When is $\sigma_n(x) = \sigma_r n \wedge \sigma_r$ ?

2. $\sigma_n(x) = U \langle k \rangle$
   
   \( \text{Re } \text{Heck}_n(x) \) - degree \( r \) zero dual scheme on \( x \)

   So $\sigma_n(x) \subset \sigma_n(x)$ since

   $\sigma_n(x) = U \langle k \rangle$
   
   \( \text{Re } \text{Heck}_n(x) \) - prin. irreducible component of $H/\Lambda \wedge k$

3. Technical fact. $\sigma_n(x) = U \langle k \rangle$
   
   \( \text{Re } \text{Heck}_n(x) \) \& L-fun.

   we can restrict to Gorenstein schemes

4. No real hard - it's in B-6-L

5. Main technical result: (B-83)
   
   $d > 2r$ and $c \leq d - r$
14A) Thus the "obvious" defines a variety that is generally big: then $\sigma_n(x)$.

14B) Result is more precise: if $p \in \text{Sym}^d(V)$ and $p \in \mathcal{X}$, then we can explicitly construct a saturated Cohen-Macaulay ideal $J$ such that $V(J)$ is a degree $r$-zero ideal scheme with $p \in \langle V(J) \rangle$.

15) Result are proved using a polarity.

Given an Artinian Cohen-Macaulay $R$ and $p \in \text{Sym}^d(V)$.

$R = \text{Sym}^d(V) / \text{Ann}(p)$.
21) So program x amounts to finding "non-expected" eqns.

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Some key questions

Class

1. Can one find a new set of equations that define $\sigma_1(kd)$ $C_\sigma(kd)$ if $d >> 0$?

2. Can the results of B-6-L $\nu$-B be extended to be scheme theoretic?

21. In fact, one should think of both classes of results as syzygy statements.

Can one extend the results (for $d$ even larger) to the higher syzygies of $\sigma_1(kd)$?

In particular if we do so, we had
16) Since $\delta'_n = \delta_n(X)$ is smoothable, we get the fact to hold
in $\sigma_n(X)$ is smoothable. Every zero-dimensional scheme
is smoothable.

17) $\sigma_n(X) \Rightarrow \delta_n(X) = \sigma_n(X)$

18 bis) B-B) $\sigma_n(X) \Rightarrow \delta_n(X) = \sigma_n(X)$ and,

$x > 2m + \sigma_n(X)$; $\delta_n(X)$ then $x$ holds.

19) $x$ holds if $n \leq 3$ or $r \leq 10$

20) $x$ known to fail if $n \geq 5$ or $r \geq 14$

$\text{II - K + others}$

21) In these cases

$\sigma_n(X) \not\subseteq \bigcup_{\mathcal{A}} \mathcal{A}(X)$
\( K_{f, f}(X_d) = K_{f, f}(B_d) \) in some range then perhaps combining Snowden's work w/ an answer to question #3 we can answer question #1. MORAL: #1 seems VERY HARD.

#3): Is there an exact relationship between the syzygies of \( X_b \) \( \sigma(X)b \)

\[ d \sigma(d(B)) = d(f) \]

Specifically if \( \sigma \) 0 then

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the \( b \)-th syzygy module of \( X_d \) is the \( (n-1) \)th syzygy module of \( \sigma_1(X_d) \) \( \sigma f d(X_d) b \).

MSJ