Secant varieties of Segre-Veronese varieties

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1 joint with Daniel Erman and Luke Oeding
Tensor rank for matrices

\( U, V \) : finite dimensional vector spaces
\( x \in U \otimes V \)

The rank of \( x \) is the smallest integer \( r \) such that \( x \) can be written

\[ x = u_1 \otimes v_1 + \cdots + u_r \otimes v_r \text{ where } u_i \in U, v_i \in V. \]
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- The rank can jump down but never up at special points.
- The rank is the same if we pass to a bigger field.
- The set of possible decompositions is a homogeneous space.
Partially symmetric tensors

$U$: $m$-dimensional $\mathbb{C}$-vector space
$V$: $n$-dimensional $\mathbb{C}$-vector space

$x \in U \otimes S^2 V$

The rank of $x$ is the smallest integer $r$ such that $x$ can be written in the form:

$$x = u_1 \otimes v_1 \otimes v_1 + \cdots + u_r \otimes v_r \otimes v_r$$
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The rank can jump both down and up for special tensors.
Partially symmetric tensors

\[ U: \text{ } m\text{-dimensional } \mathbb{C}\text{-vector space} \]
\[ V: \text{ } n\text{-dimensional } \mathbb{C}\text{-vector space} \]
\[ x \in U \otimes S^2 V \]

The **border rank** of \( x \) is the smallest integer \( r \) such that \( x \) can be approximated arbitrarily closely by expressions of the form:

\[ x \approx u_1 \otimes v_1 \otimes v_1 + \cdots + u_r \otimes v_r \otimes v_r \]
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\[ x \approx u_1 \otimes v_1 \otimes v_1 + \cdots + u_r \otimes v_r \otimes v_r \]

The set of such decompositions will in general be a finite set of points, possibly defined over a larger field than \( x \).
Equations for bounded border rank

When $U = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, we can write

$$x = e_1 \otimes A + e_2 \otimes B$$

where $A$ and $B$ are symmetric matrices.
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**Theorem**

*The ideal of partially symmetric tensors whose border rank is at most \( r \) is generated by the \((r + 1) \times (r + 1)\)-minors of the block matrix*

\[
\begin{pmatrix}
A & B
\end{pmatrix}
\]
Equations for small border rank

When $U = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$, we can write

$$x = e_1 \otimes A + e_2 \otimes B + e_3 \otimes C$$
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Theorem (C.-Erman-Oeding 2010)

If \( r \leq 5 \), the ideal of tensors whose border rank is at most \( r \) is generated by the \((r + 1) \times (r + 1)\)-minors and \((2r + 2) \times (2r + 2)\)-Pfaffians respectively of

\[
\begin{pmatrix} A & B & C \end{pmatrix}
\text{ and }
\begin{pmatrix} 0 & A & -B \\ -A & 0 & C \\ B & -C & 0 \end{pmatrix}
\]

Remark

The \( n = 4 \), \( r = 5 \) case is due to Emil Toeplitz in 1869.
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**Remark**

*The \( n = 4, r = 5 \) case is due to Emil Toeplitz in 1869.*
Outline of the proof

Assume $n = r$ and $A$ is the identity matrix. Then

\[
\begin{pmatrix}
0 & I & -B \\
-l & 0 & C \\
B & -C & 0
\end{pmatrix}
\sim
\begin{pmatrix}
0 & I & 0 \\
l & 0 & 0 \\
0 & 0 & BC - CB
\end{pmatrix}
\]

The $2r + 2$-Pfaffians of this matrix are the entries of the commutator $BC - CB$, which is a prime, Gorenstein ideal, defining the variety of commuting symmetric matrices.
Outline of the proof

- Assume \( n = r \) and \( A \) is the identity matrix. Then

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- Now just assume \( n = r \). We need to bound the dimension of the set of tensors where \( A \) is singular. This is computational and is only true for \( r \leq 5 \).
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- Now just assume $n = r$. We need to bound the dimension of the set of tensors where $A$ is singular. This is computational and is only true for $r \leq 5$.

- Arbitrary $n$. Here the minors come in.
Unifying framework for these equations

Given decomposable \( u \otimes v \otimes v \in U \otimes S^2 V \), we have linear map

\[
\psi_{j, u \otimes v \otimes v} : V^* \otimes \bigwedge^j U \to V \otimes \bigwedge^{j+1} U
\]

\[
v^* \otimes (u'_1 \wedge \cdots \wedge u'_j) \mapsto \langle v^*, v \rangle v \otimes u'_1 \wedge \cdots \wedge u'_j \wedge u
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For arbitrary \( x \in U \otimes S^2 V \), define \( \psi_{j,x} \) by extending linearly.
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For arbitrary $x \in U \otimes S^2 V$, define $\psi_{j,x}$ by extending linearly.

If $U$ is 3-dimensional,

- The $j = 0$ and $j = 2$ cases give the rectangular matrix
- The $j = 1$ case gives the skew-symmetric square matrix
Robust testing of determinantal equations

Let
\[ \sigma_1 \geq \cdots \geq \sigma_4 \quad \text{and} \quad \sigma'_1 \geq \cdots \geq \sigma'_{12} \]
be the singular values of
\[
\begin{pmatrix}
A & B & C
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
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respectively.
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\begin{pmatrix}
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& & \\
& & \\
& & \\
& & \\
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\]

respectively.
The functions

\[
\gamma_r^2 = \sum_{i=r+1}^{4} \sigma_i^2 \\
\delta_r^2 = \sum_{i=2r+1}^{12} (\sigma'_i)^2
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The functions

$$\gamma_r^2 = \sum_{i=r+1}^{4} \sigma_i^2 \quad \text{and} \quad \delta_r^2 = \sum_{i=2r+1}^{12} (\sigma'_i)^2$$

are continuous, non-negative functions which are both zero if and only if the tensor has rank at most $r$. 
Bounded real rank is a semi-algebraic set

Tensors with *real* border rank at most $r$ characterized by same equalities, but additional inequalities
Bounded real rank is a semi-algebraic set

Equalities are more important than inequalities for detecting deviations
Thank you