1. Consider the functions defined by

\[ g_a(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1} \quad \text{and} \quad g_b(z) = \frac{e^{\pi z/2} - 1}{e^{\pi z/2} + 1}. \]

Show that \( g_a \) maps the set

\[ \Omega_a := \{ z \in \mathbb{C} \mid -1 < \text{Re} z < 1 \} \]

to \( D(0, 1) \) while \( g_b \) maps the set

\[ \Omega_b := \{ z \in \mathbb{C} \mid -1 < \text{Im} z < 1 \} \]

to \( D(0, 1) \). Hence or otherwise, prove the following.

\textbf{Solution.} Note that

\[ |g_b(z)| < 1 \iff \left| \frac{e^{\pi z/2} - 1}{e^{\pi z/2} + 1} \right| < 1 \]
\[ \iff |e^{\pi z/2} - 1|^2 < |e^{\pi z/2} + 1|^2 \]
\[ \iff [\text{Re}(e^{\pi z/2}) - 1]^2 + [\text{Im}(e^{\pi z/2})] < [\text{Re}(e^{\pi z/2}) + 1]^2 + [\text{Im}(e^{\pi z/2})]^2 \]
\[ \iff \text{Re}(e^{\pi z/2}) - 2 \text{Re}(e^{\pi z/2}) + 1 < \text{Re}(e^{\pi z/2}) + 2 \text{Re}(e^{\pi z/2}) + 1 \]
\[ \iff \text{Re}(e^{\pi z/2}) > 0. \]

But \( \text{Re}(e^{\pi z/2}) = \text{Re}(e^{i\pi/2 \text{Re} z + i\pi/2 \text{Im} z}) = \exp(i\pi/2 \text{Re} z) \cos(i\pi/2 \text{Im} z) > 0 \) iff \( \cos(i\pi/2 \text{Im} z) > 0 \) — in particular this holds when \(-1 < \text{Im} z < 1\), i.e. when \( z \in \Omega_b \). To deduce the analogous result for \( g_a \) and \( \Omega_a \), just note that

\[ g_a(z) = g_b(iz) \quad \text{and} \quad \Omega_a = i\Omega_b. \]

(a) Let \( f : D(0, 1) \rightarrow \mathbb{C} \) be an analytic function that satisfies \( f(0) = 0 \). Suppose

\[ |\text{Re} f(z)| < 1 \]
for all \( z \in D(0, 1) \). By considering the function \( g_a \circ f \) or otherwise, prove that

\[ |f'(0)| \leq \frac{4}{\pi}. \]

\textbf{Solution.} Note that \( g_a \) maps \( \Omega_a \) onto \( D(0, 1) \), \( g_a \) is analytic on \( \Omega_a \), and \( g_a(0) = 0 \). The composition \( F = g_a \circ f : D(0, 1) \rightarrow \mathbb{C} \) is analytic on \( D(0, 1) \); \( F(0) = g_a(f(0)) = g_a(0) = 0 \); for any \( z \in D(0, 1) \), \( |\text{Re} f(z)| < 1 \) and so \( f(z) \in \Omega_a \) and so \( g_a(f(z)) \in D(0, 1) \), i.e.

\[ |F(z)| = |g_a(f(z))| < 1 \]
for all \( z \in D(0, 1) \). By Schwarz’s Lemma, we have \( |F'(0)| \leq 1 \). Since

\[ F'(z) = g_a'(f(z))f'(z) = \frac{\pi i e^{\pi i f(z)/2} f'(z)}{(e^{\pi i f(z)/2} + 1)^2} \]

and \( f(0) = 0 \), we get

\[ |f'(0)| \leq \frac{4}{\pi}. \]
(b) Let \( S \) be the set of functions defined by
\[
S = \{ f : \Omega_b \to \mathbb{C} \mid f \text{ analytic, } |f| < 1 \text{ on } \Omega_b, \text{ and } f(0) = 0 \}.
\]
By considering the function \( f \circ g_b^{-1} \) or otherwise, prove that
\[
\sup_{f \in S}|f(1)| = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}.
\]

Solution. Note that \( g_b \) maps \( \Omega_b \) into \( D(0, 1) \), \( g_b \) is analytic on \( \Omega_b \), and \( g_b(0) = 0 \). Furthermore, note that \( g_b \) is injective on \( \Omega_b \) \( \longrightarrow g_b(z) = g_b(w) \) iff
\[
(e^{\pi z/2} - 1)(e^{\pi w/2} + 1) = (e^{\pi z/2} + 1)(e^{\pi w/2} - 1)
\]
iff \( e^{\pi(z-w)/2} = 1 \) iff \( \text{Re}(z - w) = 0 \) and \( \text{Im}(z - w) \) is an integer multiple of 4; so when \( z, w \in \Omega_b \), this is only possible if \( z = w \). Hence \( g_b \) is an invertible map and \( g_b^{-1} \) maps \( D(0, 1) \) onto \( \Omega_b \), \( g_b^{-1} \) is analytic on \( D(0, 1) \), and \( g_b^{-1}(0) = 0 \). Given any \( f \in S \), the composition \( F = f \circ g_b^{-1} : D(0, 1) \to \mathbb{C} \) is analytic on \( D(0, 1) \); \( F(0) = f(g_b^{-1}(0)) = f(0) = 0 \); for any \( z \in D(0, 1), g_b^{-1}(z) \in \Omega_b \) and so \( f(g_b^{-1}(z)) \in D(0, 1) \), ie.
\[
|F(z)| < 1
\]
for all \( z \in D(0, 1) \). By Schwarz’s Lemma, we have \( |F(z)| \leq |z| \) for all \( z \in D(0, 1) \). In particular, if we pick
\[
z_0 = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1} \in D(0, 1),
\]
we get
\[
|f(1)| = |f(g_b^{-1}(z_0))| = |F(z_0)| \leq |z_0| = z_0.
\]
Since this is true for arbitrary \( f \in S \), we have that
\[
\sup_{f \in S}|f(1)| \leq \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}.
\]
But note that equality is attained by
\[
f(z) = e^{i\theta}g_b(z)
\]
for any \( \theta \in \mathbb{R} \). So
\[
\sup_{f \in S}|f(1)| = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}.
\]

2. Let \( f : D(0, 1) \to \mathbb{C} \) be analytic and \( |f(z)| < 1 \) for all \( z \in D(0, 1) \). Recall that \( \varphi_a(z) = (z - \alpha)/(1 - \overline{\alpha}z) \).
(a) By considering the function \( \varphi_{f(a)} \circ f \circ \varphi_{-a} \) or otherwise, show that
\[
\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.
\]

Solution. Let \( a \in D(0, 1) \). Then \( f(a) \in D(0, 1) \). We consider \( \varphi_{-a} \) and \( \varphi_{f(a)} \). Define \( g = \varphi_{f(a)} \circ f \circ \varphi_{-a} : D(0, 1) \to \mathbb{C} \). Note that \( g \) is analytic. If \( z \in D(0, 1) \), then \( \varphi_{-a}(z) \in D(0, 1) \), so \( f(\varphi_{-a}(z)) \in D(0, 1) \), and so \( \varphi_{f(a)}(f(\varphi_{-a}(z))) \in D(0, 1) \), ie.
\[
|g(z)| < 1
\]
for all \( z \in D(0, 1) \). Also \( g(0) = \varphi_{f(a)}(f(\varphi_{-a}(0))) = \varphi_{f(a)}(f(a)) = 0 \). Schwartz’s Lemma then implies that \( |g'(0)| \leq 1 \). But using chain rule and Lemma 4.18 in the lectures, we get
\[
|g'(0)| = |\varphi_{f(a)}(f(a))f'(a)\varphi_{-a}'(0)| = \frac{|f'(a)|}{1 - |f(a)|^2} \times (1 - |a|^2)
\]
and so
\[
\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.
\]

Now just observe that this works for arbitrary \( a \in D(0, 1) \).

(b) Suppose there exist two distinct points \( a, b \in D(0, 1) \) such that \( f(a) = a \) and \( f(b) = b \). By considering the function \( \varphi_a \circ f \circ \varphi_a \) or otherwise, show that \( f(z) = z \) for all \( z \in D(0, 1) \).

**SOLUTION.** Since \( a \neq b \), at least one of them must be non-zero, wlog, let \( a \neq 0 \). Define \( g = \varphi_a \circ f \circ \varphi_a : D(0, 1) \to C \). Note that \( g \) is analytic. If \( z \in D(0, 1) \), then \( \varphi_a(z) \in D(0, 1) \), so \( f(\varphi_a(z)) \in D(0, 1) \), and so \( \varphi_a(f(\varphi_a(z))) \in D(0, 1) \), ie.

\[
|g(z)| < 1
\]

for all \( z \in D(0, 1) \). Also \( g(0) = \varphi_a(f(\varphi_a(0))) = \varphi_a(f(a)) = \varphi_a(a) = 0 \). Schwartz’s Lemma then implies that \( |g(z)| \leq |z| \) for all \( z \in D(0, 1) \). Before we proceed further, we observe that for \( \alpha \in D(0, 1) \), \( \varphi_a(\varphi_a(z)) = \alpha \). \( \varphi_a(z) \) for all \( z \in D(0, 1) \), ie. \( \varphi_a \) is an invertible map and \( \varphi_a^{-1} = \varphi_a \). Let \( c = \varphi_a(b) \in D(0, 1) \). So \( b = \varphi_a(c) \). Now note that

\[
g(c) = \varphi_a(f(\varphi_a(c))) = \varphi_a(f(b)) = \varphi_a(c) = c
\]

and so equality is attained in \( |g(z)| \leq |z| \) by \( z = c \in D(0, 1) \). By Schwartz’s Lemma again, we get \( g(z) = e^{i\theta}z \) for all \( z \in D(0, 1) \). But \( e^{i\theta}c = g(c) = c \) and so \( e^{i\theta} = 1 \). Hence

\[
\varphi_a \circ f \circ \varphi_a = \text{id}
\]

and so

\[
f = \varphi_a \circ \text{id} \circ \varphi_a = \text{id}.
\]

(c) Suppose there exist \( a \in D(0, 1) \), \( a \neq 0 \), such that \( f(a) = 0 = f(-a) \). By considering the function

\[
g(z) = \frac{f(z)}{\varphi_a(z)\varphi_a(-z)}
\]

or otherwise, show that \( |f(0)| \leq |a|^2 \). What can you conclude if \( |f(0)| = |a|^2 \)?

**SOLUTION.** Note that \( g \) is analytic on \( D(0, 1) \) except possibly at \( z = a \) and \( z = -a \) where the denominator vanishes. However since

\[
\lim_{z \to a} (z - a)g(z) = \lim_{z \to a} \frac{(1 - \overline{a}z)(1 + \overline{a}z)}{(z + a)} f(z) = \frac{1}{2a} f(a) = 0,
\]

\[
\lim_{z \to -a} (z + a)g(z) = \lim_{z \to -a} \frac{(1 - \overline{a}z)(1 + \overline{a}z)}{(z - a)} f(z) = \frac{|a|^2 - 1}{2a} f(a) = 0,
\]

\( g \) has removable singularity by Theorem 5.2. We thus may assume that \( g \) is analytic on \( D(0, 1) \). If \( |z| = 1 \), ie. \( z = e^{i\theta} \) for some \( \theta \), then

\[
\left| \frac{1}{\varphi_a(z)\varphi_a(-z)} \right| = \frac{|1 - \overline{a}z|(1 + \overline{a}z)}{|z - a|(z + a)}
\]

\[
= \frac{1 - \overline{a}z}{e^{2i\theta} - a^2}
\]

\[
= \frac{e^{2i\theta}(e^{-2i\theta} - \overline{a}^2)}{e^{2i\theta} - a^2}
\]

\[
= \frac{|e^{-2i\theta} - \overline{a}^2|}{|e^{2i\theta} - a^2|}
\]

\[
= 1.
\]

Since \( |f(z)| < 1 \) for all \( z \in D(0, 1) \) and \( f \) is analytic, we must have \( |f(z)| \leq 1 \) for \( |z| = 1 \).

Hence

\[
|F(z)| \leq 1
\]
for \(|z| = 1\) and thus for \(|z| \leq 1\) by Maximum Modulus Theorem. If we take \(z = 0\), \(|F(0)| \leq 1\) gives
\[|f(0)| \leq |\varphi_a(0)\varphi_{-a}(0)| = |a|^2.\]
If equality is attained, then \(|F(0)| = 1\) means that \(F\) attains its maximum on \(\overline{D(0,1)}\) at the interior point \(z = 0\). Maximum Modulus Theorem then implies that \(F\) is constant throughout \(D(0,1)\) and so \(F \equiv e^{i\theta}\) for some \(\theta \in \mathbb{R}\). Hence
\[f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z} \frac{z + a}{1 + \overline{a}z}\]
for all \(z \in D(0,1)\).

3. Recall the \(L\)-shaped path that we used in lectures to establish the existence of antiderivative for an analytic function, i.e. \(L(z_1, z_2)\) is a path that goes from \(z_1\) to \(z_2\) comprising two line segments one parallel to the imaginary axis and the other to the real axis.

(a) Let \(f : \Omega \to \mathbb{C}\) be a continuous function on an open set \(\Omega\). Suppose
\[\int_{\partial R} f(z) \, dz = 0\]
along the boundary of any closed rectangle \(R \subseteq \Omega\). Let \(D(z_0, r) \subseteq \Omega\). Show that the function \(F : D(z_0, r) \to \mathbb{C}\) defined by
\[F(z) = \int_{L(z_0, z)} f(\zeta) \, d\zeta\]
is an antiderivative of \(f\). Hence deduce that \(f\) is analytic in \(\Omega\).

**Solution.** See proof of Morera’s Theorem on pp. 85–86 in the textbook.

(b) Let \(f : D(0,1) \to \mathbb{C}\) be analytic and \(f(z) \neq 0\) for all \(z \in D(0,1)\). By considering the function \(h : D(0,1) \to \mathbb{C}\) defined by
\[h(z) = \int_{L(0,z)} \frac{f' (\zeta)}{f(\zeta)} \, d\zeta,\]
show that there exists an analytic function \(g : D(0,1) \to \mathbb{C}\) such that
\[f(z) = e^{g(z)}\]
for all \(z \in D(0,1)\).

**Solution.** Since \(f(z) \neq 0\) for all \(z \in D(0,1)\), \(f'(z)/f(z)\) is analytic on \(D(0,1)\). Let \(L(0,z)\) is an \(L\)-shaped curve from 0 to \(z\). Recall from the proof of Theorem 3.13 that the function \(h : D(0,1) \to \mathbb{C}\) defined by
\[h(z) = \int_{L(0,z)} \frac{f' (\zeta)}{f(\zeta)} \, d\zeta\]
is analytic and is an antiderivative of the integrand, i.e.
\[h'(z) = \frac{f'(z)}{f(z)}\]
for all \(z \in D(0,1)\). Now apply chain rule to the analytic function \(f(z)e^{-h(z)}\) to get
\[
\frac{d}{dz} f(z)e^{-h(z)} = e^{-h(z)} \left[ f'(z) - h'(z)f(z) \right] \\
= e^{-h(z)} \left[ f'(z) - f'(z) \right] \\
= 0.
\]
In other words \(f(z)e^{-h(z)}\) is constant and so
\[f(z)e^{-h(z)} = f(0)e^{-h(0)} = f(0)\] (3.1)
for all $z \in D(0,1)$. Since $f(0) \neq 0$, we have $f(0) = re^{i\theta}$ for some $r > 0$ and some $\theta \in [0,2\pi)$. Since $r > 0$, we may write $r = e^{\ln r}$ and thus $f(0) = e^{\ln r + i\theta}$. Let $c = \ln r + i\theta \in \mathbb{C}$. Then (3.1) becomes

$$f(z) = f(0)e^{h(z)} = e^{c+h(z)}.$$ 

Hence $g(z) = h(z) + c$ is an analytic function satisfying

$$f(z) = e^{g(z)}$$

for all $z \in D(0,1)$. Note that (a) is a converse to Cauchy’s theorem while (b) establishes the existence of logarithms.

4. Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$. Suppose there exists $n \in \mathbb{N}$ such that $g(z) = [f(z)]^n$ and $h(z) = [f(z)]^{n+1}$ are both analytic on $\Omega$. Show that $f$ is analytic on $\Omega$.

**Solution.** Note that $g(z) = 0$ if and only if $h(z) = 0$. So the zeros of $g$ and $h$ are in common. Let $Z \subseteq \Omega$ be these common zeros. If $Z$ contains a limit point, then $g$ and $h$ are identically zero by the identity theorem and $f \equiv 0$ is of course analytic. We will assume that all zeros in $Z$ are isolated. Let $z_0 \in Z$. So there exist analytic functions $g_1$ and $h_1$ and some $\varepsilon > 0$ such that for all $z \in D(z_0, \varepsilon)$,

$$g(z) = (z - z_0)^k g_1(z), \quad g_1(z_0) \neq 0,$$

$$h(z) = (z - z_0)^l h_1(z), \quad h_1(z_0) \neq 0,$$

where $k$ and $l$ are the orders of zero of $g$ and $h$ respectively at $z_0$. But $g^{n+1} = f^{n(n+1)} = h^n$ and so

$$(z - z_0)^{(n+1)k} g_1(z)^{n+1} = (z - z_0)^{nl} h_1(z)^n.$$ 

So we get

$$\frac{g_1(z)^{n+1}}{h_1(z)^n} = (z - z_0)^{(n+1)k-nl}.$$ 

Since the LHS is non-zero for $z = z_0$, the RHS is also non-zero for $z = z_0$ and this is only possible if $(n+1)k - nl = 0$. In other words, $l > k$, i.e. the order of zero of $h$ is larger than the order of zero of $g$. So $h/g$ has a removable singularity at $z_0$. Since this is true for all $z_0 \in Z$, $h/g$ may be extended to an analytic function $\tilde{f}$ on $\Omega$. But since for all $z \in \Omega \setminus Z$,

$$\tilde{f}(z) = \frac{h(z)}{g(z)} = f(z),$$

and $\Omega \setminus Z$ clearly has limit points, $\tilde{f} \equiv f$ by the identity theorem.

5. Show that each of the following series define a meromorphic function on $\mathbb{C}$ and determine the set of poles and their orders.

$$f(z) = \sum_{n=0}^{\infty} \sin(nz) \frac{1}{n!(z^2 + n^2)}, \quad g(z) = \sum_{n=0}^{\infty} \left[ \frac{1}{(z-z_n)^2} - \frac{1}{z_n^2} \right]$$

where $\{z_n\}_{n=0}^{\infty} \subseteq \mathbb{C}^{\times}$ is a sequence of distinct complex numbers (i.e. $z_n \neq z_m$ if $n \neq m$) such that

$$\sum_{n=0}^{\infty} \frac{1}{|z_n|^3} < \infty. \quad (5.2)$$

**Solution.** We claim that $f$ is meromorphic on $\mathbb{C}$ with poles of order 1 at

$$z \in P_f := i\mathbb{Z}^{\times} = \{\ldots, -3i, -2i, -i, i, 2i, 3i, \ldots\},$$

5
and a removable singularity at $z = 0$. For any fixed $R > 0$, there exists an integer $N$ such that $N > 2R$. We will consider $f$ on the disc $D(0, R)$. We may write

$$f(z) = \sum_{n=0}^{N} \frac{\sin(nz)}{n!(z^2 + n^2)} + \sum_{n=N+1}^{\infty} \frac{\sin(nz)}{n!(z^2 + n^2)}. \quad (5.3)$$

Note that the zeroes of $\sin(nz)$ are

$$\{..., -\frac{2\pi}{n}, -\frac{\pi}{n}, 0, \frac{\pi}{n}, \frac{2\pi}{n}, ..., \}$$

and so the zeroes of $\sin(nz)$ and $z^2 + n^2 = (z + in)(z - in)$ are disjoint for all $n \geq 1$. So the first sum in (5.3) has poles of order 1 for each $\pm nz$ and so the zeroes of $\sin(nz)$ are $\{0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots\}$ and a removable singularity at $z = 0$. We will consider $g$ on the disc $D(0, R)$ with poles of order 1 at $z = 0$. For any fixed $R > 0$, we have

$$\left|\sin(nz)\right| \leq e^{n|z|}$$

by considering the power series expansions of sin and exp; and we also have

$$\left|\frac{1}{z^2 + n^2}\right| \leq \frac{1}{n^2 - |z|^2} \leq \frac{1}{n^2 - R^2} = \frac{1}{n^2} \times \frac{1}{1 - (R/n)^2} \leq \frac{4}{3n^2}$$

since $n > N > 2R$ (and so $1 - R^2/n^2 \geq 1 - 1/4 = 3/4$). Hence

$$\left|\frac{\sin(nz)}{n!(z^2 + n^2)}\right| \leq \frac{4 e^{n|z|}}{3n!n^2} =: M_n.$$

Since

$$\sum_{n=N+1}^{\infty} M_n = \frac{4}{3} \sum_{n=N+1}^{\infty} \frac{e^{nR}}{n!n^2}$$

is convergent converge by the ratio test, the series of function

$$\sum_{n=N+1}^{\infty} \frac{\sin(nz)}{n!(z^2 + n^2)}$$

is uniformly convergent. Since this argument is true for arbitrary $R > 0$, $f$ is a meromorphic on $\mathbb{C}$ with poles of order 1 at $z \in P_f$.

Let $R > 0$ be fixed but arbitrary. We will show that $g$ defines a meromorphic function on the $D(0, R)$. By (5.2), we must have $\lim_{n \to \infty} |z_n|^{-3} = 0$ and therefore $\lim_{n \to \infty} |z_n| = \infty$. So since the $z_n$’s are all distinct, there can only be finitely many $n$ such that $z_n \in D(0, R)$. Hence there exists $N \in \mathbb{N}$ such that $|z_n| > 2R$ for all $n > N$. Each $z_n \in D(0, R)$ is a pole of order 2 of $g$. We write

$$g(z) = \sum_{n=0}^{N} \left[ \frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right] + \sum_{n=N+1}^{\infty} \left[ \frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right].$$

The first sum contains all the poles of $g$ in $D(0, R)$. We claim that the second sum is analytic on $D(0, R)$. For $n > N$,

$$\left|\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2}\right| = \left|\frac{z_n^2 - 2zz_n}{z_n^2(z - z_n)^2}\right| = \frac{1}{|z_n|^3} \frac{|z^2 + 2z|}{|z_n| |z_n - 1|^2} \leq \frac{1}{|z_n|^3} \frac{R^2/2R + 2R}{1 - R/2R} = \frac{5R}{|z_n|^3} =: M_n.$$
Since
\[ \sum_{n=N+1}^{\infty} M_n = 5R \sum_{n=N+1}^{\infty} \frac{1}{|z_n|^\beta} < \infty. \]

The Weierstrass M-test implies that the second sum converges uniformly and is thus analytic on $D(0, R)$. So $g$ is meromorphic on $D(0, R)$. Since $R$ is arbitrary, $g$ is meromorphic on $\mathbb{C}$.