1. Let $a \in \mathbb{R}$ and $z \in \mathbb{C}$.
   (a) Evaluate the following integrals
   $$\int_0^1 e^{it} \cos at \, dt \quad \text{and} \quad \int_{-1}^1 \frac{dt}{t^2 + i}.$$  

   (b) Show that if $\text{Re} \, z > -1$, then the integral $\int_0^1 t^z \, dt$ exists and
   $$\left| \int_0^1 t^z \, dt \right| \leq \frac{1}{1 + \text{Re} \, z}.$$  

   (c) Show that if $a < 1$, then
   $$\left| \int_{-1}^1 \frac{\cos it}{t^a} \, dt \right| \leq 2 \int_{-1}^1 \frac{dt}{t^a}$$  
   and thus the (improper) integral $\int_{-1}^1 t^{-a} \cos it \, dt$ converges absolutely.

2. (a) For $k = 1, 2, 3$, evaluate the following integrals
   $$\int_{\Gamma_k} \text{Re}(z) \, dz, \quad \int_{\Gamma_k} z^2 \, dz, \quad \int_{\Gamma_k} \frac{dz}{z}$$  
   along the curves from the point $z_0 = 1$ to $z_1 = i$ in the counter clockwise direction as described in Figure 1.

   ![Figure 1](image)
   
   **Figure 1.** *Left:* $\Gamma_1$ is along the boundary of the square: $\{x + iy \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$. *Center:* $\Gamma_2$ is along the boundary of the circle: $\{e^{it} \mid 0 \leq t \leq \pi/2\}$. *Right:* $\Gamma_3$ is along the line segment: $\{(1 - t) + it \mid 0 \leq t \leq 1\}.$

   (b) Let $a, b > 0$. By considering a path along the ellipse $\{a \cos t + ib \sin t \mid 0 \leq t \leq 2\pi\}$ or otherwise, show that
   $$\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}.$$  
   *(Hint: Recall from lecture that $\int_{\partial D(0,1)} z^{-1} \, dz = 2\pi i$ and emulate a trick we used in the proof of Cauchy’s theorem.)*
(c) Evaluate the following integrals

\[ \int_{D_1} |z| \overline{z} \, dz \quad \text{and} \quad \int_{D_2} \overline{z} \, dz \]

where \(D_1\) and \(D_2\) are the curves in Figure 2.

Figure 2. Left: \(D_1\) is a closed curve in the counter clockwise direction along the semicircle \(\{e^{it} \mid 0 \leq t \leq \pi\}\) and the line segment \(\{z \mid -R \leq \text{Re} z \leq R, \text{Im} z = 0\}\).
Right: \(D_2\) is a closed curve in the counter clockwise direction along the semicircle \(\{e^{it} \mid -\pi/2 \leq t \leq \pi/2\}\) and the line segment \(\{z \mid -1 \leq \text{Im} z \leq 1, \text{Re} z = 0\}\).

3. Let \(\Gamma\) be a smooth curve.
   (a) Suppose \(\Gamma = [a, b]\). Prove that for any integrable function \(f : [a, b] \to \mathbb{C}\),

   \[ \int_a^b f(t) \, dt = \int_a^b f(t) \, dt. \]

   What can you deduce about the integrability of \(\overline{f}\) if \(f\) is integrable?
   (b) Let \(f\) be a function that is continuous on \(\Gamma\). Consider \(\overline{\Gamma}\), the image of \(\Gamma\) under complex conjugation \(z \mapsto \overline{z}\), i.e. \(\Gamma\) reflected about the real axis. Show that the function \(z \mapsto \overline{f(z)}\) is also continuous on \(\overline{\Gamma}\) and

   \[ \int_{\overline{\Gamma}} f(z) \, dz = \int_{\Gamma} \overline{f(z)} \, dz. \]

   (c) Suppose \(\Gamma\) is the positively oriented (i.e. going counter clockwise) circle \(z : [0, 2\pi] \to \mathbb{C}\), \(z(t) = e^{it}\). Show that

   \[ \int_{\Gamma} f(z) \, dz = -\int_{\Gamma} \frac{f(z)}{z} \, dz. \]

4. Let \(r > 0\) and \(\Gamma\) be the curve \(z : [0, 2\pi] \to \mathbb{C}\), \(z(t) = re^{it}\). Let \(f : \overline{D(0, r)} \to \mathbb{C}\) be continuous.
   (a) For \(n \in \mathbb{N}\), let \(\Gamma_n\) be the curve \(z_n : [0, 2\pi] \to \mathbb{C}\), \(z_n(t) = (1 - 1/n)re^{it}\). Prove that

   \[ \int_{\Gamma_n} f(z) \, dz = \lim_{n \to \infty} \int_{\Gamma_n} f(z) \, dz. \]

   (Hint: The result we discussed in class is of the form \(\int_{\Gamma} f = \lim_{n \to \infty} \int_{\Gamma_n} f_n\). Find a way to use this.)
   (b) Show that

   \[ \lim_{r \to 0} \int_0^{2\pi} f(re^{it}) \, dt = 2\pi f(0) \quad \text{and} \quad \lim_{r \to 0} \int_{\Gamma} \frac{f(z)}{z} \, dz = 2\pi i f(0). \]

   (Hint: Emulate the proof of the aforementioned result.)
5. Let $R > 0$ and $f : D(0, R) \to \mathbb{C}$ be analytic.
   (a) Suppose at least one of the following four conditions is true
   (i) $\Re f'(z) > 0$ for all $z \in D(0, R)$;
   (ii) $\Re f'(z) < 0$ for all $z \in D(0, R)$;
   (iii) $\Im f'(z) > 0$ for all $z \in D(0, R)$;
   (iv) $\Im f'(z) < 0$ for all $z \in D(0, R)$.
   Show that $f$ is injective on $D(0, R)$.

   (b) Suppose

   $[\Re f'(z)] [\Im f'(z)] \neq 0$

   for all $z \in D(0, R)$. Show that $f$ is injective on $D(0, R)$. 