For a real-valued function of two real variables, \( u : \Omega_\mathbb{R} \to \mathbb{R} \), we say that \( u \) is \textit{twice continuously differentiable} if all second-order partial derivatives \( u_{xx}, u_{yy}, u_{xy}, u_{yx} \) exist and are continuous on \( \Omega_\mathbb{R} \). The set of all twice continuously differentiable functions on \( \Omega_\mathbb{R} \) is denoted \( C^2(\Omega_\mathbb{R}) \).

1. We mentioned Tauberian theorems in class. Here is an example of an easy one (easy relative to other Tauberian theorems). Let \( \sum_{n=0}^{\infty} a_n z^n \) be a power series with radius of convergence 1 and suppose 
\[
\lim_{n \to \infty} n a_n = 0.
\]

(a) Show that
\[
\lim_{m \to \infty} \sum_{n=0}^{m} n |a_n| = 0.
\]

(Hint: Problem 4(a), Problem Set 3, Math 104, Spring 2009.)

Solution. Let \( \varepsilon > 0 \) be given. Since \( \lim_{n \to \infty} n a_n = 0 \), there exists \( N_1 \in \mathbb{N} \) such that
\[
|a_1| + 2|a_2| + \cdots + N_1|a_{N_1}| < \frac{\varepsilon}{2}
\]
whenever \( n > N_1 \). Now by the Archimedean property, there exists \( N_2 \in \mathbb{N} \) such that
\[
\frac{|a_1| + 2|a_2| + \cdots + N_1|a_{N_1}|}{N_2} < \frac{\varepsilon}{2}
\]
Hence
\[
\frac{|a_1| + 2|a_2| + \cdots + N_1|a_{N_1}|}{m} < \frac{\varepsilon}{2}
\]
whenever \( m > N_2 \). Now for \( m > \max\{N_1, N_2\} \),
\[
\sum_{n=0}^{m} n |a_n| = \sum_{n=0}^{N_1} n |a_n| + \sum_{n=N_1+1}^{m} n |a_n| < \varepsilon + \frac{m - N_1}{2} \varepsilon < \varepsilon.
\]
Hence
\[
\lim_{m \to \infty} \sum_{n=0}^{m} n |a_n| = 0.
\]

(b) Define a function \( f \) by
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } |z| < 1.
\]

Let \( x \) be a real variable and suppose the following left limit exists
\[
\lim_{x \to 1^-} f(x) = A.
\]

Show that the series \( \sum_{n=0}^{\infty} a_n \) converges to \( A \).
SOLUTION. Let \( x \in \mathbb{R} \) and \( 0 \leq x < 1 \). We have
\[
\left| \sum_{n=0}^{m} a_n - f(x) \right| = \left| \sum_{n=0}^{m} a_n(1 - x^n) - \sum_{n=m+1}^{\infty} a_n x^n \right|
\leq (1 - x) \sum_{n=0}^{m} |a_n| (1 + x + \cdots + x^{n-1}) + \sum_{n=m+1}^{\infty} |a_n| x^n
\leq m(1 - x) \sum_{n=0}^{m} |a_n| + \sum_{n=m+1}^{\infty} n |a_n| \frac{x^n}{n}.
\]
Let \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} n a_n = 0 \), there exists \( M_1 \in \mathbb{N} \) such that
\[
m |a_m| < \frac{\varepsilon}{2}
\]
whenever \( m > M_1 \). By (a), there exists \( M_2 \in \mathbb{N} \) such that
\[
\sum_{n=0}^{m} n |a_n| < \frac{\varepsilon}{2}
\]
for \( m > M_2 \). Hence when \( m > \max\{M_1, M_2\} \),
\[
\left| \sum_{n=0}^{m} a_n - f(x) \right| < m(1 - x) \frac{\varepsilon}{2} + \frac{\varepsilon}{2m} \sum_{n=m+1}^{\infty} x^n
= \frac{\varepsilon}{2} \left[ m(1 - x) + \frac{1}{m} \left( \frac{x^{m+1}}{1 - x} \right) \right]
< \frac{\varepsilon}{2} \left[ m(1 - x) + \frac{1}{m(1 - x)} \right].
\]
Then for \( 1 - x = 1/m \) and \( m > \max\{M_1, M_2\} \), we get
\[
\left| \sum_{n=0}^{m} a_n - f \left( \frac{1}{m} \right) \right| < \varepsilon,
\]
which implies that
\[
\lim_{m \to \infty} \sum_{n=0}^{m} a_n = \lim_{m \to \infty} f \left( \frac{1}{m} \right).
\]
Since \( \lim_{x \to 1^-} f(x) = A \) exists, we must have
\[
\lim_{m \to \infty} f \left( \frac{1}{m} \right) = A
\]
and so we get
\[
\sum_{n=0}^{\infty} a_n = A.
\]

2. Recall that \( \mathbb{C} \) is both a real vector space of dimension 2 and a complex vector space of dimension 1. A function \( \varphi : \mathbb{C} \to \mathbb{C} \) is called \( \mathbb{R} \)-linear if \( \varphi \) is a linear transformation of real vector spaces, ie.
\[
\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2) \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } z_1, z_2 \in \mathbb{C}. \quad (2.1)
\]
It is called \( \mathbb{C} \)-linear if \( \varphi \) is a linear transformation of complex vector spaces, ie.
\[
\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2) \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C} \text{ and } z_1, z_2 \in \mathbb{C}. \quad (2.2)
\]
(a) Prove that if \( \varphi \) is \( \mathbb{C} \)-linear, then it is \( \mathbb{R} \)-linear. Give an example to show that the converse is false.

SOLUTION. This is obvious since \( \mathbb{R} \subset \mathbb{C} \) and so (2.1) is a special case of (2.2). For a counterexample to the converse, consider the complex conjugate function, \( \varphi : \mathbb{C} \to \mathbb{C} \),
\(\varphi(z) = \overline{z}\). For \(\lambda_1, \lambda_2 \in \mathbb{R}\),
\[
\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \overline{\lambda_1 z_1 + \lambda_2 z_2} = \overline{\lambda_1 \overline{z}_1 + \lambda_2 \overline{z}_2} = \lambda_1 \overline{z}_1 + \lambda_2 \overline{z}_2 = \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2)
\]
and so \(\varphi\) is \(\mathbb{R}\)-linear. However, for \(\lambda_1 = i, z_1 = 1, \lambda_2 = z_2 = 0\), we see that
\[
\varphi(i) = -i \neq i = i \varphi(1)
\]
and so it is not \(\mathbb{C}\)-linear.

(b) Let \(\varphi : \mathbb{C} \rightarrow \mathbb{C}\). Prove that the following statements are equivalent.
(i) \(\varphi\) is \(\mathbb{R}\)-linear.
(ii) \(\varphi\) satisfies
\[
\varphi(z) = \varphi(1)x + \varphi(i)y
\]
for all \(z = x + iy \in \mathbb{C}\).
(iii) \(\varphi\) satisfies
\[
\varphi(z) = \left[\frac{\varphi(1) - i\varphi(i)}{2}\right] z + \left[\frac{\varphi(1) + i\varphi(i)}{2}\right] \overline{z}
\]
for all \(z = x + iy \in \mathbb{C}\).
(iv) \(\varphi\) is given by
\[
\varphi(x + iy) = (ax + by) + i(cx + dy)
\]
for some \([a, b, c, d] \in \mathbb{R}^{2 \times 2}\).

Solution. (i) \(\Rightarrow\) (ii): If we let \(\lambda_1 = x, z_1 = 1, \lambda_2 = y, z_2 = i\) in (2.1), we get
\[
\varphi(z) = \varphi(x + yi) = x\varphi(1) + y\varphi(i) = \varphi(1)x + \varphi(i)y
\]
as required.
(ii) \(\Rightarrow\) (iii): Note that if \(z = x + iy\), then \(x = (z + \overline{z})/2\) and \(y = (z - \overline{z})/2i\). Hence
\[
\varphi(z) = \varphi(1)x + \varphi(i)y
\]
\[
= \varphi(1) \left[\frac{z + \overline{z}}{2}\right] + \varphi(i) \left[\frac{z - \overline{z}}{2i}\right]
\]
\[
= \left[\frac{\varphi(1) - i\varphi(i)}{2}\right] z + \left[\frac{\varphi(1) + i\varphi(i)}{2}\right] \overline{z}.
\]
(iii) \(\Rightarrow\) (iv): Let \(a = \text{Re} \varphi(1), c = \text{Im} \varphi(1), b = \text{Re} \varphi(i), d = \text{Im} \varphi(i)\). Then
\[
\varphi(x + iy) = \left[\frac{\varphi(1) - i\varphi(i)}{2}\right] (x + iy) + \left[\frac{\varphi(1) + i\varphi(i)}{2}\right] (x - iy)
\]
\[
= \varphi(1)x + \varphi(i)y
\]
\[
= (a + ic)x + (b + id)y
\]
\[
= (ax + by) + i(cx + dy).
\]
(iv) \(\Rightarrow\) (i): With respect to the standard basis \(B = \{1, i\}\) of \(\mathbb{C}\) as a real vector space, (2.5) implies that \(\varphi\) has the matrix representation
\[
[\varphi]_{B,B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
and is thus a \(\mathbb{R}\)-linear function.

(c) Let \(\varphi : \mathbb{C} \rightarrow \mathbb{C}\). Prove that the following statements are equivalent.
(i) \(\varphi\) is \(\mathbb{C}\)-linear.
(ii) \(\varphi\) is \(\mathbb{R}\)-linear and \(\varphi(i) = i\varphi(1)\).
3. Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$. We will call $f$ **complex differentiable** at $z \in \Omega$ if it is differentiable as defined in the lectures, i.e. the limit
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]
exists. We will call $f$ **real differentiable** at $z \in \Omega$ if there exists a $\mathbb{R}$-linear function $\varphi : \mathbb{C} \to \mathbb{C}$ such that
\[
\lim_{h \to 0} \frac{f(z + h) - f(z) - \varphi(h)}{h} = 0.
\]

(a) Prove that if $f$ is complex differentiable at $z \in \Omega$, then $f$ is real differentiable at $z$.

**Solution.** Let the limit in (3.9) be $\alpha$. Then
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \alpha.
\]
Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $|h| < \delta$,
\[
\left| \frac{f(z + h) - f(z)}{h} - \alpha \right| < \varepsilon,
\]
that is
\[
\left| \frac{f(z + h) - f(z) - \alpha h}{h} \right| < \varepsilon.
\]
Let $\varphi : \mathbb{C} \to \mathbb{C}$ be defined by $\varphi(z) = \alpha z$ for all $z \in \mathbb{C}$. This is $\mathbb{C}$-linear and is thus $\mathbb{R}$-linear by (a). Hence (3.10) holds with this choice of $\varphi$.

(b) Give an example to show that the converse of (a) is false.

**Solution.** Let $g : \Omega \to \mathbb{C}$ be $g(z) = \bar{z}$. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be defined by $\varphi(x + iy) = x - iy$.

Note that $\varphi$ is $\mathbb{R}$-linear by (2.5) with $a = 1, b = c = 0, d = -1$. It easy to see that $g$ is real differentiable at any $z \in \Omega$ with respect to $\varphi$ since
\[
\lim_{h \to 0} \frac{g(z + h) - g(z) - \varphi(h)}{h} = \lim_{h \to 0} \frac{z + \bar{h} - \bar{z} - \bar{h}}{h} = 0.
\]
Now \( g \) is not complex differentiable since if we write \( h = \xi + i\eta \) and let \( h \to 0 \) along the lines \( \eta = 0 \) and \( \xi = 0 \), we get

\[
\lim_{\xi \to 0} \frac{g(z + \xi) - g(z)}{\xi} = \lim_{\xi \to 0} \frac{\bar{z} + \xi - \bar{z}}{\xi} = 1
\]

and

\[
\lim_{\eta \to 0} \frac{g(z + i\eta) - g(z)}{i\eta} = \lim_{\xi \to 0} \frac{\bar{z} - i\eta - \bar{z}}{i\eta} = -1.
\]

So the limit

\[
\lim_{h \to 0} \frac{g(z + h) - g(z)}{h}
\]

cannot exist.

(c) Let \( f \) be real differentiable at \( z \in \Omega \). If the \( \mathbb{R} \)-linear function \( \varphi : \mathbb{C} \to \mathbb{C} \) in (3.10) is also \( \mathbb{C} \)-linear, prove that \( f \) is complex differentiable at \( z \). In this case, how is \( \varphi \) related to the limit in (3.9)?

**SOLUTION.** If the \( \varphi \) is also \( \mathbb{C} \)-linear, then by Problem 2(c), we have

\[
\varphi(h) = \varphi(1)h.
\]

Then

\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{f(z + h) - f(z) - \varphi(h) + \varphi(h)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(z + h) - f(z) - \varphi(h)}{h} + \lim_{h \to 0} \frac{\varphi(h)}{h}
\]

\[
= 0 + \lim_{h \to 0} \frac{\varphi(1)h}{h} = \varphi(1).
\]

Hence the limit in (3.9) exists and so \( f \) is complex differentiable at \( z \). Note that this also gives the relation between \( \varphi \) and the limit in (3.9).

(d) Let \( f \) be real differentiable at \( z \in \Omega \). Show that if the limit

\[
\lim_{h \to 0} \left| \frac{f(z + h) - f(z)}{h} \right|
\]

exists\(^1\), then either \( f \) or \( \bar{f} \) must be complex differentiable at \( z \). Give an example to show that \( f \) is not necessarily complex differentiable at \( z \). Here the function \( \bar{f} : \Omega \to \mathbb{C} \) is defined by \( f(z) = \bar{f}(z) \) for all \( z \in \Omega \).

**SOLUTION.** Since \( f \) is real differentiable, there exists an \( \mathbb{R} \)-linear \( \varphi \) for which (3.10) is satisfied. By triangle inequality,

\[
0 \leq \left| \frac{f(z + h) - f(z)}{h} - \frac{\varphi(h)}{h} \right| = \left| \frac{f(z + h) - f(z) - \varphi(h)}{h} \right|.
\]

Since the limit of the RHS is 0, by Sandwich Lemma, the limit \( \lim_{h \to 0} |\varphi(h)|/|h| \) exists (and equals the limit in (3.11)). Since \( \varphi \) is \( \mathbb{R} \)-linear, by (2.4), we have

\[
\varphi(h) = \lambda h + \mu \bar{h}
\]

where

\[
\lambda := \frac{\varphi(1) - i\varphi(i)}{2} \quad \text{and} \quad \mu := \frac{\varphi(1) + i\varphi(i)}{2}.
\]

Now

\[
\left| \frac{\varphi(h)}{h} \right|^2 = \left| \frac{\lambda h + \mu \bar{h}}{h} \right|^2 = |\lambda|^2 + |\mu|^2 + 2 \Re \left( \lambda \bar{\mu} \frac{h}{h} \right).
\]

\(^1\)Note the difference between (3.9) and (3.11).
Since the limit of the LHS exists as $h \to 0$, the limit
\[
\lim_{h \to 0} \text{Re} \left( \frac{\lambda \bar{\mu} h}{h} \right)
\]
must also exist. Write $h = \xi + i\eta$. First we let $h \to 0$ along the lines $\eta = 0$ and $\xi = 0$ respectively, we get
\[
\lim_{\xi \to 0} \text{Re} \left( \lambda \bar{\mu} \left( \frac{\xi}{\xi} \right) \right) = \lim_{\eta \to 0} \text{Re} \left( \lambda \bar{\mu} \left( \frac{i\eta}{-i\eta} \right) \right)
\]
by the uniqueness of limit. This gives $\text{Re}(\lambda \bar{\mu}) = -\text{Re}(\lambda \bar{\mu})$ and thus
\[
\text{Re}(\lambda \bar{\mu}) = 0.
\] (3.13)

Now we let $h \to 0$ along the lines $\xi = \eta$ and $\xi = -\eta$ respectively, we get
\[
\lim_{\xi \to 0} \text{Re} \left( \lambda \bar{\mu} \left( \frac{\xi + i\xi}{\xi - i\xi} \right) \right) = \lim_{\eta \to 0} \text{Re} \left( \lambda \bar{\mu} \left( \frac{-\eta + i\eta}{-\eta - i\eta} \right) \right)
\]
by the uniqueness of limit. This gives $\text{Re}(i\lambda \bar{\mu}) = -\text{Re}(i\lambda \bar{\mu})$, ie. $-\text{Im}(\lambda \bar{\mu}) = \text{Im}(\lambda \bar{\mu})$, and thus
\[
\text{Im}(\lambda \bar{\mu}) = 0.
\] (3.14)

By (3.13) and (3.14),
\[
\lambda \bar{\mu} = 0,
\]
ie. we must either have $\lambda = 0$ or $\bar{\mu} = 0$. So by (3.12), either
\[
\varphi(1) = i\bar{\varphi}(i) \quad \text{or} \quad \bar{\varphi}(1) = i\bar{\varphi}(i).
\]

In the first case, $\varphi$ is $\mathbb{C}$-linear (by Problem 2(c)) and therefore $f$ must be complex differentiable (by Problem 3(c)). In the second case, $\bar{\varphi}$ is $\mathbb{C}$-linear and there $\bar{f}$ must be complex differentiable. In the second case, we also need to observe that if $f$ is real differentiable with respect to $\varphi$, then $f$ is real differentiable with respect to $\bar{\varphi}$ since
\[
\lim_{h \to 0} \frac{f(z + h) - f(z) - \varphi(h)}{h} = 0
\]
iff
\[
\lim_{h \to 0} \frac{|f(z + h) - f(z) - \varphi(h)|}{|h|} = 0
\]
iff
\[
\lim_{h \to 0} \frac{|f(z + h) - f(z) - \varphi(h)|}{|h|} = 0
\]
iff
\[
\lim_{h \to 0} \frac{\bar{f}(z + h) - \bar{f}(z) - \bar{\varphi}(h)}{h} = 0.
\]

(e) Show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by
\[
f(z) = \sqrt{|z^2 - \pi^2|}
\]
satisfies the Cauchy-Riemann equation at $z = 0$ but is not differentiable at $z = 0$.

SOLUTION. Note that $f$ is identically zero on the real and imaginary axes and so trivially satisfies the Cauchy-Riemann equation at $z = 0$, i.e.
\[
f_x(0) = \lim_{\xi \to 0, \xi \in \mathbb{R}} \frac{f(0 + \xi) - f(0)}{\xi} = \lim_{\xi \to 0, \xi \in \mathbb{R}} \frac{0 - 0}{\xi} = 0,
\]
\[
f_y(0) = \lim_{\eta \to 0, \eta \in \mathbb{R}} \frac{f(0 + i\eta) - f(0)}{\eta} = \lim_{\eta \to 0, \eta \in \mathbb{R}} \frac{0 - 0}{\eta} = 0
\]
and so $f_y(0) = -f_x(0)$. To see that it is not differentiable, take $h = r(\cos \theta + i \sin \theta)$ and take limit as $r \to 0$, we get
\[
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{r \to 0} \frac{f(0 + re^{i\theta}) - f(0)}{re^{i\theta}}
= \lim_{r \to 0} \frac{\sqrt{4 \cos \theta \sin \theta} - 0}{\cos \theta + i \sin \theta}
= \sqrt{\frac{4 \cos \theta \sin \theta}{\cos \theta + i \sin \theta}}.
\]
Since the last expression depends on $\theta$, taking $\theta = 0$ and $\theta = \pi/4$, we get two different values and thus the limit of the LHS does not exist.

(f) Let $\Omega \subseteq \mathbb{C}$ be a region such that the function
\[
f(x + iy) = |x^2 - y^2| + 2i|xy|
\]
is analytic on $\Omega$ but is not analytic on any larger region $\Omega'$ containing $\Omega$. Find all possible $\Omega$ with this property.

SOLUTION. The function $f$ is analytic in each of the following regions

\[
\Omega_1 = \{z \in \mathbb{C} \mid 0 < \arg(z) < \pi/4\}, \quad \Omega_2 = \{z \in \mathbb{C} \mid \pi < \arg(z) < 5\pi/4\},
\Omega_3 = \{z \in \mathbb{C} \mid \pi/2 < \arg(z) < 3\pi/4\}, \quad \Omega_4 = \{z \in \mathbb{C} \mid 3\pi/2 < \arg(z) < 7\pi/4\}.
\]
On $\Omega_1$ or $\Omega_2$, we have
\[
f(z) = z^2.
\]
On $\Omega_3$ or $\Omega_4$, we have
\[
f(z) = -z^2.
\]

(g) Find constants $a, b, c \in \mathbb{R}$ such that the functions $f, g : \mathbb{C} \to \mathbb{C}$ defined by
\[
f(x + iy) = x + ay + i(bx + cy),
g(x + iy) = \cos x(\cosh y + a \sinh y) + i \sin x(\cosh y + b \sinh y)
\]
are analytic on $\mathbb{C}$.

SOLUTION. Applying the Cauchy-Riemann equations
\[
u_x = v_y, \quad u_y = -v_x,
\]
we see that $c = 1$ and $b = -a$ in $f$ and so
\[
f(z) = (1 - ai)z.
\]
Likewise $a = b = -1$ in $g$ and so
\[
g(z) = e^{iz}.
\]

4. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \to \mathbb{C}$ be analytic and $u(x, y) = \text{Re} f(x + iy)$, $v(x, y) = \text{Im} f(x + iy)$.

(a) Suppose $u, v \in C^2(\Omega_R)$. Show that $u$ and $v$ are harmonic functions, i.e. solutions of the Laplace equation
\[
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,
\]
on $\Omega_R$.

SOLUTION. Taking partial derivatives of the Cauchy-Riemann equations
\[
u_x = v_y, \quad u_y = -v_x
\]
gives
\[
u_{xx} = v_{yx}, \quad u_{yy} = -v_{xx},
u_{xy} = v_{yxx}, \quad u_{yx} = -v_{xy}.
\]
Since the second order partial derivatives are continuous, we have that 
\( u_{xy} = u_{yx} \) and \( v_{xy} = v_{yx} \). Hence
\[
\begin{align*}
    u_{xx} + u_{yy} &= v_{yx} - v_{xy} = 0, \\
v_{xx} + v_{yy} &= -u_{yx} + u_{xy} = 0.
\end{align*}
\]

(b) Let \( a \in \mathbb{R} \). Suppose \( f \) is analytic on \( D(0,1) \). Which of the following can occur as the real or imaginary part of \( f \)?
\[
\begin{align*}
x^2 - axy + y^2, & \quad x^3 - x^2 + y^3, & \quad x^2 + y^2 - 5x, & \quad \frac{x^2 - y^2}{(x^2 + y^2)^2}.
\end{align*}
\]

**SOLUTION.** Note that all these functions are in \( C^2(\mathbb{R}^2) \) and so the result in (a) applies. For \( w(x,y) = x^2 - axy + y^2 \), we have
\[
\begin{align*}
w_{xx} &= 2 \quad \text{and} \quad w_{yy} = 2
\end{align*}
\]
and so
\[
w_{xx} + w_{yy} = 2 + 2 \neq 0.
\]
Thus \( w \) cannot be the real or imaginary part of an analytic function since it is not harmonic. Likewise for \( x^3 - x^2 + y^3 \) and \( x^2 + y^2 - 5x \). The function \( (x^2 - y^2)/(x^2 + y^2)^2 \) is not even continuous at 0 and so not a candidate. But on the other hand, if we allow the point 0 to be excluded, then for
\[
u(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2},
\]
we see that
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{2x(3y^2 - x^2)}{(x^2 + y^2)^3}, & \frac{\partial u}{\partial y} &= -\frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= \frac{6}{(x^2 + y^2)^4} \left( x^4 - 6x^2y^2 + y^4 \right), & \frac{\partial^2 u}{\partial y^2} &= -\frac{6}{(x^2 + y^2)^4} \left( x^4 - 6x^2y^2 + y^4 \right)
\end{align*}
\]
and thus
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]
We want \( v(x,y) \) such that the Cauchy-Riemann equations
\[
\begin{align*}
    \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = \frac{2x(3y^2 - x^2)}{(x^2 + y^2)^3}, & \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = \frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}
\end{align*}
\]
are satisfied and by inspection we see that
\[
v(x,y) = \frac{-2xy}{(x^2 + y^2)^2}
\]
is a possible candidate. Since \( u \) and \( v \) are both continuously differentiable in \( C^\infty \), the function
\[
f(x,y) = \frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2}
\]
is analytic in \( C^\infty \) by Theorem 2.4 (partial converse of Cauchy-Riemann equations) in the lectures.

**5.** We may rewrite any complex function \( f \) of two real variables \( x \) and \( y \) as a function of \( z \) and \( \bar{z} \) via
\[
x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}.
\]
(a) Considering \( z \) and \( \overline{z} \) as independent variables, show that
\[
\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).
\]

**Solution.** Treating \( z \) and \( \overline{z} \) as independent variables, the differentiation rules give, formally
\[
\frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = -\frac{i}{2}, \quad \frac{\partial y}{\partial \overline{z}} = \frac{i}{2}
\]
and the chain rule then implies that
\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right),
\]
\[
\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).
\]

(b) Show that the Cauchy-Riemann equation may be expressed as
\[
\frac{\partial f}{\partial \overline{z}} = 0.
\]
This may be interpreted as saying that complex differentiable functions must be independent\(^2\) of \( \overline{z} \) and depend only on \( z \).

**Solution.** By (a),
\[
\frac{\partial f}{\partial z} = 0
\]
iff
\[
\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0
\]
iff
\[
\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}
\]
which is one form of the Cauchy-Riemann equations that we have shown in lectures.

(c) Which of the following complex functions of two real variables can be expressed in terms of a polynomial in \( z = x + iy \)?
\[
f_1(x, y) = x^2 - y^2 - ixy, \quad f_2(x, y) = x^2 + y^2 - 2i xy.
\]

**Solution.** By (b), a complex function can be expressed in terms of a polynomial in \( z \) (independent of \( \overline{z} \)) iff it satisfies the Cauchy-Riemann equations. Write \( f_1 = u + iv \), since
\[
u_x = 2x \neq -x = v_y,
\]
\( f_1 \) cannot be expressed in terms of \( z \) only. Now write \( f_2 = u + iv \), since
\[
u_x = 2x \neq -2x = v_y,
\]
\( f_2 \) cannot be expressed in terms of \( z \) only either.

\(^2\)In fact you may also view this as a reason why there isn’t a ‘quaternion analysis’ similar to complex analysis. For a quaternion \( q = x + yi + zj + wk \), its quaternionic conjugate \( \overline{q} = x - yi - zj - wk \) can always be expressed in terms of \( q \):
\[
\overline{q} = -\frac{1}{2} (q + iqi + jqj + kqk),
\]
and so we don’t have functions dependent on \( q \) but not on \( \overline{q} \).