1. Let \((z_n)_{n=1}^\infty\) be a sequence of complex numbers.

(a) Show that if \(\lim_{n \to \infty} z_n = z\), then \(\lim_{n \to \infty} |z_n| = |z|\) but that the converse is not true in general.

**Solution.** We will first prove the inequality

\[ ||u| - |v|| \leq |u - v| \]

for \(u, v \in \mathbb{C}\). Since \(u = (u - v) + v\), we have \(|u| \leq |u - v| + |v|\) and so

\[ |u| - |v| \leq |u - v|. \]

Since \(v = (v - u) + u\), we have \(|v| \leq |v - u| + |u|\) and so

\[ |v| - |u| \leq |u - v|. \]

Hence

\[ -|u - v| \leq |u| - |v| \leq |u - v| \]

as required.

Since \(\lim_{n \to \infty} z_n = z\), we have that for every \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(|z_n - z| < \varepsilon\) whenever \(n > N\). Now using the inequality that we proved, we see that

\[ ||z_n| - |z|| \leq |z_n - z| < \varepsilon \]

whenever \(n > N\). Hence \(\lim_{n \to \infty} |z_n| = |z|\) as required.

The converse is not true. Let \(z_n = (-1)^n\). Then \(\lim_{n \to \infty} |z_n| = 1\) but \(\lim_{n \to \infty} z_n\) does not exist.

(b) Is it true that if \(\lim_{n \to \infty} z_n = z\), then \(\lim_{n \to \infty} \arg(z_n) = \arg(z)\)?

**Solution.** No. Let \(z_n = -1 + (-1)^n/2n\). Then \(\lim_{n \to \infty} z_n = -1\) but since \(\arg(z_{2n}) = \pi - \tan^{-1} 1/2n\) and \(\arg(z_{2n+1}) = -\pi + \tan^{-1} 1/(2n + 1)\), \(\lim_{n \to \infty} \arg(z_n)\) does not exist.

[Note: I take \(\arg(z)\) to be the angle that \(z\) makes with the positive real axis; if you use some other conventions, you could construct a similar counter example].

(c) Show that if \(\lim_{n \to \infty} |z_n| = r\) and \(\lim_{n \to \infty} \arg(z_n) = \theta\), then \(\lim_{n \to \infty} z_n = re^{i\theta}\).

**Solution.** Let \(z_n = x_n + iy_n\). Then

\[ x_n = \Re z_n = |z_n| \cos \arg(z_n), \quad y_n = \Im z_n = |z_n| \sin \arg(z_n). \]

Since \(\cos\) and \(\sin\) are continuous functions,

\[ \lim_{n \to \infty} x_n = \lim_{n \to \infty} |z_n| \times \lim_{n \to \infty} \cos \arg(z_n) = r \cos \theta, \]

\[ \lim_{n \to \infty} y_n = \lim_{n \to \infty} |z_n| \times \lim_{n \to \infty} \sin \arg(z_n) = r \sin \theta. \]

Hence

\[ \lim_{n \to \infty} z_n = r(\cos \theta + i \sin \theta) = re^{i\theta}. \]
2. Which of the following limit exists? Prove your answer.

\[ \lim_{n \to \infty} \left( \frac{1+i}{1-i} \right)^n, \quad \sum_{n=1}^{\infty} i^n \log \left( \frac{n}{n+1} \right), \quad \lim_{z \to 1} \frac{1 - \overline{z}}{1 - z}. \]

**SOLUTION.** Note that

\[ \left( \frac{1+i}{1-i} \right)^n = \begin{cases} 
1 & \text{if } n \equiv 0 \mod 4, \\
i & \text{if } n \equiv 1 \mod 4, \\
-1 & \text{if } n \equiv 2 \mod 4, \\
-i & \text{if } n \equiv 3 \mod 4.
\]

Since limit of a sequence, if exists, must be unique (Why?), \( \lim_{n \to \infty} \left( \frac{1+i}{1-i} \right)^n \) doesn’t exist.

Note that the usual way of summing a geometric progression yields

\[ \sum_{n=1}^{m} i^n = 1 - \frac{i^{m+1}}{1-i} = \begin{cases} 
1 & \text{if } m \equiv 0 \mod 4, \\
1+i & \text{if } m \equiv 1 \mod 4, \\
i & \text{if } m \equiv 2 \mod 4, \\
0 & \text{if } m \equiv 3 \mod 4,
\]

and so

\[ \left| \sum_{n=1}^{m} i^n \right| \leq \sqrt{2} \text{ for all } m \in \mathbb{N}. \]

Since we also have

\[ \left| \log \left( \frac{n}{n+1} \right) \right| = \left| -\log \left( \frac{n}{n+1} \right) \right| = \left| \log \left( \frac{1}{n+1} \right) \right| \]

converges monotonically to 0 as \( n \to \infty \), the series converges.

Note that for \( z = 1 - x \) where \( x \in \mathbb{R} \),

\[ \lim_{z \to 1} \frac{1 - \overline{z}}{1 - z} = \lim_{x \to 0} \frac{1 - 1 + x}{1 - 1 + x} = 1, \]

but for \( z = 1 + iy \) where \( y \in \mathbb{R} \),

\[ \lim_{z \to 1} \frac{1 - \overline{z}}{1 - z} = \lim_{y \to 0} \frac{1 - 1 - iy}{1 - 1 + iy} = -1. \]

Since the limit of a function, if exists, must be unique (Why?), \( \lim_{z \to 1}(1 - \overline{z})/(1 - z) \) doesn’t exist.

3. Let \( \Omega \subseteq \mathbb{C} \) be a region. Let \( f : \Omega \to \mathbb{C} \) and \( z_0 \in \Omega \).

(a) Suppose \( \lim_{z \to z_0} f(z) = w \). Prove that

\[ \lim_{z \to z_0} \overline{f(z)} = \overline{w}, \quad \lim_{z \to z_0} \text{Re } f(z) = \text{Re } w, \quad \lim_{z \to z_0} \text{Im } f(z) = \text{Im } w, \quad \lim_{z \to z_0} |f(z)| = |w|. \]

**SOLUTION.** All we need to show is that the following functions \( \psi, \rho, \iota, \mu : \mathbb{C} \to \mathbb{C} \) are all continuous on \( \mathbb{C} \),

\[ \psi(z) = \overline{z}, \quad \rho(z) = \text{Re } z, \quad \iota(z) = \text{Im } z, \quad \mu(z) = |z|, \]

and the required limits then follows from composing these functions with \( f \), i.e. \( \psi \circ f, \rho \circ f, \iota \circ f, \mu \circ f \). But the required continuity (in fact uniform continuity) on \( \mathbb{C} \) follows easily
from the equalities/inequalities

\[ |z - z_0| = |\overline{z} - \overline{z}_0| = |z - z_0|, \]
\[ |\text{Re } z - \text{Re } z_0| = |\text{Re}(z - z_0)| \leq |z - z_0|, \]
\[ |\text{Im } z - \text{Im } z_0| = |\text{Im}(z - z_0)| \leq |z - z_0|, \]
\[ ||z| - |z_0|| \leq |z - z_0|. \]

(b) Suppose \( \lim_{z \to z_0} |f(z)| = |w| \). For which value of \( w \) is it always true that \( \lim_{z \to z_0} f(z) = w \)? You will need to prove that it is true for that value and false for all other values.

**SOLUTION.** It is always true if \( w = 0 \): For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[ |f(z) - 0| = |f(z)| = ||f(z)| - 0|| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta. \]
We will show that this is the only case in general. Suppose \( \lim_{z \to z_0} f(z) = |w| > 0 \). If \( \lim_{z \to z_0} f(z) = v \) for some \( v \in \mathbb{C}^\times \), then by part (a), we must have \( |v| = |w| \). We shall examine when equality is attained in

\[ ||f(z)| - |w|| = ||f(z)| - |v|| \leq |f(z) - v| \]

(since then \( \varepsilon > ||f(z)| - |w|| = |f(z) - v| \) for \( |z - z_0| < \delta(z) \)). The equality would imply
\[ (||f(z)| - |v||)^2 = (|f(z) - v|)(|f(z)| - |v|) = |f(z)|^2 + |v|^2 - 2 \text{Re } f(z)v, \]
and thus
\[ \text{Re } f(z)v = |f(z)v| = |f(z)|v, \]
which implies that
\[ \text{Re } f(z)v \geq 0 \quad \text{and} \quad \text{Im } f(z)v = 0. \]

In other words \( f(z)v \) have to be real and nonnegative for all \( z \in \mathbb{C} \), which is clearly impossible\(^1\) for a general complex function \( f \).

4. The functions \( f, g, h : \mathbb{C} \to \mathbb{C} \) are defined as follows

\[ f(z) = \begin{cases} \frac{\text{Re}(z)}{z} & \text{if } z \neq 0, \\ \alpha & \text{if } z = 0, \end{cases} \quad g(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ \beta & \text{if } z = 0, \end{cases} \quad h(z) = \begin{cases} \frac{z \text{Re}(z)}{|z|} & \text{if } z \neq 0, \\ \gamma & \text{if } z = 0, \end{cases} \]

where \( \alpha, \beta, \gamma \in \mathbb{C} \) are constants. Show that \( f, g, h \) are continuous on \( \mathbb{C}^\times \). Are there values of \( \alpha, \beta, \gamma \) for which \( f, g, h \) are continuous on \( \mathbb{C} \)?

**SOLUTION.** As noted in the solution of Problem 3(a), \( z \mapsto |z| \) and \( z \mapsto \text{Re } z \) are both continuous functions on \( \mathbb{C} \); clearly so is the identity function \( z \mapsto z \). Hence the product of two continuous functions \( z \mapsto z \text{Re } z \) is continuous on \( \mathbb{C} \) and the quotient of continuous functions \( f, g, h \) are continuous when the denominator is non-zero, i.e. on \( \mathbb{C}^\times \). For \( f, g, h \) to be continuous at \( z = 0 \), we must have

\[ \lim_{z \to 0} \frac{\text{Re } z}{z} = \alpha, \quad \lim_{z \to 0} \frac{z}{|z|} = \beta, \quad \lim_{z \to 0} \frac{z \text{Re } z}{|z|} = \gamma. \]

Note that the first two limits do not exist:
\[ \lim_{z \to 0} \frac{\text{Re } z}{z} \neq 1 = \lim_{z \to 0} \frac{\text{Re } z}{z}, \quad \lim_{z \to 0} \frac{z}{|z|} = 1 \neq -1 = \lim_{z \to 0} \frac{z}{|z|}. \]

The last limit is 0 since
\[ \left| \frac{z \text{Re } z}{|z|} \right| = |\text{Re } z| \leq |z| = |z - 0| \]

and we may pick \( \delta = \varepsilon \).

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\(^1\)This is possible if \( f(z) = v \) for all \( z \in \mathbb{C} \) or when \( f \) is real-valued with constant sign but the problem did not impose such assumptions on \( f \).
5. Let $f : \mathbb{C}^\times \to \mathbb{C}$ be the reciprocal function

$$f(z) = \frac{1}{z}.$$ 

Define the sequence of functions $(f_n)_{n=1}^{\infty}$, $f_n : \mathbb{C}^\times \to \mathbb{C}$, by

$$f_n(z) = \frac{1}{nz}.$$ 

Let $g : \mathbb{C}^\times \to \mathbb{C}$ be the zero function, i.e. $g(z) = 0$ for all $z \in \mathbb{C}^\times$. Let $\Omega = \{z \in \mathbb{C} \mid r \leq |z| \leq R\}$ where $0 < r < R < \infty$.

(a) Show that $f$ is continuous but not uniformly continuous on $\mathbb{C}^\times$.

**SOLUTION.** Let $\alpha \in \mathbb{C}^\times$ be fixed. We need to show that

$$\lim_{z \to \alpha} f(z) = f(\alpha).$$ 

Let $\varepsilon > 0$ be given. We want $\delta > 0$ so that when $|z - \alpha| < \delta$,

$$\left| \frac{1}{z} - \frac{1}{\alpha} \right| = \frac{1}{|z\alpha|}|z - \alpha| < \varepsilon.$$ 

Note that the denominator becomes large when $z$ is near 0, so we will first want to pick $\delta$ to prevent that. The standard trick to do this is to pick

$$\delta \leq \frac{|\alpha|}{2}$$ 

since when

$$|z - \alpha| < \frac{|\alpha|}{2},$$

then by the triangle inequality,

$$|z| > \frac{|\alpha|}{2}$$

and so

$$\left| \frac{1}{z} - \frac{1}{\alpha} \right| = \frac{1}{|z\alpha|}|z - \alpha| < \frac{2}{|\alpha|^2}|z - \alpha|.$$ 

However, we also want the last term above to be not more than $\varepsilon$ and so we need to pick

$$\delta \leq \frac{\varepsilon|\alpha|^2}{2}.$$ 

(5.2)

Now for (5.1) and (5.2) to be both satisfied, we let

$$\delta = \min \left\{ \frac{|\alpha|}{2}, \frac{\varepsilon|\alpha|^2}{2} \right\}.$$ 

Hence when $|z - \alpha| < \delta$,

$$\left| \frac{1}{z} - \frac{1}{\alpha} \right| = \frac{1}{|z\alpha|}|z - \alpha| < \frac{2}{|\alpha|^2}|z - \alpha| \leq \varepsilon.$$ 

To see that $f$ is not uniformly continuous on $\mathbb{C}^\times$, let $\varepsilon = 1/2$. For any $\delta > 0$, pick $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \delta$$

and let

$$z = \frac{1}{n}, \quad w = \frac{1}{n+1}.$$ 

Then

$$|z - w| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \delta.$$
but
\[ |f(z) - f(w)| = \left| \frac{1}{z} - \frac{1}{w} \right| = 1 > \varepsilon. \]

Hence for \( \varepsilon = 1/2 \), there is no \( \delta > 0 \) such that will satisfy the requirement for uniform continuity.

(b) Show that \( f \) is uniformly continuous on \( \Omega \).
Solution. Let \( \varepsilon > 0 \) be given. We want \( \delta > 0 \) so that when \( z, w \in \Omega \) satisfies \( |z - w| < \delta \), we will have
\[ \left| \frac{1}{z} - \frac{1}{w} \right| = \frac{1}{|zw|} |z - w| < \varepsilon. \]
Note that \( z, w \in \Omega \) implies that \( |z| \geq r \) and \( |w| \geq r \). So
\[ \frac{1}{|zw|} \leq \frac{1}{r^2}. \]
Hence we just need to choose a \( \delta \) so that whenever \( |z - w| < \delta \), we will have
\[ \frac{1}{r^2} |z - w| < \varepsilon. \]
It is clear that
\[ \delta = r^2 \varepsilon \] (5.3)

(c) Show that \( f_n \) converges pointwise but not uniformly to \( g \) on \( \mathbb{C}^\times \).
Solution. Let \( \varepsilon > 0 \) be given. Let \( \alpha \in \mathbb{C}^\times \) be fixed. We want \( N \in \mathbb{N} \) so that when \( n > N \),
\[ |f_n(\alpha) - g(\alpha)| = \left| \frac{1}{n\alpha} - 0 \right| = \left| \frac{1}{n\alpha} \right| = \left| \frac{1}{n|\alpha|} \right| < \varepsilon. \]
It is clear that
\[ N = \left\lceil \frac{1}{\varepsilon |\alpha|} \right\rceil \] (5.4)
will achieve this. As a sanity check, observe that when \( n > N \),
\[ |f_n(\alpha) - g(\alpha)| = \frac{1}{n|\alpha|} < \frac{1}{N|\alpha|} \leq \varepsilon. \]
To see that \( f_n \) does not converge uniformly to \( g \) on \( \mathbb{C}^\times \), let \( \varepsilon = 1/2 \). Note that for \( z = 1/n \in \mathbb{C}^\times \),
\[ |f_n(z) - g(z)| = \left| \frac{1}{nz} - 0 \right| = \left| \frac{1}{nz} \right| = 1 > \varepsilon. \]
Hence for \( \varepsilon = 1/2 \), there is no \( N \in \mathbb{N} \) that could give us \( |f_n(z) - g(z)| < 1/2 \) for all \( z \in \mathbb{C}^\times \) and all \( n > N \).

(d) Show that \( f_n \) converges uniformly to \( g \) on \( \Omega \).
Solution. Let \( \varepsilon > 0 \) be given. We want \( N \in \mathbb{N} \) so that when \( n > N \),
\[ |f_n(z) - g(z)| = \left| \frac{1}{nz} - 0 \right| = \left| \frac{1}{nz} \right| = \left| \frac{1}{n|z|} \right| < \varepsilon \]
for all \( z \in \Omega \). Note that \( z \in \Omega \) implies that \( |z| \geq r \). So
\[ \frac{1}{|z|} \leq \frac{1}{r}. \]
Hence we just need to choose an \( N \) so that whenever \( n > N \), we will have
\[ \frac{1}{nr} < \varepsilon. \]
It is clear that
\[ N = \left\lceil \frac{1}{\varepsilon r} \right\rceil \] (5.5)
will achieve this. As a sanity check, observe that when \( n > N \),
\[
|f_n(z) - g(z)| = \frac{1}{n|z|} < \frac{1}{Nr} \leq \varepsilon.
\]

**Remark.** Notice that in (5.2) and (5.4), the choice of \( \delta \) and \( N \) are dependent on the point \( \alpha \in \mathbb{C}^\times \) that we fixed at the beginning; but in (5.3) and (5.5), the choice of \( \delta \) and \( N \) are independent of any particular point of \( \Omega \).

6. Let \( R_a \) and \( R_b \) be the radii of convergence of
\[
\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n
\]
respectively.

(a) Show that the radii of convergence of
\[
\sum_{n=0}^{\infty} (a_n + b_n) z^n \quad \text{and} \quad \sum_{n=0}^{\infty} a_nb_n z^n
\]
are at least \( \min(R_a, R_b) \) and \( R_a R_b \) respectively.

**Solution.** The triangle inequality implies that for any \( m \in \mathbb{N} \),
\[
\sum_{n=0}^{m} |(a_n + b_n) z^n| \leq \sum_{n=0}^{m} |a_n z^n| + \sum_{n=0}^{m} |b_n z^n|.
\]
We have shown in the lectures that a power series is absolutely convergent within its radius of convergence and so if \( |z| \leq \min(R_a, R_b) \), then
\[
\sum_{n=0}^{m} |a_n z^n| \leq \sum_{n=0}^{\infty} |a_n z^n| =: M_a < \infty
\]
and
\[
\sum_{n=0}^{m} |b_n z^n| \leq \sum_{n=0}^{\infty} |b_n z^n| =: M_b < \infty.
\]
Hence
\[
\sum_{n=0}^{m} |(a_n + b_n) z^n| \leq M_a + M_b \tag{6.6}
\]
for all \( m \in \mathbb{N} \). Since the LHS of \( \sum_{n=0}^{\infty} a_n b_n z^n \) is a monotone increasing sequence bounded by the RHS, it must converge. In other words, \( \sum_{n=0}^{\infty} (a_n + b_n) z^n \) is absolutely convergent and thus convergent. Let \( R_c \) be its radius of convergence. We have shown in the lecture that the set of points for which \( \sum_{n=0}^{\infty} (a_n + b_n) z^n \) converge must be a subset of \( D(0, R_c) \), so
\[
D(0, \min(R_a, R_b)) \subseteq D(0, R_c)
\]
and so we must have
\[
R_c \geq \min(R_a, R_b).
\]
Let \( R_d \) be the radius of convergence of \( \sum_{n=0}^{\infty} a_nb_n z^n \). Recall from Math 104 the fact that the limit superior of a product is bounded by the product of the limit superiors. So
\[
\limsup_{n \to \infty} |a_nb_n|^{1/n} = \limsup_{n \to \infty} |a_n|^{1/n} |b_n|^{1/n} \leq \limsup_{n \to \infty} |a_n|^{1/n} \limsup_{n \to \infty} |b_n|^{1/n}.
\]
Hence \( R_d \geq R_a R_b \).
(b) Suppose $0 < R_a < \infty$ and $p > 0$. Find the radii of convergence of the following power series in terms of $R_a$ and $p$:
\[
\sum_{n=0}^{\infty} a_n^p z^n, \quad \sum_{n=0}^{\infty} n^p a_n z^n, \quad \sum_{n=0}^{\infty} n^p a_n z^n, \quad \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.
\]

SOLUTION. As we have discussed in the lecture, if $(x_n)_{n=1}^{\infty}$ is such that $x_n \in \mathbb{R}$ and $x_n > 0$ for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} \sqrt[n]{x_n} = \lim_{n \to \infty} x_{n+1}/x_n$ if the RHS exists.
\[
\limsup_{n \to \infty} |a_n|^{p/n} = \left( \limsup_{n \to \infty} |a_n|^{1/n} \right)^p = \frac{1}{R_a^p},
\]
\[
\limsup_{n \to \infty} |n^p a_n|^{1/n} = \lim_{n \to \infty} n^p/n \limsup_{n \to \infty} |a_n|^{1/n} = \lim_{n \to \infty} \frac{(n+1)^p}{n^p} \times \frac{1}{R_a} = \frac{1}{R_a},
\]
\[
\limsup_{n \to \infty} |n^p a_n|^{1/n} = \limsup_{n \to \infty} n/a_n \left( \liminf_{n \to \infty} n \right) \times \left( \limsup_{n \to \infty} |a_n|^{1/n} \right) = \infty,
\]
\[
\limsup_{n \to \infty} |v_n|^{1/n} = \lim_{n \to \infty} (1/n!) \times \limsup_{n \to \infty} |a_n|^{1/n} = \lim_{n \to \infty} \frac{n!}{(n+1)!} \times \frac{1}{R_a} = 0.
\]
So the radius of convergence of $g$ is $R$ and the radius of convergence of $h$ is $\infty$.

7. Use the power series representation of $\exp(z)$ for this problem.
(a) Prove that
\[
\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| \leq |e^z| - \sum_{k=0}^{n} \frac{|z|^k}{k!} \leq |z|^{n+1} e^{|z|}
\]
for all $n \in \mathbb{N}$. Hence deduce that
\[
|e^z - 1| \leq |e^z| - 1 \leq |z| e^{|z|}.
\]

SOLUTION. Noting that the power series representation of $e^z$ converges absolutely and uniformly to $e^z$ on $\overline{D(0, R)}$ for any $R > 0$ (Theorem 2.4 in lecture), we may write
\[
\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} = \left| e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!} \right|
\]
and also
\[
\left| e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!} \right| = \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} = |z|^{n+1} \sum_{m=0}^{\infty} \frac{|z|^m}{(n + m + 1)!} \leq |z|^{n+1} \sum_{m=0}^{\infty} \frac{|z|^m}{m!} = |z|^{n+1} e^{|z|}.
\]
The special case is obtained with $n = 0$.
(b) Suppose
\[
0 < \limsup_{n \to \infty} |a_n|^{1/n} < \alpha \leq \infty,
\]
show that there exists $\beta > 0$ such that
\[
\sum_{k=0}^{\infty} \frac{a_n}{n!} z^n \leq \beta e^{\alpha |z|}
\]
for all $z \in \mathbb{C}$.

SOLUTION. Since
\[
\limsup_{n \to \infty} |a_n|^{1/n} < \alpha,
\]
there exists an $N \in \mathbb{N}$ such that
\[
|a_n| \leq \alpha^n
\]
whenever $n > N$. Let
\[
\beta := \max\{|a_n|\alpha^{-n} \mid n = 0, \ldots, N\} + 1.
\]
So
\[
|a_n| \leq \beta \alpha^n
\]
for all $n \leq N$ (and clearly also for all $n > N$). Hence
\[
\left|\sum_{k=0}^{\infty} \frac{a_n}{n!} z^n\right| \leq \sum_{n=0}^{\infty} \frac{|a_n|}{n!} |z|^n \leq \beta \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n |z|^n = \beta e^{|z|}.
\]