MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 2

Throughout the problem set, \( i = \sqrt{-1} \); and whenever we write \( x + yi \), it is implicit that \( x, y \in \mathbb{R} \). For \( z \in \mathbb{C} \), recall that the argument \( \text{arg}(z) \), is any \( \theta \in \mathbb{R} \) such that \( z = |z|e^{i\theta} \). We write \( \mathbb{C}^\times := \mathbb{C} \setminus \{0\} \).

1. Let \((z_n)_{n=1}^\infty\) be a sequence of complex numbers.
   (a) Show that if \( \lim_{n \to \infty} z_n = z \), then \( \lim_{n \to \infty} |z_n| = |z| \) but that the converse is not true in general.
   (b) Is it true that if \( \lim_{n \to \infty} z_n = z \), then \( \lim_{n \to \infty} \text{arg}(z_n) = \text{arg}(z) \)?
   (c) Show that if \( \lim_{n \to \infty} |z_n| = r \) and \( \lim_{n \to \infty} \text{arg}(z_n) = \theta \), then \( \lim_{n \to \infty} z_n = re^{i\theta} \).

2. Which of the following limits exists? Prove your answers.
   \[ \lim_{n \to \infty} \left( \frac{1+i}{1-i} \right)^n, \quad \sum_{n=1}^{\infty} i^n \log \left( \frac{n}{n+1} \right), \quad \lim_{n \to \infty} \frac{1-z}{1-z}. \]

3. Let \( \Omega \subseteq \mathbb{C} \) be a region. Let \( f : \Omega \to \mathbb{C} \) and \( z_0 \in \Omega \).
   (a) Suppose \( \lim_{z \to z_0} f(z) = w \). Prove that
      \[ \lim_{z \to z_0} \overline{f(z)} = \overline{w}, \quad \lim_{z \to z_0} \text{Re}(f(z)) = \text{Re}(w), \quad \lim_{z \to z_0} \text{Im}(f(z)) = \text{Im}(w), \quad \lim_{z \to z_0} |f(z)| = |w|. \]
   (b) Suppose \( \lim_{z \to z_0} |f(z)| = |w| \). For which value of \( w \) is it always true that \( \lim_{z \to z_0} f(z) = w \)? You will need to prove that it is true for that value and false for all other values.

4. The functions \( f, g, h : \mathbb{C} \to \mathbb{C} \) are defined as follows
   \[ f(z) = \begin{cases} \frac{\text{Re}(z)}{z} & \text{if } z \neq 0, \\ \alpha & \text{if } z = 0, \end{cases} \quad g(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ \beta & \text{if } z = 0, \end{cases} \quad h(z) = \begin{cases} \frac{\text{Re}(z)}{|z|} & \text{if } z \neq 0, \\ \gamma & \text{if } z = 0, \end{cases} \]
   where \( \alpha, \beta, \gamma \in \mathbb{C} \) are constants. Show that \( f, g, h \) are continuous on \( \mathbb{C}^\times \). Are there values of \( \alpha, \beta, \gamma \) for which \( f, g, h \) are continuous on \( \mathbb{C} \)?

5. Let \( f : \mathbb{C}^\times \to \mathbb{C} \) be the reciprocal function
   \[ f(z) = \frac{1}{z}. \]
   Define the sequence of function \((f_n)_{n=1}^\infty, f_n : \mathbb{C}^\times \to \mathbb{C}\), by
   \[ f_n(z) = \frac{1}{nz}. \]
   Let \( g : \mathbb{C}^\times \to \mathbb{C} \) be the zero function, ie. \( g(z) = 0 \) for all \( z \in \mathbb{C}^\times \). Let \( \Omega = \{ z \in \mathbb{C} \mid r \leq |z| \leq R \} \) where \( 0 < r < R < \infty \).
   (a) Show that \( f \) is continuous but not uniformly continuous on \( \mathbb{C}^\times \).
   (b) Show that \( f \) is uniformly continuous on \( \Omega \).
   (c) Show that \( f_n \) converges pointwise but not uniformly to \( g \) on \( \mathbb{C}^\times \).
   (d) Show that \( f_n \) converges uniformly to \( g \) on \( \Omega \).

Date: September 17, 2009 (Version 1.0); due: September 25, 2009.
6. Let $R_a$ and $R_b$ be the radii of convergence of
$$\sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} b_n z^n$$
respectively.
(a) Show that the radii of convergence of
$$\sum_{n=0}^{\infty} (a_n + b_n) z^n \text{ and } \sum_{n=0}^{\infty} a_n b_n z^n$$
are at least $\min(R_a, R_b)$ and $R_a R_b$ respectively.
(b) Suppose $0 < R_a < \infty$ and $p > 0$. Find the radii of convergence of the following power series in terms of $R_a$ and $p$:
$$\sum_{n=0}^{\infty} a_n^p z^n, \sum_{n=0}^{\infty} n^p a_n z^n, \sum_{n=0}^{\infty} n^n a_n z^n, \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

7. Use the power series representation of exp($z$) for this problem.
(a) Prove that
$$\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| \leq |e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!}| \leq |z|^{n+1} e^{|z|}$$
for all $n \in \mathbb{N}$. Hence deduce that
$$|e^z - 1| \leq |e^{|z|} - 1| \leq |z| e^{|z|}.$$
(b) Suppose
$$0 < \limsup_{n \to \infty} |a_n|^{1/n} < \alpha < \infty,$$
show that there exists $\beta > 0$ such that
$$\left| \sum_{k=0}^{\infty} \frac{a_n}{n!} z^n \right| \leq \beta e^{\alpha |z|}$$
for all $z \in \mathbb{C}$. 

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