MATH 185: COMPLEX ANALYSIS
FALL 2008/09
PROBLEM SET 9

For $f : \Omega \to \mathbb{C}$ and $n \in \mathbb{N}$, recall that $g = f^n$ is the function defined by $g(z) = |f(z)|^2$ for all $z \in \Omega$. A function on $\Omega \subseteq \mathbb{C}$ is said to be **meromorphic** if it has only removable singularities or poles in $\Omega$ (i.e. no essential singularities or non-isolated singularities).

1. Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$. Suppose $g = f^2$ and $h = f^3$ are both analytic on $\Omega$. Show that $f$ is analytic on $\Omega$.

   **Solution.** Note that $g(z) = 0$ if and only if $h(z) = 0$. So the zeros of $g$ and $h$ are in common. Let $Z \subseteq \Omega$ be these common zeros. If $Z$ contains a limit point, then $g$ and $h$ are identically zero by the identity theorem and $f \equiv 0$ is of course analytic. We will assume that all zeros in $Z$ are isolated. Let $z_0 \in Z$. So there exist analytic functions $g_1$ and $h_1$ and some $\epsilon > 0$ such that for all $z \in D(z_0, \epsilon)$,

   
   \[
   g(z) = (z - z_0)^k g_1(z), \quad g_1(z_0) \neq 0, \\
   h(z) = (z - z_0)^l h_1(z), \quad h_1(z_0) \neq 0,
   \]

   where $k$ and $l$ are the orders of zero of $g$ and $h$ respectively at $z_0$. But $g^3 = f^6 = h^2$ and so

   \[
   (z - z_0)^{3k} g_1(z)^3 = (z - z_0)^{2l} h_1(z)^2.
   \]

   So we get

   \[
   \frac{g_1(z)^3}{h_1(z)^2} = (z - z_0)^{3k - 2l}.
   \]

   Since the LHS is non-zero for $z = z_0$, the RHS is also non-zero for $z = z_0$ and this is only possible if $3k - 2l = 0$. In other words, $l > k$, i.e. the order of zero of $h$ is larger than the order of zero of $g$. So $h/g$ has a removable singularity at $z_0$. Since this is true for all $z_0 \in Z$, $h/g$ may be extended to an analytic function $\tilde{f}$ on $\Omega$. But since for all $z \in \Omega \setminus Z$,

   \[
   \tilde{f}(z) = \frac{h(z)}{g(z)} = f(z),
   \]

   and $\Omega \setminus Z$ clearly has limit points, $\tilde{f} \equiv f$ by the identity theorem.

2. Is there a polynomial $p(z)$ such that $p(z)e^{1/z}$ is an entire function?

   **Solution.** By the Taylor expansion of $\exp$,

   \[
   e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n.
   \]

   Let $p(z) = a_d z^d + \cdots + a_0$. For $n > 0$, the coefficient of $z^{-n}$ in the expansion of $p(z)e^{1/z}$ is

   \[
   c_n := \frac{a_0}{n!} + \frac{a_1}{(n + 1)!} + \cdots + \frac{a_d}{(n + d)!}.
   \]

   So the Laurent expansion of $p(z)e^{1/z}$ near $0$ will have terms of the form $c_n z^{-n}$ where $a_n \neq 0$ and $n > 0$. If $r$ is the smallest nonnegative integer such that $a_r \neq 0$, then we see that we can rewrite

   \[
   c_n = \left[ \frac{1}{(n + r)!} \right] \left[ a_r + \frac{a_{r+1}}{n + r + 1} + \cdots + \frac{a_d}{(n + r + 1) \cdots (n + d)} \right].
   \]

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Note that
\[\lim_{n \to \infty} \left[ \frac{a_{r+1}}{n + r + 1} + \cdots + \frac{a_d}{(n + r + 1) \cdots (n + d)} \right] = 0\]
and so there exists an \( N \in \mathbb{N} \) such that
\[|a_r| > \left| \frac{a_{r+1}}{n + r + 1} + \cdots + \frac{a_d}{(n + r + 1) \cdots (n + d)} \right|\]
for all \( n > N \). Hence \( c_n \) is nonzero for all \( n > N \). In other words, \( p(z)e^{1/z} \) must have singularities since its Laurent expansion contains terms of negative powers. So apart from \( p(z) = 0 \), there no polynomials such that \( p(z)e^{1/z} \) is entire.

3. Let \( f : \mathbb{C} \to \mathbb{C} \) be a nonconstant entire function. Prove that \( f(\mathbb{C}) \) is dense in \( \mathbb{C} \).

**SOLUTION.** Suppose \( f(\mathbb{C}) \) is not dense. Then for some \( w \in \mathbb{C} \) and \( \varepsilon > 0 \), we have \(|f(z) - w| \geq \varepsilon\) for all \( z \in \mathbb{C} \). The function
\[\frac{1}{f(z) - w}\]
is then entire since the denominator is never zero. It is bounded since
\[\left| \frac{1}{f(z) - w} \right| \leq \frac{1}{\varepsilon} .\]
Hence by Liouville’s theorem, the function is constant and therefore \( f \) is constant.

4. Show that each of the following series define a meromorphic function on \( \mathbb{C} \) and determine the set of poles and their orders.

\[f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z + n)},\quad g(z) = \sum_{n=0}^{\infty} \frac{\sin(nz)}{n!(z^2 + n^2)},\quad h(z) = \frac{1}{z} + \sum_{n \neq 0, n = -\infty}^{\infty} \left[ \frac{1}{z - n} + \frac{1}{n} \right].\]

**SOLUTION.** We claim that \( f \) is meromorphic on \( \mathbb{C} \) with poles of order 1 at
\[z \in P_f := -\mathbb{N} \cup \{0\} = \{0, -1, -2, -3, \ldots\} .\]
For any fixed \( R > 0 \), there exists an integer \( N \) such that \( N > 2R \). We will consider \( f \) on the disc \( D(0, R) \). We may write
\[f(z) = \sum_{n=0}^{N} \frac{(-1)^n}{n!(z + n)} + \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!(z + n)} .\]
The first sum has poles of order 1 for each nonpositive integer of absolute value \( \leq N \). The second sum defines an analytic function for all \( z \in D(0, R) \) since it is uniformly convergent by the Weierstrass M-test: For \(|z| < R \) and \( n > N \), we have
\[\left| \frac{(-1)^n}{n!(n + z)} \right| \leq \frac{1}{n!(n - |z|)} \leq \frac{1}{n!R} =: M_n .\]
Since
\[\sum_{n=N+1}^{\infty} M_n = \frac{1}{R} \sum_{n=N+1}^{\infty} \frac{1}{n!} ,\]
is convergent, the series of function
\[\sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!(z + n)} \]
is uniformly convergent. Since this argument is true for arbitrary \( R > 0 \), \( f \) is a meromorphic on \( \mathbb{C} \) with poles of order 1 at \( z \in P_f \).

We claim that \( g \) is meromorphic on \( \mathbb{C} \) with poles of order 1 at
\[z \in P_g := i\mathbb{Z}^\times = \{ \ldots, -3i, -2i, -i, i, 2i, 3i, \ldots \} .\]
and a removable singularity at \( z = 0 \). For any fixed \( R > 0 \), there exists an integer \( N \) such that \( N > 2R \). We will consider \( g \) on the disc \( D(0, R) \). We may write

\[
g(z) = \sum_{n=0}^{N} \frac{\sin(nz)}{n!(z^2 + n^2)} + \sum_{n=N+1}^{\infty} \frac{\sin(nz)}{n!(z^2 + n^2)}. \tag{4.1}
\]

Note that the zeroes of \( \sin(nz) \) are

\[
\left\{ \ldots, -2\pi n, -\pi n, \pi n, 2\pi n, \ldots \right\}
\]

and so the zeroes of \( \sin(nz) \) and \( z^2 + n^2 = (z + in)(z - in) \) are disjoint for all \( n \geq 1 \). So the first sum in (4.1) has poles of order 1 for each \( \pm in \) where \( 0 < n \leq N \). The second sum defines an analytic function for all \( z \in D(0, R) \) since it is uniformly convergent by the Weierstrass M-test: For \( |z| < R \) and \( n > N \), we have

\[
|\sin(nz)| \leq e^{n|z|}
\]

by considering the power series expansions of \( \sin \) and \( \exp \); and we also have

\[
\left| \frac{1}{z^2 + n^2} \right| \leq \frac{1}{n^2 - |z|^2} \leq \frac{1}{n^2 - R^2} = \frac{1}{n^2} \times \frac{1}{1 - (R/n)^2} \leq \frac{4}{3n^2}
\]

since \( n > N > 2R \) (and so \( 1 - R^2/n^2 \geq 1 - 1/4 = 3/4 \)). Hence

\[
\left| \frac{\sin(nz)}{n!(z^2 + n^2)} \right| \leq \frac{4e^{n|z|}}{3n!n^2} =: M_n.
\]

Since

\[
\sum_{n=N+1}^{\infty} M_n = \frac{4}{3} \sum_{n=N+1}^{\infty} \frac{e^{nR}}{n!n^2}
\]

is convergent by the ratio test, the series of function

\[
\sum_{n=N+1}^{\infty} \frac{\sin(nz)}{n!(z^2 + n^2)}
\]

is uniformly convergent. Since this argument is true for arbitrary \( R > 0 \), \( g \) is a meromorphic on \( \mathbb{C} \) with poles of order 1 at \( z \in P_g \).

We claim that \( h \) is meromorphic on \( \mathbb{C} \) with poles of order 1 at

\[
z \in P_h := \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}.
\]

For any fixed \( R > 0 \), there exists an integer \( N \) such that \( N > 2R \). We will consider \( h \) on the disc \( D(0, R) \). First note that \( h \) may be rewritten as

\[
h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z - n)} + \sum_{m=1}^{\infty} \frac{-z}{m(z + m)}.
\]

We may then write

\[
h(z) = \left[ \frac{1}{z} + \sum_{n=1}^{N} \frac{z}{n(z - n)} - \frac{z}{n(z + n)} \right] + \left[ \sum_{n=N+1}^{\infty} \frac{z}{n(z - n)} - \frac{z}{n(z + n)} \right].
\]

The term in the first bracket has poles of order 1 for each integer of absolute value \( \leq N \). The term in the second bracket defines an analytic function for all \( z \in D(0, R) \) since it is uniformly convergent by the Weierstrass M-test: For \( |z| < R \) and \( n > N \), we have

\[
\left| \frac{z}{n(z - n)} - \frac{z}{n(z + n)} \right| = \left| \frac{2z}{z^2 - n^2} \right| \leq \frac{2}{n^2} \times \frac{R}{1 - |z/n|^2} \leq \frac{8R}{3n^2} =: M_n.
\]

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since \( n > N > 2R > 2|z|^2 \) (and so \( 1 - |z/n|^2 \geq 1 - 1/4 = 3/4 \)). Since
\[
\sum_{n=N+1}^{\infty} M_n = \frac{8R}{3} \sum_{n=N+1}^{\infty} \frac{1}{n^2}
\]
is convergent, the series of function
\[
\sum_{n=N+1}^{\infty} \frac{z}{n(z - n)} - \frac{z}{n(z + n)}
\]
is uniformly convergent. Since this argument is true for arbitrary \( R > 0 \), \( h \) is a meromorphic on \( \mathbb{C} \) with poles of order 1 at \( z \in P_h \).

(a) Let \( f \) be given by
\[
f(z) = \frac{z}{1 + z^3}.
\]
Expand \( f \) in a series of positive powers of \( z \) and in a series of negative powers of \( z \). In each case, specify the region in which the expansion is valid.

**Solution.** If \( |z| < 1 \), we have
\[
f(z) = z \frac{1}{1 - (-z^3)} = z[1 + (-z^3) + (-z^3)^2 + \ldots] = \sum_{n=0}^{\infty} (-1)^n z^{3n+1}.
\]
If \( |z| > 1 \), we have
\[
f(z) = \frac{1}{z^2} \frac{1}{1 + 1/z^3} = \frac{1}{z^2} \left[ 1 + \left( \frac{-1}{z^3} \right) + \left( \frac{-1}{z^3} \right)^2 + \ldots \right] = \sum_{n=0}^{\infty} (-1)^n z^{-3n-2}.
\]

(b) Find the Laurent series for
\[
g(z) = \frac{1}{(z - 1)(z - 2)}
\]
in (i) the disc \( |z| < 1 \); (ii) the annulus \( 1 < |z| < 2 \); (iii) the region \( 2 < |z| \).

**Solution.** We can write
\[
g(z) = \frac{1}{(z - 1)(z - 2)} = \frac{-1}{z - 1} + \frac{1}{z - 2}.
\]
If \( |z| < 1 \), we have
\[
\frac{-1}{z - 1} = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n
\]
and
\[
\frac{1}{z - 2} = -\frac{1}{2} \frac{1}{1 - (z/2)} = -\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n
\]
so
\[
g(z) = \sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n
\]
\[
= \sum_{n=0}^{\infty} \left( 1 - \frac{1}{2^{n+1}} \right) z^n.
\]
If \( 1 < |z| < 2 \), then
\[
\frac{-1}{z - 1} = \frac{1}{z} \frac{1}{1 - (1/z)} = \frac{-1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}
\]
and
\[ \frac{1}{z-2} = -\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n \]
so
\[ g(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n \]
\[ = \sum_{n=1}^{\infty} (-1)z^{-n} + \sum_{n=0}^{\infty} (-2^{n+1})z^n. \]
If \(|z| > 2\), then
\[ \frac{1}{z-2} = \frac{1}{z} \frac{1}{1 - (2/z)} = \frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{2}{z} \right)^n, \]
so
\[ g(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{2}{z} \right)^n \]
\[ = \sum_{n=1}^{\infty} (-1 + 2^{n-1})z^{-n}. \]

(c) Find the Laurent series for 
\[ h(z) = \frac{z + 1}{z - 1} \]
in (i) the disc \(|z| < 1\); (ii) the region \(|z| > 1\).

**Solution.** If \(|z| < 1\), we have
\[ h(z) = -\frac{z + 1}{1-z} = -(1 + z)(1 + z^2 + \cdots) = -1 + \sum_{n=1}^{\infty} (-2)z^n. \]
If \(|z| > 1\), we have
\[ h(z) = \frac{1}{z} \frac{z + 1}{1-\frac{1}{z}} = \left( 1 + \frac{1}{z} \right) \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \right) = 1 + \sum_{n=1}^{\infty} 2z^{-n}. \]