For $z \in \mathbb{C}$, recall that the principle argument of $z$, denoted $\text{Arg}(z)$, is the unique $\theta \in [-\pi, \pi)$ such that $z = |z|e^{i\theta}$. By convention $\text{Arg}(0) = 0$.

1. Let $S$ denote the sector given by
   \[ \{ z \in \mathbb{C} \mid -\pi/4 < \text{Arg}(z) < \pi/4 \} . \]
   Let $f : \bar{S} \to \mathbb{C}$ be a continuous function such that $f$ is analytic on $S$. Suppose
   \begin{itemize}
     \item[(i)] $|f(z)| \leq 1$ for all $z \in \partial S$;
     \item[(ii)] $|f(x + iy)| \leq e^{\sqrt{x}}$ for all $x + iy \in S$.
   \end{itemize}
   Prove that $|f(z)| \leq 1$ for all $z \in S$.
   
   **SOLUTION.** Let $\epsilon > 0$. Consider the function
   \[ F(z) = e^{-\epsilon z}f(z). \]
   Then $F$ is also continuous on $\bar{S}$ and analytic in $S$. By (i),
   \[ |F(z)| = e^{-\epsilon z}|f(z)| \leq 1 \]
   for $z \in \partial S$. By (ii),
   \[ \lim_{z \in \bar{S}, |z| \to \infty} |F(z)| \leq \lim_{x \to \infty} e^{-\epsilon x}e^{\sqrt{x}} = \lim_{x \to \infty} e^{-\epsilon x + \sqrt{x}} = 0. \]
   Hence there exists $R > 0$ such that
   \[ |F(z)| \leq 1 \]
   for all $z \in \bar{S}$, $|z| \geq R$, i.e.
   \[ \max_{z \in \bar{S} \cap (\mathbb{C} \setminus \overline{D}(0,R))} |F(z)| \leq 1. \]
   Since $F$ is continuous, it attains its maximum on the compact set $\bar{S} \cap \overline{D}(0,R)$ and so we may apply the maximum modulus theorem to $F$ to conclude that
   \[ \max_{z \in \bar{S} \cap \overline{D}(0,R)} |F(z)| = \max_{z \in \partial(S \cap \overline{D}(0,R))} |F(z)| \]
   \[ = \max \left\{ \max_{z \in \partial(S \cap \overline{D}(0,R))} |F(z)|, \max_{z \in \bar{S} \cap \partial D(0,R)} |F(z)| \right\} \]
   \[ \leq 1. \]
   Hence
   \[ \max_{z \in \bar{S}} |F(z)| \leq 1 \]
   and so for all $z \in \bar{S}$,
   \[ |f(z)| \leq |e^{\epsilon z}| = e^{\epsilon \text{Re}(z)}. \]
   Since $\epsilon > 0$ is arbitrary, for each fixed $z \in \bar{S}$, we may take right limit to get
   \[ |f(z)| \leq \lim_{\epsilon \to 0^+} e^{\epsilon \text{Re}(z)} = 1. \]
   Hence $|f(z)| \leq 1$ for all $z \in \bar{S}$, as required.

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2. (a) Show that the functions $\eta_a$ and $\eta_b$ maps $D(0, 1)$ to $H_a$ and $H_b$ where

$$\eta_a(z) = \frac{1 + z}{1 - z}, \quad \eta_b(z) = i \left( \frac{1 + z}{1 - z} \right),$$

and

$$H_a = \{ z \in \mathbb{C} \mid \text{Re} \, z > 0 \}, \quad H_b = \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \}.$$

**SOLUTION.** Straightforward as in (b).

(b) Show that the functions $\sigma_a$ and $\sigma_b$ maps $S_a$ and $S_b$ to $D(0, 1)$ where

$$\sigma_a(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1}, \quad \sigma_b(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1}$$

and

$$S_a = \{ z \in \mathbb{C} \mid -1 < \text{Re} \, z < 1 \}, \quad S_b = \{ z \in \mathbb{C} \mid -1 < \text{Im} \, z < 1 \}.$$

**SOLUTION.** Note that

$$|\sigma_b(z)| < 1 \iff \left| \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1} \right| < 1$$

$$\iff |e^{i\pi z/2} - 1|^2 < |e^{i\pi z/2} + 1|^2$$

$$\iff |\text{Re}(e^{i\pi z/2}) - 1|^2 + |\text{Im}(e^{i\pi z/2})|^2 < |\text{Re}(e^{i\pi z/2}) + 1|^2 + |\text{Im}(e^{i\pi z/2})|^2$$

$$\iff \text{Re}(e^{i\pi z/2}) - 2 \text{Re}(e^{i\pi z/2}) + 1 < \text{Re}(e^{i\pi z/2}) + 2 \text{Re}(e^{i\pi z/2}) + 1$$

$$\iff \text{Re}(e^{i\pi z/2}) > 0.$$ 

But $\text{Re}(e^{i\pi z/2}) = \text{Re}(\exp(\frac{\pi}{2} \text{Re} \, z + i\frac{\pi}{2} \text{Im} \, z)) = \exp(\frac{\pi}{2} \text{Re} \, z) \cos(\frac{\pi}{2} \text{Im} \, z) > 0$ if $\cos(\frac{\pi}{2} \text{Im} \, z) > 0$ — in particular this holds when $-1 < \text{Im} \, z < 1$, i.e. when $z \in S_b$. To deduce the analogous result for $\sigma_a$ and $S_a$, just note that

$$\sigma_a(z) = \sigma_b(iz) \quad \text{and} \quad S_a = iS_b.$$

3. For a region $\Omega \subseteq \mathbb{C}$ and a point $\alpha \in \Omega$, let $\mathcal{F}(\Omega, \alpha)$ be the set of functions defined by

$$\mathcal{F}(\Omega, \alpha) := \{ f : \Omega \to \mathbb{C} \mid f \text{ analytic, } |f| < 1 \text{ on } \Omega, \text{ and } f(\alpha) = 0 \}.$$

(a) Let $\Omega = H_a$ and $\alpha = 1$. Show that

$$\sup_{f \in \mathcal{F}(H_a, 1)} |f'(1)| = \frac{1}{2}.$$

**SOLUTION.** By Problem 2(a), $\eta_a$ maps $H_a$ to $D(0, 1)$. Also $\eta_a$ is analytic on $H_a$ and $\eta_a(0) = 1$. For any $f \in \mathcal{F}(H_a, 1)$, $F = f \circ \eta_a$ maps $D(0, 1)$ into $D(0, 1)$ and $F(0) = f(\eta_a(0)) = f(1) = 0$. By Schwartz’s Lemma, we get

$$|F'(0)| \leq 1$$

and with Chain Rule, we get

$$|f'(\eta_a(0))\eta_a'(0)| \leq 1$$

and so

$$|f'(1)| \leq \frac{1}{2}.$$ 

Since this is true for arbitrary $f \in \mathcal{F}(H_a, 1)$, we have that

$$\sup_{f \in \mathcal{F}(H_a, 1)} |f'(1)| \leq \frac{1}{2}.$$ 

But note that $\eta_a : H_a \to D(0, 1)$ is invertible and that

$$\eta_a^{-1}(z) = \frac{z - 1}{z + 1}.$$
Also, \( \eta_a^{-1}(1) = 0 \) and so \( \eta_a^{-1} \in \mathcal{F}(H_a, 1) \). Note that \( \eta_a^{-1} \) attains the upper bound:

\[ |(\eta_a^{-1})'(1)| = \frac{1}{2}. \]

So

\[ \sup_{f \in \mathcal{F}(H_a, 1)} |f'(1)| = \frac{1}{2}. \]

(b) Let \( \Omega = H_b \) and \( \alpha = i \). Show that

\[ \sup_{f \in \mathcal{F}(H_b, i)} |f(2i)| = \frac{1}{3}. \]

**Solution.** By Problem 2(a), \( \eta_b \) maps \( H_b \) to \( D(0, 1) \). Also \( \eta_b \) is analytic on \( H_b \) and \( \eta_b(0) = i \). For any \( f \in \mathcal{F}(H_b, i) \), \( F = f \circ \eta_b \) maps \( D(0, 1) \) into \( D(0, 1) \) and \( F(0) = f(\eta_b(0)) = f(i) = 0 \). By Schwarz’s Lemma, we get

\[ |F(z)| \leq |z|. \]

Now solving \( \eta_b(z) = 2i \), we get \( z = 1/3 \) and so

\[ |f(2i)| = |f(\eta_b(1/3))| = |F(1/3)| \leq |1/3|. \]

Since this is true for arbitrary \( f \in \mathcal{F}(H_b, i) \), we have that

\[ \sup_{f \in \mathcal{F}(H_b, i)} |f(2i)| \leq \frac{1}{3}. \]

But note that \( \eta_b : H_b \to D(0, 1) \) is invertible and that

\[ \eta_b^{-1}(z) = \frac{z - i}{z + i}. \]

Also, \( \eta_b^{-1}(i) = 0 \) and so \( \eta_b^{-1} \in \mathcal{F}(H_b, i) \). Note that \( \eta_b^{-1} \) attains the upper bound:

\[ |\eta_b^{-1}(2i)| = \frac{1}{3}. \]

So

\[ \sup_{f \in \mathcal{F}(H_b, i)} |f(2i)| = \frac{1}{3}. \]

(c) Let \( \Omega = S_b \) and \( \alpha = 0 \). Show that

\[ \sup_{f \in \mathcal{F}(S_b, 0)} |f(1)| = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}. \]

**Solution.** By Problem 2(b), \( \sigma_b \) maps \( S_b \) into \( D(0, 1) \). Also \( \sigma_b \) is analytic on \( S_b \) and \( \sigma_b(0) = 0 \). Furthermore, note that \( \sigma_b \) is injective on \( S_b \) and \( \sigma_b(z) = \sigma_b(w) \) iff

\[ (e^{\pi z/2} - 1)(e^{\pi w/2} + 1) = (e^{\pi z/2} + 1)(e^{\pi w/2} - 1) \]

iff \( e^{\pi(z-w)/2} = 1 \) iff \( \text{Re}(z-w) = 0 \) and \( \text{Im}(z-w) \) is an integer multiple of 4; so when \( z, w \in S_b \), this is only possible if \( z = w \). Hence \( \sigma_b \) is an invertible map and \( \sigma_b^{-1} \) maps \( D(0, 1) \) onto \( S_b \), \( \sigma_b^{-1} \) is analytic on \( D(0, 1) \), and \( \sigma_b^{-1}(0) = 0 \). Given any \( f \in \mathcal{F}(S_b, 0) \), the composition \( F = f \circ \sigma_b^{-1} : D(0, 1) \to \mathbb{C} \) is analytic on \( D(0, 1) \); \( F(0) = f(\sigma_b^{-1}(0)) = f(0) = 0 \); for any \( z \in D(0, 1), \sigma_b^{-1}(z) \in S_b \) and so \( f(\sigma_b^{-1}(z)) \in D(0, 1) \), ie.

\[ |F(z)| < 1 \]

for all \( z \in D(0, 1) \). By Schwarz’s Lemma, we have

\[ |F(z)| \leq |z| \]
for all \( z \in D(0, 1) \). In particular, if we pick
\[
z_0 = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1} \in D(0, 1),
\]
we get
\[
|f(1)| = |f(\sigma_b^{-1}(z_0))| = |F(z_0)| \leq |z_0| = z_0.
\]
Since this is true for arbitrary \( f \in F(S_b, 0) \), we have that
\[
\sup_{f \in F(S_b, 0)} |f(1)| \leq \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}.
\]
But note that \( \sigma \in F(S_b, 0) \) and it attains the upper bound:
\[
|\sigma(1)| = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}.
\]

4. Let \( f : D(0, 1) \rightarrow \mathbb{C} \) be analytic and \( |f(z)| < 1 \) for all \( z \in D(0, 1) \).
   (a) Show that
\[
|f'(z)| \leq \frac{1}{1 - |f(z)|^2}.
\]
**Solution.** Let \( \alpha \in D(0, 1) \). Then \( f(\alpha) \in D(0, 1) \). We consider \( \varphi_{-\alpha} \) and \( \varphi_{f(\alpha)} \). Define 
\[ g = \varphi_{f(\alpha)} \circ f \circ \varphi_{-\alpha} : D(0, 1) \rightarrow \mathbb{C}. \]
Note that \( g \) is analytic. If \( z \in D(0, 1) \), then \( \varphi_{-\alpha}(z) \in D(0, 1) \), so \( f(\varphi_{-\alpha}(z)) \in D(0, 1) \), and so \( \varphi_{f(\alpha)}(f(\varphi_{-\alpha}(z))) \in D(0, 1) \), i.e.
\[
|g(z)| < 1
\]
for all \( z \in D(0, 1) \). Also \( g(0) = \varphi_{f(\alpha)}(f(\varphi_{-\alpha}(0))) = \varphi_{f(\alpha)}(f(\alpha)) = 0 \). Schwartz’s Lemma then implies that \( |g'(0)| \leq 1 \). But using chain rule and Lemma 4.18 in the lectures, we get
\[
|g'(0)| = |\varphi_{f(\alpha)}(f(\alpha))f'(\alpha)\varphi_{-\alpha}'(0)| = \frac{|f'(\alpha)|}{1 - |f(\alpha)|^2} \times (1 - |\alpha|^2)
\]
and so
\[
\frac{|f'(\alpha)|}{1 - |f(\alpha)|^2} \leq \frac{1}{1 - |\alpha|^2}.
\]
Now just observe that this works for arbitrary \( \alpha \in D(0, 1) \).
(b) Suppose \( f(0) = 0 \). Show that
\[
|f(z) + f(-z)| \leq 2|z|^2.
\]
**Solution.** By Schwarz Lemma, we know that
\[
|f(z)| \leq |z|
\]
for all \( z \in D(0, 1) \) and \( |f'(0)| \leq 1 \). Now consider the function
\[
g(z) = \begin{cases} 
  \frac{f(z) + f(-z)}{2z} & \text{if } z \neq 0, \\
  f'(0) & \text{if } z = 0.
\end{cases}
\]
Note that \( g \) is analytic by Corollary 4.4 (why?). Furthermore for \( z \neq 0 \),
\[
|g(z)| \leq \frac{|f(z)| + |f(-z)|}{2|z|} \leq \frac{1}{2} \left[ \frac{|f(z)| + |f(-z)|}{|z|} \right] \leq 1,
\]
and \( |g(0)| = |f'(0)| \leq 1 \). So
\[
|g(z)| \leq 1
\]
for all $z \in D(0, 1)$. Since $f$ is analytic on $D(0, 1)$, it has a power series expansion

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

note that $a_0 = f(0) = 0$. Hence for $z \neq 0$,

$$g(z) = \frac{f(z) + f(-z)}{2} = a_2 z + a_4 z^3 + a_6 z^5 + \cdots$$

and it follows that

$$g(0) = \lim_{z \to 0} g(z) = 0$$

since $g$ is analytic (thus continuous at 0). So we may apply Schwarz Lemma to $g$ to get

$$|g(z)| \leq |z|$$

for all $z \in D(0, 1)$, i.e.

$$|f(z) + f(-z)| \leq 2|z|^2$$

for all $z \in D(0, 1)$ as required.