For \( z \in \mathbb{C}^x \), recall that the principle argument of \( z \), denoted \( \text{Arg}(z) \), is the unique \( \theta \in [-\pi, \pi) \) such that \( z = |z|e^{i\theta} \). By convention \( \text{Arg}(0) = 0 \).

1. Let \( S \) denote the sector given by
   \[ \{ z \in \mathbb{C} \mid -\pi/4 < \text{Arg}(z) < \pi/4 \} . \]
   Let \( f : \overline{S} \to \mathbb{C} \) be a continuous function such that \( f \) is analytic on \( S \). Suppose
   (i) \( |f(z)| \leq 1 \) for all \( z \in \partial S \);
   (ii) \( |f(x + iy)| \leq e^{\sqrt{x}} \) for all \( x + iy \in S \).
   Prove that \( |f(z)| \leq 1 \) for all \( z \in S \).

2. (a) Show that the functions \( \eta_a \) and \( \eta_b \) maps \( D(0,1) \) to \( H_a \) and \( H_b \) where
   \[ \eta_a(z) = \frac{1 + z}{1 - z}, \quad \eta_b(z) = i \left( \frac{1 + z}{1 - z} \right) , \]
   and
   \[ H_a = \{ z \in \mathbb{C} \mid \text{Re} z > 0 \}, \quad H_b = \{ z \in \mathbb{C} \mid \text{Im} z > 0 \} . \]
   (b) Show that the functions \( \sigma_a \) and \( \sigma_b \) maps \( S_a \) and \( S_b \) to \( D(0,1) \) where
   \[ \sigma_a(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1}, \quad \sigma_b(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1} \]
   and
   \[ S_a = \{ z \in \mathbb{C} \mid -1 < \text{Re} z < 1 \}, \quad S_b = \{ z \in \mathbb{C} \mid -1 < \text{Im} z < 1 \} . \]

3. For a region \( \Omega \subseteq \mathbb{C} \) and a point \( \alpha \in \Omega \), let \( \mathcal{F}(\Omega, \alpha) \) be the set of functions defined by
   \[ \mathcal{F}(\Omega, \alpha) := \{ f : \Omega \to \mathbb{C} \mid f \text{ analytic}, |f| < 1 \text{ on } \Omega, \text{ and } f(\alpha) = 0 \} . \]
   (a) Let \( \Omega = H_a \) and \( \alpha = 1 \). Show that
   \[ \sup_{f \in \mathcal{F}(H_a,1)} |f'(1)| = \frac{1}{2} . \]
   (b) Let \( \Omega = H_b \) and \( \alpha = i \). Show that
   \[ \sup_{f \in \mathcal{F}(H_b,i)} |f(2i)| = \frac{1}{3} . \]
   (c) Let \( \Omega = S_b \) and \( \alpha = 0 \). Show that
   \[ \sup_{f \in \mathcal{F}(S_b,0)} |f(1)| = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1} . \]

4. Let \( f : D(0,1) \to \mathbb{C} \) be analytic and \( |f(z)| < 1 \) for all \( z \in D(0,1) \).
   (a) Show that
   \[ \frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} . \]

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(b) Suppose $f(0) = 0$. Show that

$$|f(z) + f(-z)| \leq 2|z|^2.$$