Ω ⊆ ℂ will always denote a region unless specified otherwise. For \( f : \Omega \to \mathbb{C} \) and \( c \in \mathbb{C} \) a constant, we write \( f \equiv c \) to mean that \( f(z) = c \) for all \( z \in \Omega \).

1. Let \( f : \Omega \to \mathbb{C} \) with \( f(x+iy) = u(x,y) + iv(x,y) \). Let \( z_0 \in \Omega \) and suppose there exists a function \( \varphi : \mathbb{C} \to \mathbb{C} \) such that

\[
\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - \varphi(h)}{h} = 0.
\]

Recall from Problem Set 2, Problem 2 that \( f \) is real differentiable if \( \varphi \) is real linear and \( f \) is complex differentiable if \( \varphi \) is complex linear. Recall from Problem Set 2, Problem 1 that a real linear \( \varphi \) satisfies

\[
\varphi(x + iy) = (ax + by) + i(cx + dy)\tag{1.1}
\]

for some \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) and a complex linear \( \varphi \) satisfies

\[
\varphi(x + iy) = (ax - cy) + i(cx + ay)\tag{1.2}
\]

for some \( \begin{bmatrix} a & -c \\ c & a \end{bmatrix} \in \mathbb{R}^{2 \times 2} \).

(a) Show that if \( f \) is real differentiable at \( z_0 = x_0 + iy_0 \in \Omega \), then the matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is given by

\[
\begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix}.
\]

**Solution.** Taking limits along \( \xi \to 0, \xi \in \mathbb{R} \), we get

\[
0 = \lim_{\xi \to 0} \frac{f(z_0 + \xi) - f(z_0) - \varphi(\xi)}{\xi}
= \lim_{\xi \to 0} \frac{f(z_0 + \xi) - f(z_0) - b\xi - ic\xi}{\xi}
= \lim_{\xi \to 0} \frac{u(x_0 + \xi, y_0) - u(x_0, y_0) - a\xi}{\xi} + i \lim_{\xi \to 0} \frac{v(x_0 + \xi, y_0) - v(x_0, y_0) - c\xi}{\xi}
= \left[ \lim_{\xi \to 0} \frac{u(x_0 + \xi, y_0) - u(x_0, y_0)}{\xi} - a \right] + i \left[ \lim_{\xi \to 0} \frac{v(x_0 + \xi, y_0) - v(x_0, y_0)}{\xi} - c \right]
= [u_x(x_0, y_0) - a] + i[v_x(x_0, y_0) - c].
\]

Hence

\[
u_x(x_0, y_0) = a \quad \text{and} \quad v_x(x_0, y_0) = c.
\]
Taking limits along $i\eta \to 0$, $i\eta \in i\mathbb{R}$, we get

$$0 = \lim_{\eta \to 0} \frac{f(z_0 + i\eta) - f(z_0) - \varphi(i\eta)}{i\eta}$$

$$= \lim_{\eta \to 0} \frac{f(z_0 + i\eta) - f(z_0) - b\eta - id\eta}{i\eta}$$

$$= -i \lim_{\eta \to 0} \frac{u(x_0, y_0 + \eta) - u(x_0, y_0) - b\eta}{\eta} + \lim_{\eta \to 0} \frac{v(x_0, y_0 + \eta) - v(x_0, y_0) - d\eta}{\eta}$$

$$= -i \left[ \lim_{\eta \to 0} \frac{u(x_0, y_0 + \eta) - u(x_0, y_0)}{\eta} - b \right] + \left[ \lim_{\eta \to 0} \frac{v(x_0, y_0 + \eta) - v(x_0, y_0)}{\eta} - d \right]$$

$$= -i[u_y(x_0, y_0) - b] + [v_y(x_0, y_0) - d].$$

Hence

$$u_y(x_0, y_0) = b \quad \text{and} \quad v_y(x_0, y_0) = d.$$  

(b) Show that if $f$ is complex differentiable at $z_0 = x_0 + iy_0 \in \Omega$, then the matrix $[\begin{smallmatrix} a & -c \\ c & a \end{smallmatrix}]$ is given by

$$\begin{bmatrix} a & -c \\ c & a \end{bmatrix} = \begin{bmatrix} u_x(x_0, y_0) & -u_y(x_0, y_0) \\ u_y(x_0, y_0) & u_x(x_0, y_0) \end{bmatrix}.$$  

Solution. By Problem Set 2, Problem 2(a), we know that complex differentiability at $z_0$ implies real differentiability at $z_0$. So part (a) holds in this case, i.e. there exists a $\varphi$ having the form in (1.1) with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix}.$$  

By the Cauchy-Riemann equations, we know that complex differentiability at $z_0$ implies

$$a = u_x(x_0, y_0) = v_y(x_0, y_0) = d$$

and

$$b = u_y(x_0, y_0) = -v_x(x_0, y_0) = -c.$$  

(c) Suppose $f$ is analytic on $\Omega$ (i.e. complex differentiable at all $z \in \Omega$) and that $u(x, y) = \varphi(x)$, $v(x, y) = \psi(y)$, i.e. $f$ takes the form

$$f(x + iy) = \varphi(x) + i\psi(y).$$

Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

Solution. By the Cauchy-Riemann equations, we have

$$\varphi'(x) = \psi'(y)$$

for all $x, y \in \mathbb{R}$. Note that the LHS is a function of $x$ and the RHS is a function of $y$. The only way two such functions can be equal for two independent variables is if they are both constant functions, i.e. there exists $\alpha \in \mathbb{R}$ such that,

$$\varphi'(x) = \alpha = \psi'(y).$$

Hence there exist $\beta, \gamma \in \mathbb{R}$ such that

$$\varphi(x) = \alpha x + \beta \quad \text{and} \quad \psi(y) = \alpha y + \gamma.$$  

In other words,

$$f(x + iy) = \varphi(x) + i\psi(y)$$

$$= \alpha(x + iy) + (\beta + i\gamma)$$

and so $f(z) = az + b$ with $a = \alpha$ and $b = \beta + i\gamma$.  

2
2. Let \( f \) be defined by

\[
f(z) = \sum_{n=0}^{\infty} \alpha_n z^n
\]

where the series has a positive radius of convergence \( R \). For each \( m \in \mathbb{N} \), let \( s_m : \mathbb{C} \to \mathbb{C} \) be the \( m \)th partial sum

\[
s_m(z) = \sum_{n=0}^{m} \alpha_n z^n.
\]

Prove that

\[
\sum_{m=0}^{\infty} |f(z) - s_m(z)| < \infty
\]

for all \( z \in D(0, R) \).

**Solution.** This is almost identical to the proof of Theorem 2.10. Note that for any \( m \in \mathbb{N} \),

\[
|f(z) - s_m(z)| = \left| \sum_{n=m+1}^{\infty} \alpha_n z^n \right| \leq \sum_{n=m+1}^{\infty} |\alpha_n z^n|.
\]

Therefore

\[
\sum_{m=0}^{\infty} |f(z) - s_m(z)| \leq \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} |\alpha_n z^n|
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} |\alpha_n z^n|
\]

\[
= \sum_{n=1}^{\infty} n|\alpha_n z^n|.
\]

Note that we have used the summation by parts formula in the second step. Since\(^1\)

\[
\limsup_{n \to \infty} \frac{|na_n|^{1/n}}{|a_n|^{1/n}} = \frac{1}{R},
\]

the power series

\[
\sum_{n=1}^{\infty} n\alpha_n z^n
\]

has the same radius of convergence as the one in (1.1). Hence it converges (and thus converges absolutely) for all \( z \in D(0, R) \); and hence

\[
\sum_{m=0}^{\infty} |f(z) - s_m(z)| = \sum_{n=1}^{\infty} n|\alpha_n z^n| < \infty
\]

for all \( z \in D(0, R) \).

3. (a) Let \( f \) be defined by

\[
f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.
\]

Prove that

\[
f''(z) - f(z) = 0.
\]

State which theorem(s) you have used here. For what values of \( z \) is this valid?

---

\(^1\)Recall that we used this same trick to prove that the formal derivative of a power series has the same radius of convergence.
SOLUTION. The ratio test implies that the radius of convergence of the series defining $f$ is $\infty$. Let

$$f_n(z) = \frac{z^{2n}}{(2n)!}.$$  

Then

$$f_n'(z) = \frac{2n}{(2n)!} z^{2n-1}$$

and

$$f_n''(z) = \frac{2n(2n-1)}{(2n)!} z^{2n-2} = f_{n-1}(z).$$

By Theorem 2.3, the series

$$f(z) = \sum_{n=0}^{\infty} f_n(z)$$

converges uniformly in $\mathbb{C}$, we may differentiate term by term of the summands to get

$$f''(z) = \sum_{n=1}^{\infty} f''_n(z) = \sum_{n=1}^{\infty} f_{n-1}(z) = \sum_{n=0}^{\infty} f_n(z) = f(z).$$

Alternatively, we may use Theorem 2.10 to justify the equality between formal derivatives and derivatives of power series.

(b) Let $g$ be defined by

$$g(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2}.$$  

Prove that

$$z^2 g''(z) + zg'(z) - 4z^2 g(z) = 0.$$  

For what values of $z$ is this valid?

SOLUTION. The ratio test implies that the radius of convergence of the series defining $g$ is $\infty$. Hence by Theorem 2.10,

$$g'(z) = \sum_{n=1}^{\infty} \frac{2n}{(n!)^2} z^{2n-1}$$

and

$$g''(z) = \sum_{n=1}^{\infty} \frac{2n(2n-1)}{(n!)^2} z^{2n-2}$$

and so

$$zg'(z) = \sum_{n=1}^{\infty} \frac{2n}{(n!)^2} z^{2n}$$

and

$$z^2 g''(z) = \sum_{n=1}^{\infty} \frac{2n(2n-1)}{(n!)^2} z^{2n} = 4 \sum_{n=1}^{\infty} \frac{n^2}{(n!)^2} z^{2n} - zg'(z).$$

Therefore

$$z^2 g''(z) + zg'(z) = 4 \sum_{n=1}^{\infty} \frac{n^2}{(n!)^2} z^{2n} = 4 \sum_{n=0}^{\infty} \frac{1}{((n-1)!)^2} z^{2n} = 4z^2 g(z)$$

as required.

[Note: When we write things like

$$\lambda z^m \times \left[ \sum_{n=0}^{\infty} a_n z^n \right] = \sum_{n=0}^{\infty} \lambda a_n z^{m+n},$$

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n + b_n) z^n,$$

we have used the results in Problem Set 2, Problem 4 implicitly.]
(c) Let \( h \) be defined by the power series
\[
h(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots.
\]
Prove that
\[
h'(z) = \frac{1}{1 + z^2}.
\]

**SOLUTION.** The radius of convergence of the series defining \( h \) is 1 since
\[
\limsup_{n \to \infty} \left( \frac{1}{n} \right) = 1.
\]

By Theorem 2.10, we get
\[
h'(z) = 1 - z^2 + z^4 - \cdots = \sum_{n=0}^{\infty} (-z^2)^n = \frac{1}{1 - (-z^2)} = \frac{1}{1 + z^2}.
\]

4. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), and \( h(z) = \sum_{n=0}^{\infty} c_n z^n \) be power series with positive radii of convergence.

(a) Is it possible for \( f \) to satisfy
\[
f \left( \frac{1}{n} \right) = \frac{1}{n^2} = f \left( -\frac{1}{n} \right)
\]
for all \( n \in \mathbb{N} \)? If so, what is \( f \)?

**SOLUTION.** Since \( f \) is continuous at 0, we have that
\[
f(0) = \lim_{n \to \infty} f \left( \frac{1}{n} \right) = \lim_{n \to \infty} \frac{1}{n^2} = 0.
\]
The given condition says that the two power series\(^2\)
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \tilde{f}(z) = z^2
\]
agree on the set
\[
S = \{ n^{-1} \mid n \in \mathbb{Z}^\times \} \cup \{ 0 \} \subseteq \mathbb{C}.
\]
Since \( S \) has an accumulation point 0, the Uniqueness Theorem for power series (ie. Corollary 2.13) implies that \( f = \tilde{f} \), ie. \( f(z) = z^2 \).

(b) Is it possible for \( g \) to satisfy
\[
g \left( \frac{1}{n} \right) = \frac{1}{n^3} = g \left( -\frac{1}{n} \right) \quad (4.3)
\]
for all \( n \in \mathbb{N} \)? If so, what is \( g \)?

**SOLUTION.** Since \( g \) is continuous at 0, we have that
\[
g(0) = \lim_{n \to \infty} g \left( \frac{1}{n} \right) = \lim_{n \to \infty} \frac{1}{n^3} = 0.
\]
The first equality says that the two power series
\[
g(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad \tilde{g}(z) = z^3
\]
agree on the set
\[
S = \{ n^{-1} \mid n \in \mathbb{N} \} \cup \{ 0 \} \subseteq \mathbb{C}.
\]
\(^2\)Note that \( \tilde{f} \) is a power series with all its coefficients 0 except the 2nd, which is 1. We use the same observation in (b) and (c).
Since $S$ has an accumulation point 0, the Uniqueness Theorem for power series implies that $g = \hat{g}$, ie.

$$g(z) = z^3 \quad \text{for all } z \in D(0, r), \quad (4.4)$$

where $r > 0$ is the radius of convergence of $g$.

However, the second equality says that the two power series

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad \hat{g}(z) = -z^3$$

agree on $S$ and the Uniqueness Theorem for power series now implies that $g = \hat{g}$, ie.

$$g(z) = -z^3 \quad \text{for all } z \in D(0, r). \quad (4.5)$$

Since (4.4) and (4.5) cannot be simultaneously satisfied. It is not possible for $g$ to satisfy (4.3).

(c) Is it possible for $h$ to satisfy

$$h \left( \frac{1}{n} \right) = \frac{(-1)^n}{n} \quad (4.6)$$

for all $n \in \mathbb{N}$? If so, what is $h$?

**SOLUTION.** Since $h$ is continuous at 0, we have that

$$h(0) = \lim_{n \to \infty} h \left( \frac{1}{n} \right) = \lim_{n \to \infty} \frac{(-1)^n}{n} = 0.$$ 

The condition for $n$ even, ie.

$$h \left( \frac{1}{2n} \right) = \frac{1}{2n}$$

says that the two power series

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad \hat{h}(z) = z$$

agree on the set

$$S_e = \{(2n)^{-1} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{C}.$$ 

Since $S_e$ has an accumulation point 0, the Uniqueness Theorem for power series implies that $h = \hat{h}$, ie.

$$h(z) = z \quad \text{for all } z \in D(0, r) \quad (4.7)$$

where $r > 0$ is the radius of convergence of $h$.

However, the condition for $n$ odd, ie.

$$h \left( \frac{1}{2n+1} \right) = -\frac{1}{2n+1}$$

says that the two power series

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad \hat{h}(z) = -z$$

agree on the set

$$S_o = \{(2n+1)^{-1} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{C}.$$ 

Since $S_o$ has an accumulation point 0, the Uniqueness Theorem for power series implies that $h = \hat{h}$, ie.

$$h(z) = -z \quad \text{for all } z \in D(0, r). \quad (4.8)$$

Since (4.7) and (4.8) cannot be simultaneously satisfied. It is not possible for $h$ to satisfy (4.6).
5. Let \( f \) be defined by

\[
   f(z) = \sum_{n=0}^{\infty} \alpha_n z^n
\]

where

\[
   \limsup_{n \to \infty} |\alpha_n|^{1/n} = 0.
\]

Suppose \( f \) satisfies the following:

\[
   f'' \left( \frac{i^n}{n^3} \right) + f \left( \frac{i^n}{n^3} \right) = 0 \quad \text{for all } n \in \mathbb{N},
\]

\[
   f(0) = a,
\]

\[
   f'(0) = b.
\]

What is \( f \)? Express \( f \) in terms of \( a, b \) and familiar functions.

**Solution.** Note that since \( f \) has infinite radius of convergence, then so does \( f'' \). Hence the power series

\[
   f''(z) + f(z) = \sum_{n=0}^{\infty} [(n+2)(n+1)\alpha_{n+2} + \alpha_n]z^n
\]

has infinite radius of convergence by Problem Set 2, Problem 4(a). In particular, \( f'' + f \) is continuous on \( \mathbb{C} \) and so

\[
   f''(0) + f(0) = \lim_{n \to \infty} \left[ f'' \left( \frac{i^n}{n^3} \right) + f \left( \frac{i^n}{n^3} \right) \right] = 0.
\]

This, together with the fact that \( f'' + f \) is zero on a sequence

\[
   z_n := \frac{i^n}{n^3}
\]

where \( \lim_{n \to \infty} z_n = 0 \) implies that

\[
   f'' + f \equiv 0
\]

on the whole of \( \mathbb{C} \). Now since \( f \) is entire, its Taylor series expansion about 0 that converges everywhere in \( \mathbb{C} \), and is given by

\[
   f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.
\]

Likewise for \( f'' \), we have

\[
   f''(z) = \sum_{n=0}^{\infty} \frac{f^{(n+2)}(0)}{n!} z^n.
\]

Hence, for all \( z \in \mathbb{C} \),

\[
   f''(z) + f(z) = \sum_{n=0}^{\infty} \left[ \frac{f^{(n+2)}(0)}{n!} + \frac{f^{(n)}(0)}{n!} \right] z^n.
\]

Now since \( f'' + f \equiv 0 \), we must have \( f^{(n+2)}(0) = -f^{(n)}(0) \) for all \( n \in \mathbb{N} \cup \{0\} \), ie.

\[
   f(0) = -f''(0) = \cdots = (-1)^n f^{(2n)}(0) = \cdots
\]

\[
   f'(0) = -f''(0) = \cdots = (-1)^n f^{(2n+1)}(0) = \cdots.
\]
Hence

\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \]

\[ = f(0) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + f'(0) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \]

\[ = a \cos z + b \sin z. \]

Note that the ‘splitting’ of the first power series into a sum of two power series is valid because of Problem Set 2, Problem 4(a) and the fact that all three power series have infinite radius of convergence.