Ω ⊆ C will always denote a region unless specified otherwise. For \( f : \Omega \to \mathbb{C} \) and \( c \in \mathbb{C} \) a constant, we write \( f \equiv c \) to mean that \( f(z) = c \) for all \( z \in \Omega \).

1. Let \( f : \Omega \to \mathbb{C} \) with \( f(x+iy) = u(x,y) + iv(x,y) \). Let \( z_0 \in \Omega \) and suppose there exists a function \( \varphi : \mathbb{C} \to \mathbb{C} \) such that
\[
\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - \varphi(h)}{h} = 0.
\]

Recall from Problem Set 2, Problem 2 that \( f \) is real differentiable if \( \varphi \) is real linear and \( f \) is complex differentiable if \( \varphi \) is complex linear. Recall from Problem Set 2, Problem 1 that a real linear \( \varphi \) satisfies
\[
\varphi(x+iy) = (ax + by) + i(cx + dy)
\]
for some \((a b) \in \mathbb{R}^{2 \times 2}\) and a complex linear \( \varphi \) satisfies
\[
\varphi(x+iy) = (ax - cy) + i(cx + ay)
\]
for some \((a c) \in \mathbb{R}^{2 \times 2}\).

(a) Show that if \( f \) is real differentiable at \( z_0 = x_0 + iy_0 \in \Omega \), then the matrix \((a b) \) is given by
\[
\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] = \left[ \begin{array}{cc} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{array} \right].
\]

(b) Show that if \( f \) is complex differentiable at \( z_0 = x_0 + iy_0 \in \Omega \), then the matrix \((-c a) \) is given by
\[
\left[ \begin{array}{cc} a & c \\ -c & a \end{array} \right] = \left[ \begin{array}{cc} u_x(x_0,y_0) & u_y(x_0,y_0) \\ -u_y(x_0,y_0) & u_x(x_0,y_0) \end{array} \right].
\]

(c) Suppose \( f \) is analytic on \( \Omega \) (i.e., complex differentiable at all \( z \in \Omega \)) and that \( u(x,y) = \varphi(x) \), \( v(x,y) = \psi(y) \), i.e. \( f \) takes the form
\[
f(x+iy) = \varphi(x) + i\psi(y).
\]

Show that \( f(z) = az + b \) for some \( a, b \in \mathbb{C} \).

2. Let \( f \) be defined by
\[
f(z) = \sum_{n=0}^{\infty} \alpha_n z^n
\]
where the series has a positive radius of convergence \( R \). For each \( m \in \mathbb{N} \), let \( s_m : \mathbb{C} \to \mathbb{C} \) be the \( m \)th partial sum
\[
s_m(z) = \sum_{n=0}^{m} \alpha_n z^n.
\]

Prove that
\[
\sum_{m=0}^{\infty} |f(z) - s_m(z)| < \infty
\]
for all \( z \in D(0,R) \).

Date: October 3, 2008 (Version 1.1); due: October 10, 2007.
3. (a) Let \( f \) be defined by
\[
f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.
\]
Prove that
\[
f''(z) - f(z) = 0.
\]
State which theorem(s) you have used here. For what values of \( z \) is this valid?
(b) Let \( g \) be defined by
\[
g(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2}.
\]
Prove that
\[
z^2 g''(z) + zg'(z) - 4z^2 g(z) = 0.
\]
For what values of \( z \) is this valid?
(c) Let \( h \) be defined by the power series
\[
h(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots.
\]
Prove that
\[
h'(z) = \frac{1}{1 + z^2}.
\]
4. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), and \( h(z) = \sum_{n=0}^{\infty} c_n z^n \) be power series with positive radii of convergence.
(a) Is it possible for \( f \) to satisfy
\[
f\left(\frac{1}{n}\right) = \frac{1}{n^2} = f\left(-\frac{1}{n}\right)
\]
for all \( n \in \mathbb{N} \)? If so, what is \( f \)?
(b) Is it possible for \( g \) to satisfy
\[
g\left(\frac{1}{n}\right) = \frac{1}{n^3} = g\left(-\frac{1}{n}\right)
\]
for all \( n \in \mathbb{N} \)? If so, what is \( g \)?
(c) Is it possible for \( h \) to satisfy
\[
h\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}
\]
for all \( n \in \mathbb{N} \)? If so, what is \( h \)?
5. Let \( f \) be defined by
\[
f(z) = \sum_{n=0}^{\infty} \alpha_n z^n
\]
where
\[
\limsup_{n \to \infty} |\alpha_n|^{1/n} = 0.
\]
Suppose \( f \) satisfies the following:
\[
f''\left(\frac{i^n}{n^3}\right) + f\left(\frac{i^n}{n^3}\right) = 0 \quad \text{for all } n \in \mathbb{N},
f(0) = a,
f'(0) = b.
\]
What is \( f \)? Express \( f \) in terms of \( a, b \) and familiar functions.