For $a \in \mathbb{C}$, $r > 0$, we write $D^*(a, r) := \{ z \in \mathbb{C} \mid 0 < |z - a| < r \}$. We write $\mathbb{C}^\times = \mathbb{C}\{0\}$. You may use without proof any results that had been proved in the lectures.

1. (a) Find a function with a non-isolated singularity at 0.
   (b) Suppose $f$ has a non-isolated singularity at $a_0 \in \mathbb{C}$ satisfying the following:
      (i) $(a_n)_{n=1}^\infty$ is a sequence of poles of $f$ that converges to $a_0$;
      (ii) $f$ is analytic on $\Omega := \mathbb{C}\{a_n \mid n = 0, 1, 2, \ldots \}$.
      Show that $f(D^*(a_0, \varepsilon) \cap \Omega)$ is dense in $\mathbb{C}$ for every $\varepsilon > 0$.

2. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $a \in \Omega$ and $f : \Omega \{a\} \to \mathbb{C}$ be a function with an isolated singularity at $a$.
   (a) Prove the converse of Casorati-Weierstraß’s theorem, ie. show that if $f(D^*(a, \varepsilon))$ is dense in $\mathbb{C}$ for every $\varepsilon > 0$ (as long as $D^*(a, \varepsilon) \subseteq \Omega$), then $f$ has an essential singularity at $a$.
   (b) Show that if $f$ has a pole or an essential singularity at $a$, then $e^f$ has an essential singularity at $a$.

3. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $a \in \Omega$ and $f : \Omega \{a\} \to \mathbb{C}$ be a function with an isolated singularity at $a$. Suppose for some $m \in \mathbb{N}$ and $\varepsilon > 0$,
   \[
   \text{Re } f(z) \leq -m \log |z - a|
   \]
   for all $z \in D^*(a, \varepsilon)$. Show that $a$ is a removable singularity of $f$.

4. Let $f : \mathbb{C}^\times \to \mathbb{C}$ be analytic on $\mathbb{C}^\times$ with a pole of order 1 at 0. Show that if $f(z) \in \mathbb{R}$ for all $|z| = 1$, then for some $\alpha \in \mathbb{C}^\times$ and $\beta \in \mathbb{R}$,
   \[
   f(z) = \alpha z + \frac{1}{\alpha \bar{z}} + \beta
   \]
   for all $z \in \mathbb{C}^\times$.

5. Let $f : D^*(0, 1) \to \mathbb{C}$ be analytic. Show that if
   \[
   |f(z)| \leq \log \frac{1}{|z|}
   \]
   for all $z \in D^*(0, 1)$, then $f \equiv 0$. 

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