1. Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. Show that if
\[
|f(z)| \leq \frac{1}{|\text{Re} z|}
\]
for all \( z \in \mathbb{C} \), then \( f \equiv 0 \). What happens if we replace the condition by
\[
|f(z)| \leq \frac{1}{|\text{Im} z|}
\]
for all \( z \in \mathbb{C} \)?

2. Consider the functions defined by
\[
g_a(z) = \frac{e^{i\pi z/2} - 1}{e^{i\pi z/2} + 1} \quad \text{and} \quad g_b(z) = \frac{e^{\pi z/2} - 1}{e^{\pi z/2} + 1}.
\]
Show that \( g_a \) maps the set
\[
\Omega_a := \{ z \in \mathbb{C} \mid -1 < \text{Re} z < 1 \}
\]
to \( D(0,1) \) while \( g_b \) maps the set
\[
\Omega_b := \{ z \in \mathbb{C} \mid -1 < \text{Im} z < 1 \}
\]
to \( D(0,1) \). Hence or otherwise, prove the following.
(a) Let \( f : D(0,1) \to \mathbb{C} \) be an analytic function that satisfies \( f(0) = 0 \). Suppose
\[
|\text{Re} f(z)| < 1
\]
for all \( z \in D(0,1) \), prove that
\[
|f'(0)| \leq \frac{4}{\pi}.
\]
(b) Let \( S \) be the set of functions defined by
\[
S = \{ f : \Omega_b \to \mathbb{C} \mid f \text{ analytic, } |f| < 1 \text{ on } \Omega_b, \text{ and } f(0) = 0 \}.
\]
Prove that
\[
\sup_{f \in S} |f(1)| = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}.
\]

3. Let \( f : D(0,1) \to \mathbb{C} \) be analytic and \( |f(z)| < 1 \) for all \( z \in D(0,1) \).
(a) Show that
\[
\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.
\]
(b) Suppose there exist two distinct points \( a, b \in D(0,1) \) such that \( f(a) = a \) and \( f(b) = b \).
Show that \( f(z) = z \) for all \( z \in D(0,1) \).
(c) Suppose there exist \( a \in D(0,1), a \neq 0, \) such that \( f(a) = 0 = f(-a) \). Show that \( |f(0)| \leq |a|^2 \). What can you conclude if \( |f(0)| = |a|^2 \)?