We write \( \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \). We say that \( f \) is identically zero on \( \Omega \), denoted \( f \equiv 0 \), if \( f(z) = 0 \) for all \( z \in \Omega \). When we write \( f(z) > 0 \) for a complex function \( f \), it is implicit that \( f(z) \in \mathbb{R} \) (likewise for \( <, \leq, \geq \), and if 0 is replaced by any other real number). You may use without proof any results that had been proved in the lectures.

1. Prove or disprove. Given any entire function \( f : \mathbb{C} \to \mathbb{C} \), there exist functions \( g, h : \mathbb{C} \to \mathbb{C} \) such that
   (i) \( g \) and \( h \) are both entire functions,
   (ii) \( f(z) = g(z) + ih(z) \) for all \( z \in \mathbb{C} \),
   (iii) \( g(x) \in \mathbb{R} \) and \( h(x) \in \mathbb{R} \) for all \( x \in \mathbb{R} \).
   
   **Solution.** By Theorem 4.3 in the lectures, \( f \) has a power series representation
   \[
   f(z) = \sum_{n=0}^{\infty} a_n z^n
   \]
   for all \( z \in \mathbb{C} \) (i.e. the radius of convergence of the RHS is \( \infty \)). Let \( a_n = \beta_n + i \gamma_n \) where \( \beta_n, \gamma_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \cup \{0\} \). We define \( g, h \) by
   \[
   g(z) = \sum_{n=0}^{\infty} \beta_n z^n \quad \text{and} \quad h(z) = \sum_{n=0}^{\infty} \gamma_n z^n.
   \]
   Note that \( |\beta_n| \leq |a_n| \) for all \( n \in \mathbb{N} \cup \{0\} \). So
   \[
   0 \leq \limsup_{n \to \infty} \sqrt[2]{|\beta_n|} \leq \limsup_{n \to \infty} \sqrt[2]{|a_n|} = 0,
   \]
   and the series defining \( g \) has an infinite radius of convergence. Likewise, the series defining \( h \) has an infinite radius of convergence. Hence \( g \) and \( h \) both entire functions. Since the series defining \( f, g, \) and \( h \) all have infinite radii of convergence, the following equation is valid for all \( z \in \mathbb{C} \):
   \[
   \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \beta_n z^n + i \sum_{n=0}^{\infty} \gamma_n z^n
   \]
   (note that this is not true in general — see Chapter 2, Exercise 10 in the textbook). Hence we have \( f(z) = g(z) + ih(z) \) for all \( z \in \mathbb{C} \). Since \( \beta_n, \gamma_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \cup \{0\} \), it is clear that \( g(x) \in \mathbb{R} \) and \( h(x) \in \mathbb{R} \) for all \( x \in \mathbb{R} \).

2. (a) Let \( J \) be defined by the power series
   \[
   J(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{z}{2} \right)^{2n}.
   \]
   Prove that
   \[
   z^2 J''(z) + zJ'(z) + z^2 J(z) = 0.
   \]
   State which theorem(s) you have used here. For what values of \( z \) is this valid?
   
   **Solution.** Observe that
   \[
   \lim_{n \to \infty} \left| \frac{(-1)^{n+1}/((n+1)!)^2}{((-1)^n/(n!)^2)^2} \right| = \lim_{n \to \infty} \frac{1}{(n+1)^2} = 0
   \]

*Date: October 21, 2007 (Version 1.0)*
and so $J$ has infinite radius of convergence. By a result in the lectures,

$$J'(z) = \sum_{n=0}^{\infty} 2n \frac{(-1)^n}{(n!)^2} 2^{2n} z^{2n-1} \quad \text{and} \quad J''(z) = \sum_{n=0}^{\infty} 2n(2n-1) \frac{(-1)^n}{(n!)^2} 2^{2n} z^{2n-2}$$

and both series have infinite radii of convergence. So

$$z^2 J''(z) + zJ'(z) + z^2 J(z) = \sum_{n=0}^{\infty} (z^2 + 2n + 2n(2n-1)) \frac{(-1)^n}{(n!)^2} 2^{2n} z^{2n}$$

$$= \sum_{n=0}^{\infty} (2 + 4n^2) \frac{(-1)^n}{(n!)^2} 2^{2n} z^{2n}.$$

Now observe that

$$(z^2 + 4n^2) \frac{(-1)^n}{(n!)^2} 2^{2n} z^{2n} = \frac{(-1)^n}{(n!)^2} 2^{2n} z^{2n} = \frac{(-1)^n - 1}{((n-1)!)^2} 2^{2(n-1)} z^{2n}.$$ (2.1)

For a fixed $z \in \mathbb{C}$, let

$$Z_n := \frac{(-1)^n}{(n!)^2} 2^{2n} z^{2(n+1)}$$

for $n \in \mathbb{N} \cup \{0\}$ and set $Z_{-1} := 0$. So by (2.1),

$$\sum_{n=0}^{\infty} (Z_n - Z_{n-1}) = Z_{-1} = 0,$$

since we have a telescopic sum. Since $z \in \mathbb{C}$ is arbitrary,

$$z^2 J''(z) + zJ'(z) + z^2 J(z) = 0$$

for all $z \in \mathbb{C}$.

(b) More generally, for any $k \in \mathbb{N} \cup \{0\}$, let $J_k$ be defined by the power series

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left( \frac{z}{2} \right)^{2n+k}.$$ 

Prove that

$$z^2 J''_k(z) + zJ'_k(z) + (z^2 - k^2)J_k(z) = 0.$$ 

For what values of $z$ is this valid?

**SOLUTION.** For $n \in \mathbb{N} \cup \{0\}$, let

$$a_n := \frac{(-1)^n}{2^{2n+k} n!(n+k)!}.$$

Observe that

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{1}{4(n+1)(n+k+1)} = 0$$

and so $J_k$ has infinite radius of convergence. By a result in the lectures,

$$J'_k(z) = \sum_{n=0}^{\infty} (2n + k)a_n z^{2n+k-1} \quad \text{and} \quad J''_k(z) = \sum_{n=0}^{\infty} (2n + k)(2n + k - 1)a_n z^{2n+k-2}$$

and both series have infinite radii of convergence. So the coefficient of $z^{2n+k}$ in the sum $z^2 J''_k(z) + zJ'_k(z) + (z^2 - k^2)J_k(z)$ is

$$(2n + k)(2n + k - 1)a_n + (2n + k)a_n + a_{n-1} - k^2 a_n = 4n(n+k)a_n + a_{n-1}.$$ 

But note that

$$\frac{a_{n-1}}{a_n} = -4n(n+k).$$
and so the coefficient of \(z^{2n+k}\) is 0. Hence

\[z^2J''_k(z) + zJ'_k(z) + (z^2 - k^2)J_k(z) = 0\]

for all \(z \in \mathbb{C}\).

3. (a) For \(i = 1, 2\) and \(j = 1, 2, 3, 4\), determine the value of

\[\int_{\Gamma_j} f_i\]

where \(f_i\) is defined by

\[f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = z^3,\]
\[f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = \overline{z};\]

and \(\Gamma_j\) is defined by

\[z_1 : [0, 1] \to \mathbb{C}, \quad z_1(t) = 1 + it,\]
\[z_2 : [0, 1] \to \mathbb{C}, \quad z_2(t) = e^{-i\pi t},\]
\[z_3 : [0, 1] \to \mathbb{C}, \quad z_3(t) = e^{i\pi t},\]
\[z_4 : [0, 1] \to \mathbb{C}, \quad z_4(t) = 1 + it + t^2.\]

SOLUTION. Routine. The answers are given by

\[\int_{\Gamma_j} f_1 = \begin{cases} (1 + i)^4/4 - 1/4 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \\ 0 & \text{if } j = 3, \\ (2 + i)^4/4 - 1/4 & \text{if } j = 4. \end{cases}\]

\[\int_{\Gamma_j} f_2 = \begin{cases} i + 1/2 & \text{if } j = 1, \\ -\pi i & \text{if } j = 2, \\ \pi i & \text{if } j = 3, \\ 2 + 2i/3 & \text{if } j = 4. \end{cases}\]

Note that Cauchy’s theorem applies for \(j = 2, 3\) (closed \(\Gamma\)) in the case \(i = 1\) (analytic \(f\)) but not in the case \(i = 2\) (non-analytic \(f\)).

(b) For \(i = 1, 2\), determine the value of

\[\int_{\Gamma} g_i\]

where \(g_i\) is defined by

\[g_1 : \mathbb{C} \to \mathbb{C}, \quad g_1(z) = ze^{z^2},\]
\[g_2 : \mathbb{C} \to \mathbb{C}, \quad g_2(z) = \sin z;\]

and \(\Gamma\) is the path from 0 to \(1 + i\), taken along the parabola \(y = x^2\).

SOLUTION. Both \(g_1\) and \(g_2\) are entire and observe that

\[G_1(z) = \frac{1}{2}e^{z^2}\]

is a primitive for \(g_1\) and

\[G_2(z) = -\cos z\]

is a primitive for \(g_2\). So

\[\int_{\Gamma} g_1 = G_1(1 + i) - G_1(0) = \frac{1}{2}(e^{2i} - 1)\]

and

\[\int_{\Gamma} g_2 = G_2(1 + i) - G_2(0) = 1 - \cos(1 + i).\]

4. Let \(S = \{x + iy \in \mathbb{C} \mid x, y \in [0, 1]\}\) be the unit square in \(\mathbb{C}\). Let \(f\) be analytic on a region \(\Omega\) that contains \(S\). Suppose the following is true:
(i) for all $z$ with $\text{Re}(z) = 0$, $0 \leq \text{Im}(z) \leq 1$,

$$f(z + 1) - f(z) \geq 0;$$

(ii) for all $z$ with $0 \leq \text{Re}(z) \leq 1$, $\text{Im}(z) = 0$,

$$f(z + i) - f(z) \geq 0.$$

Show that $f$ is a constant.

**Solution.** Since $f$ is analytic on $\Omega$ and $\Gamma = \partial S$ is a rectangular path contained in $\Omega$, we may apply Cauchy's theorem to get

$$0 = \int_{\Gamma} f(z) \, dz = \int_0^1 f(x) \, dx + i \int_0^1 f(1 + yi) \, dy - \int_0^1 f(x + i) \, dx - i \int_0^1 f(yi) \, dy \tag{4.2}$$

Hence we have

$$\int_0^1 [f(x) - f(x + i)] \, dx = 0 \quad \text{and} \quad \int_0^1 [f(1 + iy) - f(iy)] \, dy = 0.$$

By condition (ii),

$$f(x) - f(x + i) \leq 0$$

for all $0 \leq x \leq 1$; and by condition (i),

$$f(1 + iy) - f(iy) \geq 0$$

for all $0 \leq y \leq 1$. Furthermore, since $f$ is analytic and thus continuous in $\Omega$, the integrands in (4.2) must be identically zero for $x, y \in [0, 1]$. So

$$f(x) = f(x + i) \quad \text{and} \quad f(iy) = f(iy + 1)$$

for $x, y \in [0, 1]$. So

$$f(z) = f(z + i) \quad \text{and} \quad f(z) = f(z + 1)$$

for on subsets of $\Omega$ with limit points and so they must be true for all $z \in \Omega$. Now we may define a function $F : \mathbb{C} \to \mathbb{C}$ as follows:

$$F(x + iy) = f(\langle x \rangle + i(y))$$

where $\langle x \rangle$ denotes the fractional part of $x \in \mathbb{R}$. Hence,

$$F(z) = F(z + i) \quad \text{and} \quad F(z) = F(z + 1)$$

for all $z \in \mathbb{C}$ and thus

$$|F(z)| \leq \max_{z \in S}|f(z)|$$

for all $z \in \mathbb{C}$. Note that $F$ is analytic for all $z \in \mathbb{C}$ (why?). Now the RHS is bounded since $F$ is continuous and $S$ is compact. Hence $F$ is a bounded entire function and Liouville’s Theorem implies that it must be a constant function. And so $f$ is also a constant function.

5. (a) Let $f$ be an entire function. Show that if

$$\lim_{|z| \to \infty} \frac{|f(z)|}{|z|} = 0,$$

then $f$ is a constant.
SOLUTION. By Corollary 4.4 in the lectures, the function \( g : \mathbb{C} \to \mathbb{C} \) defined by
\[
g(z) = \begin{cases} 
\frac{f(z) - f(0)}{z} & z \neq 0, \\
 f'(0) & z = 0,
\end{cases}
\]
is also entire. Now for \( z \in \mathbb{C}^\times \),
\[
|g(z)| = \left| \frac{f(z) - f(0)}{z} \right| \leq \left| \frac{f(z)}{z} \right| + \left| \frac{f(0)}{z} \right|
\]
and so
\[
\lim_{|z| \to \infty} |g(z)| = 0. \tag{5.3}
\]
Hence \( g \) is bounded. So \( g \) is a constant function by Liouville’s Theorem and (5.3) further implies that \( g \equiv 0 \). It then follows that \( f(z) = f(0) \) for all \( z \in \mathbb{C} \) and so \( f \) is a constant function.

(b) Let \( f \) be an entire function. Suppose \( g : \mathbb{R} \to \mathbb{R} \) is such that \( \lim_{x \to \infty} |g(x)| = 0 \). Show that if
\[
|f(z)| \leq |z| \cdot |g(|z|)|
\]
for all \( z \in \mathbb{C}^\times \), then \( f \equiv 0 \).

SOLUTION. Observe that
\[
\lim_{|z| \to \infty} \frac{|f(z)|}{|z|} \leq \lim_{|z| \to \infty} |g(|z|)| = 0
\]
and so part (a) above applies and we have \( f(z) = f(0) \) for all \( z \in \mathbb{C} \). Now
\[
|f(0)| = \lim_{z \to 0} |f(z)| \leq \lim_{z \to 0} |z| \cdot |g(|z|)| = 0
\]
and so \( f \equiv 0 \).