MATH 110: LINEAR ALGEBRA
FALL 2007/08
PROBLEM SET 9 SOLUTIONS

In the following $V$ will denote a finite-dimensional vector space over $\mathbb{R}$ that is also an inner product space with inner product denoted by $\langle \cdot , \cdot \rangle$. The norm induced by $\langle \cdot , \cdot \rangle$ will be denoted $\| \cdot \|$. 

1. Let $S$ be a subset (not necessarily a subspace) of $V$. We define the \textit{orthogonal annihilator} of $S$, denoted $S^\perp$, to be the set 

$$S^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in S \}.$$

(a) Show that $S^\perp$ is always a subspace of $V$.

Solution. Let $v_1, v_2 \in S^\perp$. Then $\langle v_1, w \rangle = 0$ and $\langle v_2, w \rangle = 0$ for all $w \in S$. So for any $\alpha, \beta \in \mathbb{R}$, 

$$\langle \alpha v_1 + \beta v_2, w \rangle = \alpha \langle v_1, w \rangle + \beta \langle v_2, w \rangle = 0$$

for all $w \in S$. Hence $\alpha v_1 + \beta v_2 \in S^\perp$.

(b) Show that $S \subseteq (S^\perp)^\perp$.

Solution. Let $w \in S$. For any $v \in S^\perp$, we have $\langle v, w \rangle = 0$ by definition of $S^\perp$. Since this is true for all $v \in S^\perp$ and $\langle w, v \rangle = \langle v, w \rangle$, we see that $\langle w, v \rangle = 0$ for all $v \in S^\perp$, i.e. $w \in (S^\perp)^\perp$.

(c) Show that span$(S) \subseteq (S^\perp)^\perp$.

Solution. Since $(S^\perp)^\perp$ is a subspace by (a) (it is the orthogonal annihilator of $S^\perp$) and $S \subseteq (S^\perp)^\perp$ by (b), we have span$(S) \subseteq (S^\perp)^\perp$.

(d) Show that if $S_1$ and $S_2$ are subsets of $V$ and $S_1 \subseteq S_2$, then $S_2^\perp \subseteq S_1^\perp$.

Solution. Let $v \in S_2^\perp$. Then $\langle v, w \rangle = 0$ for all $w \in S_2$ and so for all $w \in S_1$ (since $S_1 \subseteq S_2$). Hence $v \in S_1^\perp$.

(e) Show that $((S^\perp)^\perp)^\perp = S^\perp$.

Solution. Applying (d) to (b) with $S_1 = S$ and $S_2 = (S^\perp)^\perp$, we get 

$$((S^\perp)^\perp)^\perp \subseteq S^\perp.$$ 

Apply (c) to $S^\perp$, we get 

$$\text{span}(S^\perp) \subseteq ((S^\perp)^\perp)^\perp$$

but by (a), $\text{span}(S^\perp) = S^\perp$. Hence we get equality.

(f) Show that either $S \cap S^\perp = \emptyset$ or the zero subspace $\{0_V\}$.

Solution. If $S \cap S^\perp \neq \emptyset$, then let $v \in S \cap S^\perp$. Since $v \in S^\perp$, we have $\langle v, w \rangle = 0$ for all $w \in S$. Since $v \in S$, in particular, $\|v\|^2 = \langle v, v \rangle = 0$, implying that $v = 0_V$. In other words, the only vector in $S \cap S^\perp$ is $0_V$. So $S \cap S^\perp = \{0_V\}$. On the other hand, if $0_V \notin S$, then $0_V \notin S \cap S^\perp$ and so $S \cap S^\perp = \emptyset$ (if not, then the previous argument gives a contradiction).

2. Let $W$ be a subspace of $V$. The subspace $W^\perp$ is called the \textit{orthogonal complement} of $W$.

(a) Show that $V = W \oplus W^\perp$.

Solution. Since $W$ and $W^\perp$ are both subspaces (the latter by Problem 1(a)), $0_V \in W$ and $0_V \in W^\perp$. So by Problem 1(f), we have that $W \cap W^\perp = \{0_V\}$. It remains to show that
Clearly \( W + W^\perp \subseteq V \). For the converse, let \( v \in V \) and let \( \{w_1, \ldots, w_r\} \) be an orthonormal basis of \( W \). Consider

\[
x := \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 + \cdots + \langle v, w_r \rangle w_r
\]

and

\[
y := v - x.
\]

Clearly \( x \in W \). We claim that \( y \in W^\perp \). For any \( w \in W \), we could write

\[
w = \langle w, w_1 \rangle w_1 + \langle w, w_2 \rangle w_2 + \cdots + \langle w, w_r \rangle w_r
\]

since \( \{w_1, \ldots, w_r\} \) is an orthonormal basis of \( W \). So

\[
\langle y, w \rangle = \langle v - x, w \rangle
\]

\[
= \langle v, w \rangle - \langle x, w \rangle
\]

\[
= \langle v, \sum_{i=1}^r \langle w, w_i \rangle w_i \rangle - \langle \sum_{i=1}^r \langle v, w_i \rangle w_i, w \rangle
\]

\[
= \sum_{i=1}^r \langle v, w_i \rangle \langle w, w_i \rangle - \sum_{i=1}^r \langle v, w_i \rangle \langle w_i, w \rangle
\]

\[
= 0
\]

since \( \langle w, w_i \rangle = \langle w_i, w \rangle \). Hence \( v = x + y \in W + W^\perp \).

(b) Show that \( W = (W^\perp)^\perp \).

**Solution.** By Problem 1(b), we have \( W \subseteq (W^\perp)^\perp \). But by (a), we have

\[
W \oplus W^\perp = V = W^\perp \oplus (W^\perp)^\perp
\]

and so

\[
dim(W) + dim(W^\perp) = dim(W^\perp) + dim((W^\perp)^\perp)
\]

and so

\[
dim(W) = dim((W^\perp)^\perp).
\]

Hence \( W = (W^\perp)^\perp \).

3. Let \( A \in \mathbb{R}^{m \times n} \) and regard \( A : \mathbb{R}^n \to \mathbb{R}^m \) as a linear transformation in the usual way. Show that

\[
\text{nullsp}(A^\top) = (\text{colsp}(A))^\perp
\]

and

\[
\text{rowsp}(A) = (\text{nullsp}(A))^\perp.
\]

Hence deduce that

\[
\mathbb{R}^m = \text{nullsp}(A^\top) \oplus \text{colsp}(A)
\]

and

\[
\mathbb{R}^n = \text{rowsp}(A) \oplus \text{nullsp}(A).
\]

**Solution.** Clearly \( \text{colsp}(A) = \{Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n\} \). So if \( v \in (\text{colsp}(A))^\perp \), then

\[
\langle v, Ax \rangle = 0
\]

for all \( x \in \mathbb{R}^n \). So

\[
A^\top v, x = (A^\top v)^\top x = v^\top Ax = \langle v, Ax \rangle = 0
\]

for all \( x \in \mathbb{R}^n \). In particular, we may pick \( x = A^\top v \) to get

\[
\|A^\top v\|^2 = \langle A^\top v, A^\top v \rangle = 0,
\]

giving us

\[
A^\top v = 0.
\]

Hence \( v \in \text{nullsp}(A^\top) \). Conversely, if \( v \in \text{nullsp}(A^\top) \), then

\[
\langle v, Ax \rangle = v^\top Ax = (A^\top v)^\top x = 0^\top x = 0
\]
for all $x \in \mathbb{R}^n$. So $v \in (\text{colsp}(A))^\perp$. This proves the first equality. For the second equality, just note that

$$\text{rowsp}(A) = \text{colsp}(A^\top)$$

and apply the first equality to get

$$(\text{colsp}(A^\top))^\perp = \text{nullsp}(A),$$

and then apply Problem 2(b) to get

$$\text{rowsp}(A) = ((\text{rowsp}(A))^\perp)^\perp = (\text{nullsp}(A))^\perp.$$  

The next two inequalities are immediate consequences of Problem 2(a).

4. Let $A, B \in \mathbb{R}^{n \times n}$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$.

(a) Show that if $\langle Ax, y \rangle = 0$ for all $x, y \in \mathbb{R}^n$, then $A = O$, the zero matrix.

SOLUTION. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Observe that if $A = [a_{ij}]_{i,j=1}^n$, then

$$a_{ij} = e_i^\top A e_j = \langle e_i, A e_j \rangle = 0$$

by letting $x$ and $y$ be $e_i$ and $e_j$ respectively. Since this is true for all $i$ and $j$, we have that $A = O$.

(b) Is it true that if $\langle Ax, x \rangle = 0$ for all $x \in \mathbb{R}^n$, then $A = O$, the zero matrix?

SOLUTION. Recall notation from Homework 4, Problem 1(d). Let $A \in \wedge^2(\mathbb{R}^n)$, i.e. $A^\top = -A$. Then

$$\langle Ax, x \rangle = x^\top A^\top x = -x^\top A x = -\langle x, Ax \rangle = -\langle Ax, x \rangle,$$

implying that $\langle Ax, x \rangle = 0$. So any non-zero skew-symmetric matrix would be a counter example. An explicit $2 \times 2$ example is given by

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$  

(c) Is it true that if $\langle Ax, x \rangle = \langle Bx, x \rangle$ for all $x \in \mathbb{R}^n$, then $A = B$?

SOLUTION. Any two distinct skew-symmetric matrices would provide a counter example. An explicit $2 \times 2$ example is given by

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  

5. Let $\mathscr{B}_1 = \{x_1, \ldots, x_n\}$ and $\mathscr{B}_2 = \{y_1, \ldots, y_n\}$ be two bases of $\mathbb{R}^n$. $\mathscr{B}_1$ and $\mathscr{B}_2$ are not necessarily orthonormal bases.

(a) Show that there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$Q x_i = y_i, \quad i = 1, \ldots, n,$$

if and only if

$$\langle x_i, x_j \rangle = \langle y_i, y_j \rangle, \quad i, j = 1, \ldots, n.$$  

SOLUTION. By a theorem we have proved in the lectures, an orthogonal matrix $Q$ preserves inner product and so satisfies

$$\langle y_i, y_j \rangle = \langle Q x_i, Q x_j \rangle = \langle x_i, x_j \rangle.$$

It remains to show that if the latter condition is satisfied, then such an orthogonal matrix may be defined. First, observe that since $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases, the equations

$$Q x_i = y_i, \quad i = 1, \ldots, n,$$
defines the matrix $Q$ uniquely\(^1\). We only need to verify that $Q$ is orthogonal. Let $x \in \mathbb{R}^n$ and let

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

be the representation of $x$ in terms of $\mathcal{B}_1$. Then

$$\|Qx\|^2 = \langle Qx, Qx \rangle$$

$$= \langle \sum_{i=1}^{n} \alpha_i Qx_i, \sum_{j=1}^{n} \alpha_j Qx_j \rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \langle x_i, x_j \rangle$$

Since this is true for all $x \in \mathbb{R}^n$, i.e. $Q$ preserves norm, the aforementioned theorem in the lectures imply that $Q$ is orthogonal.

(b) Show that if $\mathcal{B}_1$ is an orthonormal basis and if $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then

$$\{Qx_1, \ldots, Qx_n\}$$

is also an orthonormal basis.

**SOLUTION.** Clearly $Qx_1, \ldots, Qx_n$ are linearly independent since if

$$\alpha_1 Qx_1 + \cdots + \alpha_n Qx_n = 0,$$

then

$$Q(\alpha_1 x_1 + \cdots + \alpha_n x_n) = 0,$$

and so

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = Q^\top 0 = 0$$

and so $\alpha_1 = \cdots = \alpha_n = 0$ by the linear independence of $x_1, \ldots, x_n$. Hence $\{Qx_1, \ldots, Qx_n\}$ is a basis (since the set has size $n = \dim(\mathbb{R}^n)$). The orthonormality follows from the fact that $Q$ preserves inner product and that $x_1, \ldots, x_n$ form an orthonormal basis:

$$\langle Qx_i, Qx_j \rangle = \langle x_i, x_j \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

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\(^1\)In fact, $Q$ is just the matrix whose coordinate representation with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ is the identity matrix, i.e. $[Q]_{\mathcal{B}_1,\mathcal{B}_2} = I$. Explicitly, if we let $X$ and $Y$ be the matrices whose columns are $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively, then $Q = YX^{-1}$. 

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