For a matrix $A = [a_{ij}]_{i,j=1}^{m,n} \in \mathbb{F}^{m \times n}$, the transpose of $A$ is the matrix $A^\top = [a_{ji}]_{j,i=1}^{n,m} \in \mathbb{F}^{n \times m}$. A square matrix $A \in \mathbb{R}^{n \times n}$ is called symmetric if $a_{ij} = a_{ji}$ for all $i, j \in \{1, \ldots, n\}$ and is called skew-symmetric or anti-symmetric if $a_{ij} = -a_{ji}$ for all $i, j \in \{1, \ldots, n\}$.

A basis will be denoted $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ when the ordering of the basis vectors is not important and $\mathcal{B} = [u_1, u_2, \ldots, u_n]$ when it is.

1. Find a basis for the following vector spaces
   (a) $\mathbb{C}^{n \times n}$ as a vector space over $\mathbb{C}$.
   (b) $\mathbb{C}^{n \times n}$ as a vector space over $\mathbb{R}$.
   (c) $S^2(\mathbb{R}^n) = \{ A \in \mathbb{R}^{n \times n} \mid A^\top = A \}$, i.e. the real symmetric matrices, as a vector space over $\mathbb{R}$.
   (d) $\wedge^2(\mathbb{R}^n) = \{ A \in \mathbb{R}^{n \times n} \mid A^\top = -A \}$, i.e. the real skew-symmetric matrices, as a vector space over $\mathbb{R}$.

Show that
$$\mathbb{R}^{n \times n} = S^2(\mathbb{R}^n) \oplus \wedge^2(\mathbb{R}^n).$$

2. Let $V$ be a vector space and let $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ be a basis for $V$. Suppose $n \geq 2$. Let $\mathcal{B}' = \{u_1 + u_2, u_2 + u_3, \ldots, u_{n-1} + u_n, u_n + u_1\}$.

Is $\mathcal{B}'$ also a basis for $V$? How about the converse? i.e. if $\mathcal{B}'$ is a basis for $V$, is $\mathcal{B}$ necessarily a basis for $V$? [Hint: the answer depends on whether $n$ is odd or even.]

3. For the following real vector spaces $V$, find the coordinate representation of the element $v$ with respect to the ordered basis $\mathcal{B}$.
   (a) $V = \mathbb{P}_2 = \{ax + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$. Find $[v]_{\mathcal{B}}$ where $v = 6 - 5x + 2x^2$, $\mathcal{B} = [1, -1 + x, 1 - 2x + x^2]$.
   (b) $V = S^2(\mathbb{R}^2)$. Find $[v]_{\mathcal{B}}$ where $v = \begin{pmatrix} 4 & 11 \\ -11 & -7 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -1 & -5 \end{pmatrix}$.
   (c) $V = \mathbb{P}_3 = \{ax + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$. Find $[v]_{\mathcal{B}}$ where $v = a + bx + cx^2 + dx^3$, $\mathcal{B} = [1, (1 - x), (1-x)^2, (1-x)^3]$.

4. Let $W_1, W_2$ be subspaces of a vector space $V$.
   (a) Show that
   $$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2).$$
   (4.1)
   (b) Let $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ be a basis for $V$. Suppose $W$ is a $k$-dimensional subspace of $V$.

   Show that for any subset $\{u_{i_1}, u_{i_2}, \ldots, u_{i_m}\} \subseteq \mathcal{B}$ with $m > n - k$, there exists a nonzero vector $w \in W$ that is a linear combination of $u_{i_1}, u_{i_2}, \ldots, u_{i_m}$.
   (c) Find an expression for $\dim(W_1 \oplus W_2)$ similar to (4.1) (assuming that it makes sense to write $W_1 \oplus W_2$).
   (d) Let $W$ be a subspace of $V$ having the property that there exists a unique subspace $W'$ such that $V = W \oplus W'$. Show that $W = V$. 

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Date: October 7, 2007 (Version 1.0); posted: October 7, 2007; due: October 12, 2007.