1. Let $\mathbb{P}_7$ be the vector space of polynomials of degree not more than 7, ie.
\[ \mathbb{P}_7 = \{ p(x) = a_0 + a_1 x + \cdots + a_7 x^7 \mid a_i \in \mathbb{R} \text{ for all } i \}. \]
Consider the following subsets of $\mathbb{P}_7$:
(a) $W_a = \{ p(x) \in \mathbb{P}_7 \mid \text{degree of } p(x) \text{ is 3} \}$,
(b) $W_b = \{ p(x) \in \mathbb{P}_7 \mid \text{degree of } p(x) \text{ is not more than 3} \}$,
(c) $W_c = \{ p(x) \in \mathbb{P}_7 \mid 2p(0) = p(1) \}$,
(d) $W_d = \{ p(x) \in \mathbb{P}_7 \mid p(x) \geq 0 \text{ for all } x \text{ with } 0 \leq x \leq 1 \}$,
(e) $W_e = \{ p(x) \in \mathbb{P}_7 \mid p(x) = p(1-x) \text{ for all } x \}$.
Which of these are subspaces of $\mathbb{P}_7$? Justify your answer.

2. Let $W_1$ and $W_2$ be subspaces of a vector space $V$. Suppose $W_1$ is neither the zero subspace $\{0\}$ nor the vector space $V$ itself and likewise for $W_2$. Show that there exists a vector $v \in V$ such that $v \notin W_1$ and $v \notin W_2$. [If a subspace $W = \{0\}$ or $V$, we call it a trivial subspace and otherwise we call it a non-trivial subspace.]

3. Let $W_1$ and $W_2$ be subspaces of a vector space $V$. Let $W_1 + W_2$ be the subset of $V$ defined by
\[ W_1 + W_2 = \{ w_1 + w_2 \in V \mid w_1 \in W_1, w_2 \in W_2 \}. \]
(a) Prove that $W_1 + W_2$ is a subspace of $V$. [This subspace is called the sum of $W_1$ and $W_2$.]
(b) Prove that $W_1 \cup W_2 \subseteq W_1 + W_2$.
(c) Prove that $W_1 + W_2$ is the smallest subspace of $V$ that contains $W_1 \cup W_2$. In other words, show that if $U$ is any subspace of $V$ such that
\[ W_1 \cup W_2 \subseteq U, \]
then
\[ W_1 + W_2 \subseteq U. \]
(d) What is $W + W$?
(e) Under what condition is $W_1 \cup W_2$ a subspace? The condition I am looking for is not $W_1 \cup W_2 = W_1 + W_2$ (although this is true).
(f) Let $W$ be any subspace of $V$. We know from basic set theory that
\[ W \cap (W_1 \cup W_2) = (W \cap W_1) \cup (W \cap W_2). \]
But is the following true?
\[ W \cap (W_1 + W_2) = (W \cap W_1) + (W \cap W_2) \]
(g) Show that if $W_1 \cap W_2 = \{0\}$, then every $w \in W_1 + W_2$ can be expressed in a unique way as
\[ w = w_1 + w_2 \]
with $w_1 \in W_1$ and $w_2 \in W_2$. In other words, show that if
\[ w = w_1' + w_2' \]
is another such expression, then we must have $w_1 = w_1'$ and $w_2 = w_2'$. [If $W_1 \cap W_2 = \{0\}$, then $W_1 + W_2$ is written $W_1 \oplus W_2$ and is called the direct sum of $W_1$ and $W_2$.]
4. (a) Consider the following elements of $\mathbb{R}^4$, the vector space of 4-tuples,
\[
\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ -5 \\ -4 \\ 0 \end{bmatrix}.
\]
Is $\mathbf{a}$ a linear combination of $\mathbf{b}$ and $\mathbf{c}$? If so, write $\mathbf{a}$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$.
(b) Consider the following elements of $\mathbb{R}^{2\times2}$, the vector space of 2-by-2 matrices,
\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}.
\]
Is $A$ a linear combination of $B$ and $C$? If so, write $A$ as a linear combination of $B$ and $C$.
(c) Consider the following elements of $\mathbb{P}_3$, the vector space of polynomials of degree not more than 3,
\[
a(x) = 1 + 2x + 3x^2 + x^3, \quad b(x) = 3 - x + 2x^2 + 2x^3, \quad c(x) = 1 - 5x - 4x^2.
\]
Is $a(x)$ a linear combination of $b(x)$ and $c(x)$? If so, write $a(x)$ as a linear combination of $b(x)$ and $c(x)$.

5. Let $\mathbf{u} = (1, -3, 2)$ and $\mathbf{v} = (2, -1, 1) \in \mathbb{R}^3$.
(a) Is $(2, -5, 4) \in \mathbb{R}^3$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?
(b) Write $(1, 7, -4) \in \mathbb{R}^3$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
(c) Find $a \in \mathbb{R}$ so that $(1, a, 5) \in \mathbb{R}^3$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
(d) Find conditions on $a, b, c \in \mathbb{R}$ so that $(a, b, c) \in \mathbb{R}^3$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.