1.5: Show that if $f : X \rightarrow Y$ is one-to-one and $g : Y \rightarrow Z$ is one-to-one then $g \circ f : X \rightarrow Z$ is one-to-one.
Solution. $g \circ f(x_1) = g \circ f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

1.16: Prove $2^{n-1} \leq n!$ for each $n \in \mathbb{N}$.
Solution. Let $A := \{n \in \mathbb{N} \mid 2^{n-1} \leq n!\}$. Since $2^{1-1} = 1 \leq 1!$, so $1 \in A$. If $p \in A$, then $2^{p-1} \leq p!$ and so $2^{(p+1)-1} = 2 \cdot 2^{p-1} \leq 2 \cdot p! \leq (p+1) \cdot p! = (p+1)!$ as $2 \leq p+1$ for all $p \in \mathbb{N}$, hence $p+1 \in A$. By P5, $A = \mathbb{N}$.

1.20: Prove the cancelation laws in $\mathbb{Z}$:
(a) $j + k = j + l$ implies $k = l$ for all $j, k, l \in \mathbb{Z}$.
Solution. Let $j = [(a,b)], k = [(c,d)], l = [(e,f)]$ where $a, \ldots, f \in \mathbb{N}$. Then

$$[(a,b)] + [(c,d)] = [(a,b)] + [(e,f)]$$

$$[(a+c, b+d)] = [(a+e, b+f)]$$

$$(a+c) + (b+f) = (a+c) + (b+d)$$

$$(a+b) + (c+f) = (a+b) + (e+d)$$

$$(c+f) = (e+d)$$

where the last step is by the additive cancelation law in $\mathbb{N}$. Since $c + f = e + d$, we see that

$$k = [(c,d)] = [(e,f)] = l.$$ 

(b) $j \cdot k = j \cdot l$ implies $k = l$ for all $j, k, l \in \mathbb{Z}$ with $j \neq 0$.
Solution. Let $j = [(a,b)], k = [(c,d)], l = [(e,f)]$ where $a, \ldots, f \in \mathbb{N}$ and $a \neq b$. Then

$$[(a,b)] \cdot [(c,d)] = [(a,b)] \cdot [(e,f)]$$

$$[(ac + bd, ad + bc)] = [(ae + bf, af + be)]$$

$$(ac + bd) + (af + be) = (ad + bc) + (ae + bf)$$

Since $a \neq b$, we must have $a < b$ or $b < a$ by trichotomy. Suppose $a < b$. Then there exists $n \in \mathbb{N}$ such that $a + n = b$. The last equation above becomes

$$ac + ad + nd + af + ae + ne = ad + ac + nc + ae + af + nf$$

and applying the additive cancelation law in $\mathbb{N}$ we get

$$nd + ne = nc + nf$$

$$n(d + e) = n(c + f)$$

and applying the multiplicative cancelation law in $\mathbb{N}$ we get

$$c + f = d + e$$

and so $k = [(c,d)] = [(e,f)] = l$, as required.
1.25: Prove that the order properties in \( \mathbb{Q} \):
(a) \( u < v \) implies \( u + w < v + w \) for every \( w \in \mathbb{Q} \).

Solution. Let \( \mathbb{Q}_+ \) denote the subset of positive rationals. Since \( u < v \), there exists 
\( p \in \mathbb{Q}_+ \) such that \( u + p = v \). Let \( w \in \mathbb{Q} \), by associativity and commutativity, \((u + w) + p = (u + p) + w = v + w\). Hence \( u + w < v + w \) for every \( w \in \mathbb{Q} \).
(b) \( u < v \) and \( w > 0 \) implies \( u \cdot w < v \cdot w \) for every \( w \in \mathbb{Q} \).

Solution. Since \( u < v \) and \( 0 < w \), there exists \( p \in \mathbb{Q}_+ \) such that \( u + p = v \) and 
\( q \in \mathbb{Q}_+ \) such that \( 0 + q = w \). In particular \( w = q \). Now \( v \cdot w = (u + p) \cdot w = u \cdot w + p \cdot q \). Since the product of two positive rationals is positive, \( p \cdot q \in \mathbb{Q}_+ \) and hence \( u \cdot w + p \cdot q = v \cdot w \) implies that \( u \cdot w < v \cdot w \).

1.28: Prove that for any \( r \in \mathbb{Q} \) the set \( \{ x \in \mathbb{Q} \mid x > r \} \) is a ray in \( \mathbb{Q} \).

Solution. Let \( U = \{ x \in \mathbb{Q} \mid x > r \} \). Since \( r \in \mathbb{Q} \), we have \( r + 1 \in \mathbb{Q} \) and so \( r + 1 \in U \) and so \( U \neq \emptyset \); also since \( r \notin U \), \( U \neq \mathbb{Q} \). Hence \( U \) is a nonempty proper subset of \( \mathbb{Q} \). If \( x \in U \) and \( y > x \), then \( y > x > r \) and so \( y > r \) by transitivity, so \( y \in U \). Suppose \( x_0 \in U \) is a first element, then \( x_0 < x \) for all \( x \in U \) and \( x_0 > r \). Let \( y_0 = (x_0 + r)/2 \). Then \( y_0 > r \) and \( y_0 \in \mathbb{Q} \) and so \( y_0 \in U \). But \( y_0 < x_0 \) and so \( x_0 \) cannot be a first element, contradicting our assumption. Hence \( U \) has no first element. These show that \( U \) is a ray in \( \mathbb{Q} \).

1.33: Prove that a nonempty set \( S \) of real numbers is bounded if and only if there is a nonnegative real number \( K \) such that \(-K \leq x \leq K\) for every \( x \in S \).

Solution. \( \Rightarrow \) Let \( l, u \in \mathbb{R} \) be lower and upper bounds of \( S \). Then \( l \leq x \leq u \) for every \( x \in S \). Let \( K = \max\{|l|, |u|\} \). Note that \( |l| \leq K \) and so \(-K \leq l \); also \( |u| \leq K \) and so \( u \leq K \). Hence \(-K \leq l \leq x \leq u \leq K \) for every \( s \in S \), as required.

\( \Leftarrow \) If \(-K \leq x \leq K \) for every \( x \in S \), then \( l = -K \) is a lower bound for \( S \) and \( u = K \) is an upper bound for \( S \). So \( S \) is bounded.

1.37: Prove that, if they exist, the least upper bound and the greatest lower bound of a nonempty set \( S \subset \mathbb{R} \) are unique.

Solution. Let \( M_1 \) and \( M_2 \) both be least upper bounds of \( S \). Since \( M_1 \) is an upper bound, and \( M_2 \) is a least upper bound, we must have \( M_1 \leq M_2 \). Since \( M_2 \) is an upper bound, and \( M_1 \) is a least upper bound, we must have \( M_2 \leq M_1 \). Hence \( M_1 = M_2 \). Ditto for greatest lower bound.

1.38: Prove Bernoulli’s inequality: \( (1 + x)^n \geq 1 + nx \) for every real number \( x \geq -1 \) and every \( n \in \mathbb{N} \).

Solution. Let \( A = \{ n \in \mathbb{N} \mid (1 + x)^n \geq 1 + nx \} \) for all \( x \in (-1, \infty) \). Since \((1 + x)^1 \geq 1 + 1 \cdot x\) for all \( x \geq -1 \), \( 1 \in A \). If \( p \in A \), then \((1 + x)^p \geq 1 + px\) for all \( x \geq -1 \) and so \((1 + x)^{p+1} = (1 + x)(1 + x)^p \geq (1 + x)(1 + px) = 1 + (p + 1)x + px^2 \geq 1 + (p + 1)x\) for all \( x \geq -1 \), since \( px^2 \geq 0 \) for all \( p \in \mathbb{N} \), hence \( p + 1 \in A \). By \( \textbf{P5} \), \( A = \mathbb{N} \).

1.43: Prove that every interval of real numbers contains infinitely many rational and irrational numbers.

Solution. Let \( I \) be an interval of real numbers and let \( a = \inf I \) and \( b = \sup I \). Then \((a, b) \subseteq I \). If \( a = -\infty \), then \( I \) contains an infinite number of disjoint intervals \( I_n = (b - n, b - n + 1) \), \( n \in \mathbb{N} \). If \( b = \infty \), then \( I \) contains an infinite number of disjoint intervals \( I_n = (a + n - 1, a + n) \), \( n \in \mathbb{N} \). If \( -\infty < a < b < \infty \), then \( I \) contains an infinite number of disjoint intervals \( I_n = ((a + (2^n - 1)b)/2^{n-1}, (a + (2^n - 1)b)/2^n) \). In all three cases, each \( I_n \) contains a rational and an irrational by Theorems 1.9 and 1.10. Since \( I_n \cap I_m = \emptyset \),
if $n \neq m$, the rationals/irrationals in these intervals are distinct. So $I \supseteq \bigcup_{n \in \mathbb{N}} I_n$ contains infinitely many rationals and irrationals.