1. Show that $\sqrt{2} + \sqrt[3]{3}$ is
(a) an algebraic number;

**Solution.** Let $x = \sqrt{2} + \sqrt[3]{3}$. Taking powers and rearranging successively, we get

$$x - \sqrt{2} = \sqrt[3]{3}$$

$$(x - \sqrt{2})^3 = 3$$

$$x^3 - 3x^2\sqrt{2} + 6x - 2\sqrt{2} = 3$$

$$x^3 + 6x - 3 = \sqrt{2}(3x^2 + 2)$$

$$(x^3 + 6x - 3)^2 = 2(3x^2 + 2)^2$$

$$x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1 = 0$$

This calculation shows that $\sqrt{2} + \sqrt[3]{3}$ is a zero of $f(x) = x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1$. Since $f$ has coefficients in $\mathbb{Z}$ (and therefore in $\mathbb{Q}$), $\sqrt{2} + \sqrt[3]{3}$ is an algebraic number.

(b) an irrational number.

**Solution.** Suppose there exists $p, q \in \mathbb{Z}$, $q \neq 0$, $\gcd(p, q) = 1$ such that

$$\frac{p}{q} = \sqrt{2} + \sqrt[3]{3}.$$ 

By (a), $f(p/q) = 0$ and therefore $q^6f(p/q) = 0$, i.e.

$$p^6 - 6p^4q^2 - 6p^3q^3 + 12p^2q^4 - 36pq^5 + q^6 = 0.$$ 

In other words,

$$p^6 - 6p^4q^2 - 6p^3q^3 + 12p^2q^4 - 36pq^5 = -q^6.$$ 

Since $p$ divides the LHS, it divides $-q^6$, and so it must divide $q$. Since $\gcd(p, q) = 1$, this is only possible if $p = \pm 1$. But then

$$1 < \sqrt{2} + \sqrt[3]{3} \left| \frac{p}{q} \right| = \frac{1}{|q|} \leq 1$$

yields a contradiction.

2. Let $X, Y \subseteq \mathbb{R}$ be nonempty sets. Using the definitions of supremum and infimum, prove that

$$\sup(X \cup Y) = \max\{\sup X, \sup Y\},$$

$$\inf(X \cup Y) = \min\{\inf X, \inf Y\}.$$ 

**Solution.** Suppose $X$ and $Y$ are both bounded above. Let

$$A := \sup X \quad \text{and} \quad B := \sup Y.$$ 

We may assume wlog that $A \leq B$ and so

$$\max\{\sup X, \sup Y\} = \max\{A, B\} = B.$$ 

Let $z \in X \cup Y$,

- if $z \in X$, then $z \leq A \leq B$, 

*Proof:*

1. **Show that $\sqrt{2} + \sqrt[3]{3}$ is**
   (a) an algebraic number;
   
   **Solution.** Let $x = \sqrt{2} + \sqrt[3]{3}$. Taking powers and rearranging successively, we get
   
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   This calculation shows that $\sqrt{2} + \sqrt[3]{3}$ is a zero of $f(x) = x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1$. Since $f$ has coefficients in $\mathbb{Z}$ (and therefore in $\mathbb{Q}$), $\sqrt{2} + \sqrt[3]{3}$ is an algebraic number.

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   Since $p$ divides the LHS, it divides $-q^6$, and so it must divide $q$. Since $\gcd(p, q) = 1$, this is only possible if $p = \pm 1$. But then
   
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2. **Let $X, Y \subseteq \mathbb{R}$ be nonempty sets. Using the definitions of supremum and infimum, prove that**
   
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   **Solution.** Suppose $X$ and $Y$ are both bounded above. Let
   
   $$A := \sup X \quad \text{and} \quad B := \sup Y.$$ 

   We may assume wlog that $A \leq B$ and so
   
   $$\max\{\sup X, \sup Y\} = \max\{A, B\} = B.$$ 

   Let $z \in X \cup Y$,
   
   - if $z \in X$, then $z \leq A \leq B$, 

   *Proof:*
3. For nonempty sets $X, Y \subseteq \mathbb{R}$, we define

$$\neg X := \{ -x \in \mathbb{R} \mid x \in X \}, \quad X + Y := \{ x + y \in \mathbb{R} \mid x \in X, y \in Y \}$$

($X + Y$ is often called the Minkowski sum). Using the definitions of supremum and infimum, prove the following equalities:

$$\sup(-X) = -\inf X, \quad \inf(-X) = -\sup X,$$

$$\sup(X + Y) = \sup X + \sup Y, \quad \inf(X + Y) = \inf X + \inf Y.$$  

**Solution.** Suppose $X$ is bounded below. Let $a = \inf X$. Then

- $x \geq a$ for all $x \in X$;
- for any $\varepsilon > 0$, there is an $x^* \in X$ such that $x^* < a + \varepsilon$.

Multiplying the inequalities above by $-1$, we get

- $y \leq -a$ for all $y \in -X$;
- for any $\varepsilon > 0$, there is an $y^* \in -X$ such that $y^* > -a - \varepsilon$.

These two conditions together imply that

$$\sup(-X) = -a.$$  

If $X$ is not bounded below, then $-X$ is not bounded above and therefore

$$\sup(-X) = +\infty = -(-\infty) = -\inf X.$$  

The second equality can be established in a similar fashion. Suppose $X$ is bounded above.

Let $A = \sup X$. Then

- $x \leq A$ for all $x \in X$;
- for any $\varepsilon > 0$, there is an $x^* \in X$ such that $x^* > A - \varepsilon$.

Multiplying the inequalities above by $-1$, we get

- $y \geq -A$ for all $y \in -X$;
- for any $\varepsilon > 0$, there is an $y^* \in -X$ such that $y^* < -A + \varepsilon$.  


These two conditions together imply that
\[ \inf(-X) = -A. \]
If \( X \) is not bounded above, then \(-X\) is not bounded below and therefore
\[ \inf(-X) = -\infty = -(+\infty) = -\sup X. \]

Now for the third equality. Suppose \( X \) and \( Y \) are bounded above. Let \( A = \sup X \) and \( B = \sup Y \). Then \( A \) is an upper bound or \( X \) and \( B \) is an upper bound of \( Y \). Hence \( A + B \) is an upper bound of \( X + Y \). Moreover, for any \( \epsilon > 0 \), there are \( x^* \in X \) and \( y^* \in Y \) such that \( x^* > A - \epsilon/2 \) and \( y^* > B - \epsilon/2 \). Therefore \( x^* + y^* > A + B - \epsilon \). Since \( z^* = x^* + y^* \in X + Y \), this shows that
\[ \sup(X + Y) = A + B. \]
If \( X \) or \( Y \) is unbounded above, then \( X + Y \) is also unbounded above, and by definition,
\[ \sup X + \sup Y = \begin{cases} \sup X + \infty & \text{if } Y \text{ unbounded}, \\ +\infty + \sup Y & \text{if } X \text{ unbounded}, \end{cases} = +\infty = \sup(X + Y). \]
The fourth equality can be established in a similar fashion. Suppose \( X \) and \( Y \) are bounded below. Let \( a = \inf X \) and \( b = \inf Y \). Then \( a \) is a lower bound or \( X \) and \( b \) is a lower bound of \( Y \). Hence \( a + b \) is a lower bound of \( X + Y \). Moreover, for any \( \epsilon > 0 \), there are \( x^* \in X \) and \( y^* \in Y \) such that \( x^* < a + \epsilon/2 \) and \( y^* < b + \epsilon/2 \). Therefore \( x^* + y^* < a + b + \epsilon \). Since \( z^* = x^* + y^* \in X + Y \), this shows that
\[ \inf(X + Y) = a + b. \]
If \( X \) or \( Y \) is unbounded below, then \( X + Y \) is also unbounded below, and by definition,
\[ \inf X + \inf Y = \begin{cases} \inf X - \infty & \text{if } Y \text{ unbounded}, \\ -\infty + \inf Y & \text{if } X \text{ unbounded}, \end{cases} = -\infty = \inf(X + Y). \]

4. Define
\[
S_1 = \{ x \in \mathbb{Q} \mid x^2 + x + 1 > 0 \},
S_2 = \{ x \in \mathbb{R} \mid x^2 + x + 1 > 0 \},
S_3 = \{ x + x^{-1} \in \mathbb{R} \mid x > 0 \},
S_4 = \{ mn/(1 + m + n) \in \mathbb{Q} \mid m, n \in \mathbb{N} \},
S_5 = \{ \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q} \mid n \in \mathbb{N} \}.
\]
For \( i = 1, \ldots, 5 \), determine the values of \( \max S_i, \min S_i, \sup S_i, \) and \( \inf S_i \) or state why they do not exist.

**SOLUTION.** It is clear that
\[ \sup S_1 = \sup S_2 = \sup S_3 = +\infty \]
and so \( \max S_1, \max S_2, \max S_3 \) do not exist. Observe that
\[ x^2 + x + 1 = \left( x + \frac{1}{2} \right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0 \]
for all \( x \in \mathbb{Q} \subset \mathbb{R} \) and so \( S_1 = \mathbb{Q} \) and \( S_2 = \mathbb{R} \). Hence
\[ \inf S_1 = \inf S_2 = -\infty \]
and so \( \min S_1 \) and \( \min S_2 \) do not exist. For any \( x > 0 \), the arithmetic mean-geometric mean inequality implies that
\[ x + \frac{1}{x} \geq 2 \sqrt{x \cdot \frac{1}{x}} = 2 \]
and this lower bound of $S_3$ is attained when $x = 1$, we see that

$$\min S_3 = \inf S_3 = 2.$$  

Note that since $m, n \geq 1$,

$$\frac{1 + m + n}{mn} = \frac{1}{mn} + \frac{1}{n} + \frac{1}{m} \leq 1 + 1 + 1 = 3$$

and so we must have

$$\frac{mn}{1 + m + n} \geq \frac{1}{3}.$$  

This lower bound of $S_4$ is attained by $m = n = 1$. Hence

$$\min S_4 = \inf S_4 = \frac{1}{3}.$$  

Setting $m = n$, we see that

$$\lim_{n \to \infty} \frac{n^2}{1 + 2n} = \infty$$

and so $S_4$ contains an unbounded subset \( \{ n^2/(1 + 2n) \in \mathbb{Q} \mid n \in \mathbb{N} \} \) and therefore $S_4$ is unbounded above. Hence

$$\sup S_4 = +\infty$$

and so max $S_4$ does not exist.

Let

$$s_n := \sum_{k=0}^{n} \frac{1}{k!}.$$  

From Math 1B, we know that $s_n$ is a monotone increasing sequence and therefore

$$1 = s_1 = \inf S_5 = \min S_5.$$  

We also know that

$$\lim_{n \to \infty} s_n = e \notin \mathbb{Q}$$

and thus

$$\sup S_5 = e$$

and max $S_5$ does not exist.