Determinants of path matrices and applications to Gaussian graphical models & trek systems

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(weighted) path matrix

\[ X_{ij} = \sum_{P: s_i \rightarrow t_j} w^+(P) \]
\[
\begin{bmatrix}
1 & e & ef \\
b & 1 + be & d + f + bef \\
ab & a + c + abe & 1 + (a + c)(d + f) + abef
\end{bmatrix}
\]
Theorem: (Karlin-MacGregor, Lindström, Gessel-Viennot)

Each minor $\Delta_{AB}$ of the weighted path matrix of a planar network counts the systems of non-intersecting paths connecting the sources in $A$ to the sinks in $B$.

2x2 let

$A_{01}$ $B_{01}$

$A_{00}$ $B_{00}$

$A_{02}$ $B_{02}$

$A_{03}$ $B_{03}$

$\rightarrow b_2$
\[ \det = 1 \]
\[ \Delta_{13,23} = (e)(1 + ad + cd) \]

Proofs from the Book, Aigner, Ziegler
totally positive matrices:
Every minor is strictly positive.

Degenerations give totally nonnegative matrices.
positive definite matrices

degenerations: hard?
  open
(to pos semidef)
Gaussian graphical models

\[ \Sigma = (I - \Lambda)^{-1} \Omega (I - \Lambda)^{-1} \]

Entries \( \Gamma_{ij} \) "count" treks between \( i \) and \( j \).

\[ \begin{array}{c}
  \text{ok} \\
  i \quad j
\end{array} \]
avoid

Sided Intersection

A

B
Then (T - Sullivan-Draisma)

\[ \sum_{A,B} \text{identially vanishes} \]

iff every system of treks from A to B has a sided intersection.
Alternative:

$\sum_{A,B} (k \times k)$ vanishes iff we can find two sets $C_{up}$ and $C_{down}$ satisfying

$|C_{up}| + |C_{down}| = k - 1$ and every trek between $A$ and $B$ hits $C_{up}$ on its up side or $C_{down}$ on its down side.
uses Menger's Thm
Fact:

trek separation can find some constraints that d-separation doesn't.
Before: acyclic.

Two main ways to deal with graphs with cycles:

1) Do essentially the same thing. Get a matrix of formal power series.

Determinants may have infinitely many terms, but we know which path families.
2) Based Postnikov's work on TNN Grassmannian. (See Talaska thesis/papers)

counts walks with signs according to topological winding #.

get formulas $\sum$