STAT391, Lecture 4

The forward measure

The price at time $t$ of an $\mathcal{F}_T$-measurable contingent claim $X$ is under some assumptions

$$\Pi_X(t, T) = \mathbb{E} \left[ e^{-\int_t^T r_s ds} X \big| \mathcal{F}_t \right] = B_t \mathbb{E} [B_T^{-1} X | \mathcal{F}_t].$$

If $X$ and the short rate of interest process $r$ were independent under $\bar{P}$, then the value of $X$ at time $t$ would be

$$\mathbb{E} \left[ e^{-\int_t^T r_s ds} \big| \mathcal{F}_t \right] \mathbb{E} [X | \mathcal{F}_t] = P(t, T) \mathbb{E} [X | \mathcal{F}_t].$$

In general this is not true, but below we shall see that there exists an equivalent measure $P^T$ called the forward measure so that

$$\Pi_X(t, T) = P(t, T) \mathbb{E}^T [X | \mathcal{F}_t].$$

This formula is sometimes very helpful for calculations.

Assume that the bond price can be written as (See Lecture 3),

$$dP(t, T) = r_t P(t, T) dt + v(t, T) P(t, T) d\tilde{W}_t,$$

where as usual $\tilde{W}$ is a Brownian motion under the risk neutral measure $\bar{P}$. Let

$$\eta_t = e^{\int_0^t v(s, T) d\tilde{W}_s - \frac{1}{2} \int_0^t v^2(s, T) ds}.$$  \hspace{1cm} (0.2)

Note that $\eta$ solves $d\eta_t = v(t, T) \eta_t d\tilde{W}_t$ with $\eta_0 = 1$. Assume that $\eta$ is a $\bar{P}$-martingale on $[0, T]$. That is the case if e.g. the Novikov condition

$$\mathbb{E} \left[ e^{\frac{1}{2} \int_0^T v^2(s, T) ds} \right] < \infty,$$

is satisfied, and this is trivially satisfied if $v(t, T)$ is deterministic. Then it follows from Girsanov’s theorem that the process

$$W_t^T = \tilde{W}_t - \int_0^t v(s, T) ds, \quad t \leq T,$$

is a martingale under the probability measure $P^T$ given by

$$\frac{dP^T}{dP} \bigg|_{\mathcal{F}_t} = \eta_t, \quad t \leq T.$$

(0.4)
Letting \( Y_t = \log P(t, T) \), a standard use of Itô’s formula gives that the solution of (0.1) is
\[
P(t, T) = P(0, T) e^{\int_0^t (r_s - \frac{1}{2} v^2(s, T)) ds + \int_0^t v(s, T) d\tilde{W}_s}.
\] (0.5)
Consequently
\[
\frac{P(t, T)}{B_t P(0, T)} = \frac{e^{\int_0^t (r_s - \frac{1}{2} v^2(s, T)) ds + \int_0^t v(s, T) d\tilde{W}_s}}{e^{\int_0^t r_s ds}} = e^{\int_0^t v(s, T) d\tilde{W}_s - \frac{1}{2} \int_0^t v^2(s, T) ds} = \eta_t,
\] (0.6)
and in particular
\[
\eta_T = \frac{1}{B_T P(0, T)},
\]
so that
\[
B_T^{-1} = P(0, T) \eta_T.
\] (0.7)
Before we continue, we shall need the following abstract version of Bayes theorem.

**Theorem 0.1** Let \( Q \) and \( P \) be equivalent martingale measures on a measure space \( (\Omega, \mathcal{F}) \), and let
\[
\eta = \frac{dQ}{dP}
\]
be the Radon-Nikodym derivative. If \( \mathcal{G} \subset \mathcal{F} \) is a sub \( \sigma \)-algebra and \( E_Q[|\psi|] < \infty \), then
\[
E_Q[\psi|\mathcal{G}] = \frac{E_P[\psi \eta | \mathcal{G}]}{E_P[\eta | \mathcal{G}]}.
\]
Equivalently
\[
E_P[\psi \eta | \mathcal{G}] = E_P[\eta | \mathcal{G}] E_Q[\psi | \mathcal{G}].
\]
Using this theorem we get what was announced in the beginning.

**Theorem 0.2** Assume the bond price follows (0.1), with the Brownian motion \( \tilde{W} \) generating the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). Assume also that the process \( \eta \) given by (0.2) is a martingale. Let \( X \) be an \( \mathcal{F}_T \)-measurable contingent claim, and assume that \( E^T[|X|] < \infty \) where the probability measure \( P^T \) is defined by (0.4). Then the price of \( X \) at time \( t \) equals
\[
\Pi_X(t, T) = P(t, T) E^T[X | \mathcal{F}_t].
\]

**Proof** By (0.6) and (0.7) and the Bayes theorem,
\[
\Pi_X(t, T) = B_t E^T[X | \mathcal{F}_t]
\]
\[
= B_t E^T[P(0, T) \eta_T X | \mathcal{F}_t]
\]
\[
= B_t P(0, T) E^T[\eta_T X | \mathcal{F}_t]
\]
\[
= B_t P(0, T) \eta_T E^T[X | \mathcal{F}_t]
\]
\[
= P(t, T) E^T[X | \mathcal{F}_t].
\]

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Why the name forward measure? To explain, let again $X$ be an $\mathcal{F}_T$-measurable contingent claim. The forward price $F_X(t, T)$ of $X$ at time $t$ equals

$$F_X(t, T) = \frac{\Pi_X(t, T)}{P(t, T)} = E^T[X|\mathcal{F}_t],$$

(0.8)
i.e. the expected value of $X$ under the forward measure $P^T$. Now let $V_t$ be the value of a self-financing portfolio so that $V_T = X$. Then of course $V_t = \Pi_X(t, T)$. Furthermore, letting

$$Y_t = \frac{V_t}{P(t, T)},$$

we get from the above and the fact that $P(T, T) = 1$, that $Y_t = E^T[Y_T|\mathcal{F}_t]$. Therefore, by the tower property of conditional expectation, for $s < t < T$,

$$E^T[Y_t|\mathcal{F}_s] = E^T[E^T[Y_T|\mathcal{F}_t]|\mathcal{F}_s] = E^T[Y_T|\mathcal{F}_s] = Y_s,$$

e.i. $Y$ is martingale. In other words, the forward price is a $P^T$-martingale.

Assume now that the forward rate follows

$$df(t, T) = -\sigma(t, T)S(t, T)dt + \sigma(t, T)dW_t,$$

where the special form of the drift is a consequence of the Heath-Jarrow-Morton drift condition. Here

$$S(t, T) = -\int_t^T \sigma(t, s)ds.$$

In Theorem 0.1c, Lecture 2, we saw that this implies that $P(t, T)$ follows (0.1) with $v(t, T) = S(t, T)$. Therefore, by (0.3)

$$df(t, T) = -\sigma(t, T)S(t, T)dt + \sigma(t, T)(dW^T_t + S(t, T)dt) = \sigma(t, T)dW^T_t,$$

so that the forward rate is a martingale under $P^T$. Now

$$r_T = f(T, T) = f(t, T) + \int_t^T \sigma(s, T)dW^T_s,$$

hence

$$f(t, T) = E^T[r_T|\mathcal{F}_t],$$

e.i. seen from time $t$, the expected value of the spot rate at time $T$ equals the forward rate for time $T$.

Assume that the contingent claim $X$ is in fact $\mathcal{F}_S$-measurable for $S < T$, although time of maturity is $T$, i.e. it has a deferred payout. Its value at time $t < S$ then equals, again
by the tower property of conditional expectations

\[
\Pi_X(t, T) = \mathbb{E} \left[ e^{-\int_t^T r_s ds} X \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E} \left[ \mathbb{E} \left[ e^{-\int_t^T r_s ds} e^{-\int_t^S r_s ds} X \bigg| \mathcal{F}_S \right] \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E} \left[ e^{-\int_t^S r_s ds} X \mathbb{E} \left[ e^{-\int_t^T r_s ds} \bigg| \mathcal{F}_S \right] \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E} \left[ e^{-\int_t^S r_s ds} X P(S, T) \bigg| \mathcal{F}_t \right] = \Pi_{P(S,T)}(t, S).
\]

From this equation we also get

\[
\Pi_X(t, T) = P(t, S) E^S[ XP(S, T) | \mathcal{F}_t ].
\]

These results are in fact quite reasonable. Why?

**Pricing of European bond-options**

We assume the same setup as above, and in particular the bond dynamics is given by (0.1). For \( t < T < T^* \), it follows from (0.8) that the forward price for delivery of a bond with maturity \( T^* \) at time \( T \) equals

\[
F_{P(T, T^*)}(t, T) = \frac{P(t, T^*)}{P(t, T)}, \quad (0.9)
\]

since the price at time \( t \) for that bond is \( P(t, T^*) \). We know that the forward price is a \( P^T \)-martingale, implying that there exists an adapted process \( \gamma(t, T, T^*) \) so that with \( Z = P(T, T^*) \) and \( F_Z(t, T) = F_{P(T, T^*)}(t, T) \), the process \( F_Z \) satisfies

\[
dF_Z(t, T) = F_Z(t, T) \gamma(t, T, T^*) dW_t^T, \quad t < T. \quad (0.10)
\]

This is because of the martingale property, the \( dt \) term must disappear. We will now identify the \( \gamma(t, T, T^*) \) term. Note that for any \( U > t \),

\[
\int_0^t (r_s - \frac{1}{2} v^2(s, U)) ds + \int_0^t v(s, U) dW_s
\]

\[
= \int_0^t (r_s - \frac{1}{2} v^2(s, U)) ds + \int_0^t v(s, U) (dW_s^T + v(s, T) ds)
\]

\[
= \int_0^t (r_s - \frac{1}{2} v^2(s, U) + v(s, U) v(s, T)) ds + \int_0^t v(s, U) dW_s^T.
\]

Hence by (0.5) and (0.9),

\[
F_Z(t, T) = \frac{P(0, T^*)}{P(0, T)} e^{\int_0^t \alpha_s ds + \int_0^t (v(s, T^*) - v(s, T)) dW_s^T},
\]

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where $a_s$, which is not important to us, depends on $v(s, T)$ and $v(s, T^*)$. Now from (0.10)

$$F_Z(t, T) = F_Z(0, T) e^{-\frac{1}{2} \int_0^t \gamma(s, T, T^*) ds + \int_0^t \gamma(s, T, T^*) d\tilde{W}_s.$$ 

Identification of the $d\tilde{W}_s$ terms in these two expressions for $F_Z(t, T)$ gives

$$\gamma(t, T, T^*) = v(t, T^*) - v(t, T).$$

(0.11)

Incidently it also gives that

$$a_s = -\frac{1}{2} (v(s, T^*) - v(s, T))^2.$$ 

We shall now find the price for an European bond option, i.e. for the option with exercise time $T$ and value then equal to

$$X = (P(T, T^*) - K)^+ = (F_Z(T, T) - K)^+.$$ 

By Theorem 0.1, the price at time $t$ is given as

$$\Pi_X(t, T) = P(t, T) E^T[(F_Z(T, T) - K)^+ | \mathcal{F}_t]$$

$$= P(t, T) E^T[(F_Z(T, T) - K) 1_{\{F_Z(T, T) > K\}} | \mathcal{F}_t]$$

(0.12)

$$= P(t, T) E^T[F_Z(T, T) 1_{\{F_Z(T, T) > K\}} | \mathcal{F}_t] - K P^T(F_Z(T, T) > K | \mathcal{F}_t)$$

since the expectation of an indicator equals the probability of the event given by the indicator.

When $v(t, T)$ is deterministic, we can actually find the value of the option.

**Theorem 0.3** Assume that the bond price dynamics follows

$$dP(t, T) = r_t P(t, T) dt + v(t, T) P(t, T) d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian motion under the risk neutral measure $\tilde{P}$. Assume also that $v(t, T)$ is a bounded deterministic function for all $t \leq T \leq T^*$.

Then the price at time $t$ of a European call option on a bond with maturity $T^*$, exercise price $K$ and delivery time $T$, i.e. for

$$X = (P(T, T^*) - K)^+,$$

is given as

$$\Pi_X(t, T) = P(t, T^*) N(d_1) - KP(t, T) N(d_2),$$

where

$$d_1 = \frac{\log P(t, T^*) - \log P(t, T) - \log K + \frac{1}{2} b^2(t, T, T^*)}{b(t, T, T^*)},$$

$$d_2 = d_1 - b(t, T, T^*).$$

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Here
\[ b^2(t, T, T^*) = \int_t^T (v(s, T^*) - v(s, T))^2 ds \]
and
\[ N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz. \]

For the proof we need the following simple lemma

**Lemma 0.1** Let the stochastic variable \( Y \sim \mathcal{N}(\mu, \sigma^2) \). Then

a)
\[ P(Y > k) = N\left( \frac{\mu - k}{\sigma} \right) \]

b)
\[ E\left[ e^{Y} 1_{\{Y > k\}} \right] = e^{k} e^{\frac{1}{2} \sigma^2} N\left( \frac{\mu + \sigma^2 - k}{\sigma} \right). \]

**Proof of Theorem 0.1** The proof is rather simple. From (0.10) we find that
\[ F_Z(T, T) = F_Z(t, T) e^{\int_t^T \gamma(s, T, T^*) dW_s^T - \frac{1}{2} \int_t^T \gamma^2(s, T, T^*) ds}, \]
where
\[ Y = \log F_Z(t, T) - \frac{1}{2} \int_t^T \gamma^2(s, T, T^*) ds + \int_t^T \gamma(s, T, T^*) dW_s^T. \]
Since \( v(t, T) \) is deterministic, then so is \( \gamma(t, T, T^*) \) given by (0.11). Therefore, conditioned on \( \mathcal{F}_t \),
\[ Y \sim \mathcal{N}\left( \log F_Z(t, T) - \frac{1}{2} \int_t^T \gamma^2(s, T, T^*) ds, \int_t^T \gamma^2(s, T, T^*) ds \right) \]
\[ = \mathcal{N}(\log F_Z(t, T) - \frac{1}{2} b^2(t, T, T^*), b^2(t, T, T^*)) \]
w.r.t. the measure \( P^T \). Now we can use (0.9), (0.12) and Lemma 0.1 to get
\[ \Pi_X(t, T) = P(t, T) E[e^{Y} 1_{\{Y > \log K\}}] - KP(t, T) P(Y > \log K) \]
\[ = P(t, T) e^{\log F_Z(t, T)} N(d_1) - KP(t, T) N(d_2) \]
\[ = P(t, T^*) N(d_1) - KP(t, T) N(d_2). \]

This ends the proof of the Theorem.

To price a European put option on a bond, let
\[ X_C = (P(T, T^*) - K)^+ \]
be the value of the call option at delivery date, and
\[ X_P = (K - P(T, T^*))^+ \]
the corresponding put value. Then
\[ X_C - X_P = P(T, T^*) - K. \]
But at time \( t < T \), the price for a \( T^* \) bond equals \( P(t, T^*) \) and the price for cash \( K \) equals \( P(t, T)K \). Hence we have the put-call parity
\[ \Pi_{X_C}(t, T) - \Pi_{X_P}(t, T) = P(t, T^*) - KP(t, T). \]

Futures

We assume the same setup as before, \( (\Omega, \mathcal{F}, P) \) is a measure space, and the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) is generated by the Brownian motion \( W \). Also let \( \hat{P} \) be the risk neutral probability measure, and \( \tilde{W} \) is a Brownian motion under \( \hat{P} \).

Consider a tradeable asset \( Y \) which in a time interval \( dt \) pays a “dividend” \( dD_t \). Here \( dD_t \) can be both positive and negative, in the later case money has to be paid upfront. For an investment in one \( Y \) asset, total gains in a time interval \( dt \) equals
\[ dG_t = dY_t + dD_t, \]
i.e. gains from changes in the value of the asset plus dividend yield. Since \( Y \) is tradeable, in order to avoid arbitrage it is necessary that the deterministic part of the gain must be the same as the risk free interest rate when using the risk neutral measure \( \hat{P} \), i.e.
\[ dG_t = r_tY_t dt + \sigma_t d\tilde{W}_t, \]
where \( \sigma_t \) is some adapted process. We drop a formal proof of this statement, but see Exercise 2, Homework 2. Since in fact \( Y_t \) can be zero, we may for example gamble without any assets, we will not write it on the form \( \frac{dG_t}{Y_t} = r_t dt + \sigma_t d\tilde{W}_t \).

Consider now the discounted value process
\[ Z_t = B_t^{-1}Y_t + \int_0^t B_s^{-1} dD_s. \]  

(0.13)

Differentiating by parts gives
\[ dZ_t = -r_t B_t^{-1}Y_t dt + B_t^{-1} dY_t + B_t^{-1} dD_t \]
\[ = -r_t B_t^{-1}Y_t dt + B_t^{-1} dG_t \]
\[ = -r_t B_t^{-1}Y_t dt + B_t^{-1}(r_t dt + \sigma_t d\tilde{W}_t) \]
\[ = B_t^{-1}\sigma_t d\tilde{W}_t. \]  

(0.14)
Therefore, the discounted accumulated value process is a $\tilde{P}$ martingale. We shall use this to find arbitrage free prices for futures. For a future with delivery date $T$ of an $\mathcal{F}_T$-measurable claim $X$, we will write $f_X(t, T)$ for its price (or quotation) at time $t$. It is defined by:

1. At delivery date $T$, $f_X(T, T) = X$.
2. During any time interval $(s, T]$, the holder of the contract receives $f_X(t, T) - f_X(s, T)$. In particular during an interval $dt$ he receives $df_X(t, T)$.
3. The spot price at any time $t < T$ for obtaining a contract is zero. Thus the value of the contract is always zero.

In terms of the above, we have that a future on a $T$-claim $X$ consists of an asset $Y$ (value of the contract) and a dividend process $D$ (accumulated gains or losses), where

$$
Y_t = 0,
D_t = f_X(t, T),
f_X(T, T) = X.
$$

Alternatively we could have written $D_t = f_X(t, T) - f_X(0, T)$, but since we only consider increments $dD_t$, it does not matter. We then have the important result.

**Theorem 0.4** With the above setup, the price of a future at any time $t \leq T$ equals

$$
f_X(t, T) = \tilde{E}[X|\mathcal{F}_t].
$$

**Proof** The proof is easy. In (0.13), let $Y_t = 0$ and $D_t = f_X(t, T)$. Then by (0.13) and (0.14),

$$
dZ_t = B^{-1}_t df_X(t, T) = B^{-1}_t \sigma_t d\tilde{W}_t,
$$

and consequently

$$
df_X(t, T) = \sigma_t d\tilde{W}_t,
$$

hence $f_X(t, T)$ is a $\tilde{P}$ martingale. Therefore

$$
f_X(t, T) = \tilde{E}[f_X(T, T)|\mathcal{F}_t] = \tilde{E}[X|\mathcal{F}_t].
$$

This ends the proof.

So although a forward contract and a future contract are quite similar, they have different prices. For a forward contract for delivery of $X$ at time $t$, we saw above that

$$
F_X(t, T) = E^T[X|\mathcal{F}_t],
$$

while for a future contract

$$
f_X(t, T) = \tilde{E}[X|\mathcal{F}_t].
$$
If the short term rate $r_t$ is deterministic, as is the case in the Black-Scholes environment, then $P^T = \tilde{P}$ (since then $P(t, T) = e^{-\int_t^T r_s ds}$ giving $dP(t, T) = r_t P(t, T)dt$, hence $v(t, T) = 0$), and the forward price and future price coincide.

**Pricing of a bond future**

Let $t < T < T^*$, and let $X = P(T, T^*)$ be the claim to be delivered at time $T$ in a futures contract. Then from Theorem 0.4,

$$f_X(t, T) = \tilde{E}[P(T, T^*)|\mathcal{F}_t].$$

(0.15)

We know that the corresponding forward price equals

$$F_X(t, T) = \frac{P(t, T^*)}{P(t, T)} = \frac{E^T[P(T, T^*)|\mathcal{F}_t]}{E_T[P(t, T)|\mathcal{F}_t]},$$

(0.16)

since the price of a $T^*$ bond at time $t$ equals $P(t, T^*)$. It would now be nice to calculate $f_X(t, T)$ without calculating (0.15) explicitly, but just use the simple expression (0.16) for $F_X(t, T)$. In some cases it can be done.

**Theorem 0.5** Let the bond price dynamics be given as

$$dP(t, T) = r_t P(t, T)dt + v(t, T)P(t, T)dW_t, \quad t \leq T \leq T^*,$$

where the volatility function $v(t, T)$ is deterministic. Then with the $T$-claim $X = P(T, T^*)$,

$$f_X(t, T) = F_X(t, T)e^{\int_t^T (v(s, T) - v(s, T^*))v(s, T)ds},$$

where $F_X(t, T)$ is given by (0.16)

**Proof** We saw in (0.10) that

$$dF_X(t, T) = F_X(t, T)\gamma(t, T, T^*)dW^T_t,$$

so

$$F_X(T, T) = F_X(t, T)e^{\int_t^T \gamma(s, T, T^*)dW^T_s - \frac{1}{2} \int_t^T \gamma^2(s, T, T^*)ds}$$

$$= F_X(t, T)e^{\int_t^T \gamma(s, T, T^*)dW_s - \int_t^T \gamma(s, T, T^*)v(s, T)ds - \frac{1}{2} \int_t^T \gamma^2(s, T, T^*)ds}$$

$$= F_X(t, T)e^{-\int_t^T \gamma(s, T, T^*)v(s, T)ds},$$

where

$$\xi_t = e^{\int_t^T \gamma(s, T, T^*)dW_s - \frac{1}{2} \int_t^T \gamma^2(s, T, T^*)ds}$$

is a $\tilde{P}$-martingale for $u \geq t$. Here

$$\gamma(s, T, T^*) = v(s, T^*) - v(s, T).$$

Since $\xi$ is a $\tilde{P}$-martingale, we get $\tilde{E}[\xi_T|\mathcal{F}_t] = \xi_t = 1$. Therefore

$$f_X(t, T) = \tilde{E}[F_X(T, T)|\mathcal{F}_t] = \tilde{E}[X|\mathcal{F}_t] = \tilde{E}[F_X(T, T)|\mathcal{F}_t]$$

$$= F_X(t, T)e^{-\int_t^T \gamma(s, T, T^*)v(s, T)ds} \tilde{E}[\xi_T|\mathcal{F}_t],$$

and the result follows.