STAT 391 HOMEWORK ASSIGNMENT 5

Exercise 1

Consider the setup of Theorem 0.4, Lecture 4, where we price the future $f_X(t,T)$ on the $T$-claim

$$X = P(T, T^*) .$$

a) Calculate $f_X(t,T)$ for the Ho-Lee model

$$dr_t = \theta(t) dt + \sigma d\tilde{W}_t .$$

b) Calculate $f_X(t,T)$ for the Hull-White model

$$dr_t = (\theta(t) - \alpha r_t) dt + \sigma d\tilde{W}_t .$$

In both cases, $\tilde{W}$ is a Brownian motion under the risk neutral measure $\tilde{P}$.

Exercise 2

An interest rate caplet is a contract that guarantees that the floating rate interest for a time period $[S, T]$, typically LIBOR which we will use here, is not lower than a predetermined number $R$, so that at delivery time $T$ it is worth

$$X = \tau(S, T)(L(S, T) - R)^+ .$$

Here $\tau(S, T)$ is as usual the time between $S$ and $T$. We see that its value is already known at time $S$, but payments are deferred to time $T$.

a) Show that the value at delivery time can be written as

$$X = \frac{1 + \tau R}{P(S, T)} \left( \frac{1}{1 + \tau R} - P(S, T) \right)^+ ,$$

where we for brevity wrote $\tau = \tau(S, T)$.

b) Explain why the payment of $X$ at time $T$ is equal to the payment of

$$Z = (1 + \tau R) \left( \frac{1}{1 + \tau R} - P(S, T) \right)^+$$

at time $S$. 
Assume now that the bond price dynamics can be written as

\[ dP(t, T) = r_P(t, T)dt + \sigma(t, T)P(t, T)d\hat{W}_t, \]

where as usual \( \hat{W} \) is a Brownian motion under the risk neutral probability \( \hat{P} \). Also assume that \( \sigma(t, T) \) is a deterministic function.

c) Use the put-call parity formula in Lecture 4 to show that the value at time \( t \) of a European put option on a \( T^* \) bond, delivery date \( T \) and exercise price \( K \), i.e. on

\[ Y = (K - P(T, T^*))^+ \]

equals

\[ \Pi_Y(t, T) = KP(t, T)N(-d_2) - P(t, T^*)N(-d_1). \]

Here \( d_1 \) and \( d_2 \) are given in Theorem 0.3, Lecture 4.

d) Give a formula for the value of the above caplet at time \( t < S \), i.e. for \( \Pi_X(t, T) \).

**Exercise 3**

In this exercise we shall work with an \( n \) dimensional extension of Exercise 2, Homework 3. You can look there for hints of how to solve parts of this exercise. As there we assume that \( \hat{P} = P \).

Assume that the forward rate follows

\[ df(t, T) = \alpha(t, T)dt + \sigma'(t, T)dW(t), \]

where \( W \) is an \( n \) dimensional Brownian motion and

\[ \sigma(t, T) = (\phi_1(t)\psi_1(T), \ldots, \phi_n(t)\psi_n(T))^t, \]

where the \( \phi_i \) and \( \psi_i \) are nonstochastic functions. It is assumed that the \( \psi_i \)'s are differentiable.

a) Let

\[ X_i(t) = \psi_i(t) \left( \int_0^t \phi_i^2(s)(\Psi_i(s) - \bar{\Psi}_i(s))ds + \int_0^t \phi_i(s)dW_i(s) \right), \quad i = 1, \ldots, n, \]

where

\[ \bar{\Psi}_i(t) = \int_0^t \psi_i(s)ds. \]

Show that

\[ r_t = f(0, t) + \sum_{i=1}^n X_i(t). \]

Now show that

\[ dX_i(t) = (\theta_i(t) - a_i(t)X_i(t))dt + \sigma_i(t, t)dW_i(t), \]

and identify \( \theta_i(t) \) and \( a_i(t) \).
b) Show that

\[ E \left[ \int_0^T X_i(t)dt \right] = \frac{1}{2} \int_0^T \phi_i^2(s)(\Psi_i(T) - \Psi_i(s))^2ds, \]

and that

\[ \text{Var} \left[ \int_0^T X_i(t)dt \right] = \int_0^T \phi_i^2(s)(\Psi_i(T) - \Psi_i(s))^2ds. \]

Use these results to give an analytical formula for \( P(0, T) \). You will end up with an obvious result. Why? What is the lesson to be learned from this?

c) Find the price \( \Pi_Y(t, S) \) at time \( t < S \) of a European call option on a bond with maturity \( T > S \), delivery date \( S \) and exercise price \( K \), i.e. the value at delivery equals

\[ Y = (P(S, T) - K)^+. \]

d) What is the future price at time \( t < S \) for a bond with maturity \( T \)? Delivery time for the future is \( S < T \). So we ask what is \( f_X(t, S) \) where \( X = P(S, T) \)?

e) Do the relevant calculations in parts c and d for the mixed Ho-Lee and Hull-White model

\[ \sigma(t, T) = (\sigma_1, \sigma_2 e^{-a(T-t)})', \]

with \( a > 0 \). Here you should rely on calculations done in other exercises, then this exercise is quickly done.