STAT 391 HOMEWORK ASSIGNMENT 3

Exercise 1

Consider the forward rate dynamics

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t,$$

where we assume $P = \tilde{P}$, i.e. that $\gamma_t = 0$. Assume also that $\sigma(t, T) = \sigma$, a constant.

Show that this corresponds to the Ho-Lee model, i.e. it gives for the short rate dynamics

$$dr_t = (f_T(0, t) + \sigma^2)dt + \sigma dW_t,$$

cf. Exercise 1, Homework 2.

Exercise 2

In this exercise, you can solve part b by using the results from part a, even without having done part a properly.

Here we have the same forward rate dynamics as in Exercise 1, with $P = \tilde{P}$, but assume now that $\sigma(t, T) = \phi(t)\psi(T)$ for nonstochastic functions $\phi$ and $\psi$. It is assumed that $\psi$ is differentiable.

a) Use that

$$r_t = f(0, t) + \int_0^t \alpha(s, t)ds + \int_0^t \sigma(s, t)dW_s,$$

to show that $r_t$ follows the Hull-White model

$$dr_t = (\theta(t) - a(t)r_t)dt + \sigma_0(t)dW_t,$$

where

$$\theta(t) = f_T(0, t) + \psi^2(t)\int_0^t \phi^2(s)ds + a(t)f(0, t),$$

$$a(t) = \frac{\psi'(t)}{\psi(t)} = \frac{d}{dt}\log\psi(t),$$

$$\sigma_0(t) = \sigma(t, t) = \phi(t)\psi(t).$$

It is known that the volatility $\sigma(t, T)$ decreases with time to maturity, or equivalently with the time difference $T - t$. One possible way to account for this is to let

$$\sigma(t, T) = \sigma e^{-a(T-t)}.$$
b) Show that this choice of $\sigma(t, T)$ yields the Hull-White model with constant $a$ and $\sigma$. Also calculate $\theta(t)$, where $f(0, t)$ and $f_T(0, t)$ are taken as input to the model.

**Exercise 3**

If $r_t$ follows (under the risk neutral measure $\tilde{P}$),

$$dr_t = \mu(t, r_t) dt + \sigma(t, r_t) dW_t$$

with

$$\mu(t, r) = \alpha(t) r + \beta(t),$$

$$\sigma(t, r) = \sqrt{\gamma(t) r + \delta(t)},$$

we saw that the bond price $P(t, T)$ can be written as

$$P(t, T) = e^{A(t, T) - B(t, T) r_t}.$$ 

If $\gamma(t) = 0$, then we saw that $B(t, T)$ satisfies

$$B_t(t, T) + \alpha(t) B(t, T) = -1$$

with boundary condition $B(T, T) = 0$.

a) Show that the solution is given as

$$B(t, T) = \int_t^T e^{\int_t^u \alpha(s) ds} du.$$ 

b) Consider now the Hull-White model

$$dr_t = (\theta(t) - ar_t) dt + \sigma dW_t.$$ 

Calculate $A(t, T)$ and $B(t, T)$ in this case.

**Exercise 4**

In this exercise we will value a floating rate bond. To explain, let $T_0 < T_1 < \cdots < T_n$, and assume that the bond (with principal 1) at time $T_1, \ldots, T_n$ pays a coupon equal to the floating LIBOR rate, i.e. it pays

$$\tau(T_{i-1}, T_i) L(T_{i-1}, T_i) = \tau_i L(T_{i-1}, T_i),$$

where we set $\tau_i = \tau(T_{i-1}, T_i)$. Note that the coupon is known at time $T_{i-1}$, but it is not paid until time $T_i$. At time $T_n$ it pays the coupon plus the principal, i.e. it pays $\tau_n L(T_{n-1}, T_n) + 1$.

a) Show that the value of this bond at time $t \leq T_0$ equals $P(t, T_0)$.

b) Show that by investing $P(t, T_0)$ at time $t$, you can produce a self financing portfolio that replicates the payoff of the floating rate bond. This of course solves part a, but part a can also be solved using a somewhat different approach.