Exercise 1

Consider the Ho-Lee short rate model

$$dr_t = \theta(t)dt + \sigma dW_t,$$

where $\theta$ is a deterministic function and $W$ is the usual Brownian motion.

a) What is the distribution of $r_t$ given that $r_s = r$ is known for $s < t$.

b) Show that

$$\int_t^T r_s ds = (T - t) r_t + \int_t^T \theta(s)(T - s) ds + \sigma \int_t^T (T - s) dW_s,$$

and use this to show that

$$\text{Var}\left[\int_t^T r_s ds \bigg| F_t\right] = \frac{1}{3} \sigma^2 (T - t)^2.$$

What is the distribution of $\int_t^T r_s ds$ given that $r_t$ is known?

Assume that $P = \tilde{P}$, i.e. that the market price of risk $\lambda_t = 0$. Then we have that

$$P(t, T) = E^{t, r_t}\left[ e^{-\int_t^T r_s ds}\right].$$

c) Show that

$$P(t, T) = e^{A(t, T) - B(t, T) r_t},$$

where

$$A(t, T) = \int_t^T \theta(s)(s - T) ds + \frac{\sigma^2}{2} (T - t)^3,$$

$$B(t, T) = T - t$$

d) Use the relation

$$P(0, t) = e^{A(0, t) - B(0, t) r_0}$$

to show that

$$f(0, t) = \int_0^t \theta(s) ds - \frac{\sigma^2}{2} t^2 + r_0$$

and also that

$$\theta(t) = f_T(0, t) + \sigma^2 t.$$
Assume now that we know $\sigma^2$. It then follows that if we observe $f(0,t)$, denote the observed empirical forward curve by $f^*(0,t)$, then we can estimate

$$\theta^*(t) = f^*_t(0,t) + \sigma^2 t.$$ 

Of course in practice it may be difficult to find a good estimate of $f^*_t(0,t)$ based on a finite number of observations $P(0,t_1), P(0,t_2), \ldots, P(0,t_n)$.

Now let

$$P^*(0,t) = e^{-\int_0^t f^*(0,s)ds}$$

be estimated bond values (again based on the finite observations $P(0,t_1), P(0,t_2), \ldots, P(0,t_n)$) at time 0.

e) Show that at time $t$

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\{\int_t^T (T-t) f(0,t) - \frac{\sigma^2}{2}(T-t)^2 - (T-t)r_t \}.$$ 

For practical use, this means that if

$$P^*(0,t) = e^{-\int_0^t f^*(0,s)ds}$$

are estimated bond values (again based on the finite observations $P(0,t_1), P(0,t_2), \ldots, P(0,t_n)$) at time 0, then the estimated bond prices at time $t$ can be written as

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} \exp\{\int_t^T f^*(0,t) - \frac{\sigma^2}{2}(T-t)^2 - (T-t)r_t \}.$$ 

f) Explain why the relations in part e are still valid even when the measures $P$ and $\tilde{P}$ are different, as long as $r$ under $\tilde{P}$ follows the above dynamics.

Exercise 2

Assume the short rate model

$$dr_t = \mu(t,r_t) dt + \sigma(t,r_t) dW_t$$

and let $\Pi(t,T)$ be the price of a derivative that pays $\Phi(r_T)$ at time $T$. Show that $\Pi$ has the dynamics

$$d\Pi(t,T) = r_t \Pi(t,T) dt + \sigma^\Pi(t,T) \Pi(t,T) d\tilde{W}_t$$

for some volatility process $\sigma^\Pi(t,T)$. In particular, it means that the discounted process

$$Z(t,T) = \frac{\Pi(t,T)}{B_t}$$

satisfies

$$dZ(t,T) = \sigma^\Pi(t,T) Z(t,T) d\tilde{W}_t,$$

and hence it is a $\tilde{P}$ (local) martingale.
Exercise 3

In this exercise we will look at a Forward Rate Agreement (FRA). Consider a market with the following zero coupon bond values (principal value equals 100).

<table>
<thead>
<tr>
<th>Time to maturity</th>
<th>Bond prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>96.7</td>
</tr>
<tr>
<td>2 years</td>
<td>93.5</td>
</tr>
</tbody>
</table>

At time $t$, A makes an agreement to borrow 100 from B one year ahead and pay back in two years at a simply compounded rate of 4%.

What is the value of this contract at time $t$ for B? Explain how B can trade with bonds to realize that value.