ESTIMATING NETWORK MEMBERSHIPS BY SIMPLEX VERTICES HUNTING

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Consider an undirected mixed membership network with \( n \) nodes and \( K \) communities. For each node \( i \), \( 1 \leq i \leq n \), we model the membership by a Probability Mass Function (PMF) \( \pi_i = (\pi_i(1), \pi_i(2), \ldots, \pi_i(K))' \), where \( \pi_i(k) \) is the probability that node \( i \) belongs to community \( k \), \( 1 \leq k \leq K \). We call node \( i \) “pure” if \( \pi_i \) is degenerate and “mixed” otherwise. The primary interest is to estimate \( \pi_i \), \( 1 \leq i \leq n \).

We model the adjacency matrix \( A \) with a Degree Corrected Mixed Membership (DCMM) model. Let \( \hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_K \) be the eigenvectors of \( A \) associated with the \( K \) largest (in magnitude) eigenvalues. We define a matrix \( \hat{R} \in \mathbb{R}^{n,K-1} \) by \( \hat{R}(i,k) = \hat{\xi}_{k+1}(i)/\hat{\xi}_1(i) \), \( 1 \leq k \leq K-1 \), \( 1 \leq i \leq n \). The matrix can be viewed as a distorted version of its non-stochastic counterpart \( R \in \mathbb{R}^{n,K-1} \), which is unknown but contains all information we need for the memberships.

We reveal an interesting insight: there is a simplex \( S \) in \( \mathbb{R}^{K-1} \) such that row \( i \) of \( R \) corresponds to a vertex of \( S \) if node \( i \) is pure, and corresponds to an interior point of \( S \) otherwise. Vertices Hunting (i.e., estimating the vertices of \( S \)) is therefore the key to our problem.

We propose a new approach Mixed-SCORE to estimating the memberships, at the heart of which is an easy-to-use Vertices Hunting algorithm. The approach is successfully applied to 4 network data sets (a coauthorship and a citee network for statisticians, a political book network, and a football network) with encouraging results.

We analyze Mixed-SCORE and derive its rate of convergence, using delicate Random Matrix Theory. The work is closely related to Jin [22] but it deals with a different problem in a more complicated setting and contains several innovations in methods and theory.

1. Introduction. In the study of social networks, the problem of estimating the mixed memberships has received a lot of attention; see [2, 15] for example. Consider an undirected network \( \mathcal{N} = (V, E) \), where \( V = \{1, 2, \ldots, n\} \) is the set of nodes and \( E \) is the set of edges. We assume that
the network consists of $K$ perceivable disjoint communities
\begin{equation}
C_1, C_2, \ldots, C_K,
\end{equation}

and that for each node $i$, there is a Probability Mass Function (PMF) $\pi_i = (\pi_i(1), \pi_i(2), \ldots, \pi_i(K))' \in \mathbb{R}^K$ such that
\begin{equation}
P(\text{node } i \text{ belongs to } C_k) = \pi_i(k), \quad 1 \leq k \leq K, \quad 1 \leq i \leq n.
\end{equation}

We call node $i$ “pure” if $\pi_i$ is degenerate (i.e., one entry is 1, all others are 0) and “mixed” otherwise. The primary interest is to estimate $\pi_i$, $1 \leq i \leq n$.

Table 1 lists several data sets we study in this paper. Take the Polbook for example: each node is a book on politics of USA sold by Amazon.com, and there is an edge between two nodes if they are frequently co-purchased; the nodes are manually labeled as either “Conservative”, “Liberal”, or “Neutral”. We model the data with a two-community (“Conservative” and “Liberal”) mixed membership model, where a “Neutral” node is thought of as having mixed memberships. The goal is to use the network information to estimate how much weight each book puts on “Conservative” and “Liberal”.

Alternatively, one could use a non-mixing non-overlapping model (e.g., [27]) where we have three communities: “Conservative”, “Liberal”, and “Neutral”. However, we prefer to use a mixed membership model for

- A non-mixing model usually assumes more communities than necessary, and some of them may be hard to interpret or not meaningful.
- A mixed membership model allows us to assess the weight each node puts on each community, while a non-mixing model does not.

Such a viewpoint is valid for many data sets including the Polbook. See Section 2 for more discussion and also details for all the data sets.

We propose Mixed-SCORE as a new approach to membership estimation. The approach can be viewed as an extension of the recent method of SCORE by Jin [22] but is different in important ways:

\footnote{For example, in a coauthorship network for statisticians [21], each node is an author, and a community may be thought of as a research area.}
• SCORE is for non-mixing models (i.e., all $\pi_i$ are degenerate) while Mixed-SCORE is for mixing models.
• SCORE is for the problem of community detection while Mixed-SCORE is for the problem of membership estimation.

Therefore, to adapt SCORE to Mixed-SCORE, we need new insights and substantial innovations. The contributions of the paper include:

• **Key Insight.** There is a matrix $\hat{R} \in \mathbb{R}^{n,K-1}$ constructed from the network adjacency matrix which approximates its ideal counterpart $R \in \mathbb{R}^{n,K-1}$. Viewing each row of $R$ as a point in $\mathbb{R}^{K-1}$, there is a simplex in $\mathbb{R}^{K-1}$ which we call the Ideal Simplex (IS) such that row $i$ of $R$ corresponds to a vertex of IS if node $i$ is pure, and an interior point of IS (or an interior point of an edge/face of IS) otherwise. Such a low-dimensional geometric structure paves the way for membership estimation, and Vertices Hunting (i.e., estimating the vertices of IS) is the key to our problem.

• **Methods and Theory.** We propose Mixed-SCORE as a new approach to membership estimation, at the heart of which is an easy-to-use Vertices Hunting algorithm. We analyze Mixed-SCORE using delicate spectral analysis and Random Matrix Theory.

• **Scientific Contributions.** We have applied Mixed-SCORE to all data sets in Table 1. Our data analysis results (a) shed light on research patterns and topology of statisticians, (b) reveal a connection between the community structure of American college football teams and their geographical locations, and (c) further compare two modeling strategies—non-mixing non-overlapping models and mixed membership models (presumably with fewer communities)—from a scientific viewpoint.

1.1. **Degree Corrected Mixed Membership (DCMM) model.** To facilitate the analysis, we use the DCMM model. DCMM can be viewed as an extension of the Mixed Membership Stochastic Block Model (MMSB) by Airoldi et al. [2], to accommodate degree heterogeneity, and can also be viewed as an extension of the Degree Corrected Block Model (DCBM) by Karrer and Newman [27], to accommodate mixed memberships. See also [10, 22, 35, 36, 40].

Let $A \in \mathbb{R}^{n,n}$ be the (symmetric) adjacency matrix of $\mathcal{N}$ where the diagonals are 0 for we do not think a node is connected to itself. Fix a symmetric non-negative matrix $P \in \mathbb{R}^{K,K}$ such that

\begin{equation}
P \text{ is non-singular, irreducible, and has unit diagonals.}
\end{equation}

Similar to MMSB [2], DCMM models the upper triangle of $A$ (excluding diagonals) as Bernoulli random variables that are generated by parallel hier-
archival models independently (i.e., for all pairs of \((i, j)\) with \(1 \leq i < j \leq n\), the random processes underlying \(A(i, j)\) are independent of each other). Fix a positive vector \(\theta = (\theta(1), \ldots, \theta(n))'\) which models the \textit{degree heterogeneity}.

For any fixed pair of \((i, j)\) such that \(i < j\), DCMM assumes that

\[
(1.4) \quad P(A(i, j) = 1 \mid i \in C_k \& j \in C_\ell) = \theta(i)\theta(j)P(k, \ell).
\]

Combining this with (1.2) and the above assumption on independence, for all pairs of \((i, j)\) with \(1 \leq i < j \leq n\), \(A(i, j)\) are Bernoulli random variables that are independent of each other, satisfying

\[
(1.5) \quad P(A(i, j) = 1) = \theta(i)\theta(j)\sum_{k=1}^{K} \sum_{\ell=1}^{K} \pi_i(k)\pi_j(\ell)P(k, \ell).
\]

Introduce the degree heterogeneity matrix \(\Theta \in \mathbb{R}^{n,n}\) and the membership matrix \(\Pi \in \mathbb{R}^{n,K}\):

\[
(1.6) \quad \Theta = \text{diag}(\theta(1), \theta(2), \ldots, \theta(n)), \quad \Pi = [\pi_1, \pi_2, \ldots, \pi_n]' .
\]

**Definition 1.1.** We call model (1.1)-(1.5) the Degree Corrected Mixed Membership (DCMM) model, and denote it by DCMM\((n, P, \Theta, \Pi)\).

We now decompose \(A\) into the sum of a “signal” part and a “noise” part:

\[
(1.7) \quad A = [\Omega - \text{diag}(\Omega)] + W
\]

where \(\Omega\) is a symmetric matrix satisfying \(\Omega(i, j) = P(A(i, j) = 1), 1 \leq i < j \leq n\), and \(W = A - \Omega + \text{diag}(\Omega)\) is a generalized Wigner matrix [38]. By basic algebra, \(\Omega\) and \(\Theta\Pi'\Pi'\Theta\) have matching off-diagonals. Since the diagonals of \(\Omega\) are not unique and can be chosen for our convenience, we choose them in a way such that

\[
(1.8) \quad \Omega = \Theta\Pi'\Pi'\Theta.
\]

Our primary interest is the membership matrix \(\Pi\). Note that \(P\) is the matrix that directly models the community partitions, and the degree heterogeneity matrix \(\Theta\) is largely a nuisance in membership estimation (see below).

**Remark.** In the case where all \(\pi_i\) are degenerate, \(\Omega(i, j) = \theta(i)\theta(j)P(k, \ell)\) if \(i \in C_k\) and \(j \in C_\ell\), and DCMM reduces to DCBM [27]. In the case where \(\theta(1) = \theta(2) = \ldots = \theta(n) = c_0\) (say), \(\Omega(i, j) = c_0\sum_{\ell=1}^{K} \pi_i(k)\pi_j(\ell)P(k, \ell),\) and DCMM reduces to (a simplified version of) MMSB [2].

\[\text{For a vector } v, \text{ diag}(v) \text{ is the diagonal matrix with entries of } v \text{ on its diagonal. For a matrix } M, \text{ diag}(M) \text{ is the diagonal matrix with diagonal elements of } M \text{ on its diagonal.} \]
1.2. The Ideal Simplex (IS) and the Ideal Mixed-SCORE. Before we discuss any real estimator, we investigate an oracle approach. The idea is to consider the oracle situation where $\Omega$ is given (presumably by God, and so the term of oracle), and to construct an approach that

- exactly recovers the membership matrix $\Pi$ when $(\Omega, K)$ are given.
- is easily extendable to the real case where $(A, K)$ (but not $\Omega$) are given.

The main challenge is that DCMM has too many parameters, including the degree heterogeneity parameters $\theta(i)$ and the PMF $\pi_i$, $1 \leq i \leq n$. First, we recognize that $\theta(i)$ are nuisance parameters and their nuisance effects can be largely removed by SCORE [22]; SCORE can be viewed both as a normalization method and a complexity reduction method. Second, we relate $\pi_i$ to a low-dimensional simplex—the Ideal Simplex (IS)—and use IS to retrieve all $\pi_i$ in a homely yet effective way.

See Donoho [13] for a philosophical comparison between homely effective statistical methods and ambitious machine learning algorithms.

Note that $\text{rank}(\Omega) = K$. Let $\lambda_1, \lambda_2, \cdots, \lambda_K$ be all the nonzero eigenvalues of $\Omega$ (arranged in the descending order of magnitudes), and let $\xi_1, \cdots, \xi_K$ be the corresponding eigenvectors. Write $\Xi = [\xi_1, \cdots, \xi_K]$. By basic linear algebra, there is a unique non-singular matrix $B \in \mathbb{R}^{K,K}$ such that

\begin{align*}
(1.9) \quad \Xi = \Theta \Pi B; \quad &\text{note that } \Omega = \Xi \cdot \text{diag}(\lambda_1, \ldots, \lambda_K) \cdot \Xi'.
\end{align*}

We require all entries of $\xi_1$ and $b_1$ (the first column of $B$) to be positive; this is possible due to Perron’s theorem [19]. See Lemma 6.1 below.

The goal of our oracle approach is to use $\Xi$ to recover $\Pi$. \footnote{The choice of $\Xi$ is not unique and different choices are different by an orthogonal column transformation; still, they give exactly the same oracle reconstruction of $\Pi$.}

We observe

- The desired information, $\Pi$, is contained in $\Xi$ through $\Xi = \Theta \Pi B$.
- If we divide each column of $\Xi$ by its first column entry-wise, then the matrix $\Theta$ — which is diagonal — is cancelled out in the division.

The above is the key insight in SCORE [22], which recognizes that the degree heterogeneity matrix $\Theta$ is a nuisance and the nuisance effects can be largely removed by taking entry-wise ratios between its columns. In light of this, we define the Matrix of Entry-wise Ratios $R \in \mathbb{R}^{n,K-1}$ by

\begin{align*}
(1.10) \quad R(i,k) &= \xi_{k+1}(i)/\xi_1(i), \quad 1 \leq i \leq n, \quad 1 \leq k \leq K - 1.
\end{align*}

Write $R = [r_1, r_2, \ldots, r_n]'$ and $B = [b_1, b_2, \ldots, b_K]$. Define a matrix $V = [v_1, v_2, \ldots, v_K] \in \mathbb{R}^{K-1,K}$ by

\begin{align*}
(1.11) \quad v_k(\ell) &= b_{\ell+1}(k)/b_1(k), \quad 1 \leq \ell \leq K - 1, \quad 1 \leq k \leq K,
\end{align*}

where $\ell$ and $k$ are two different indices.
where we call the Ideal Simplex (IS).

The central surprise of the paper is that, there is a $B$ which combines this with the relation between $P$ and $b$ recall.

Now, in order to retrieve $\pi_i$, all we need to know is $b_1$ (in fact, once $v_1, v_2, \ldots, v_k$ and $b_1$ are known, we can first compute $w_i$ using $r_i$ and (1.12) and then compute $\pi_i$ using $(w_i, b_1)$ and (1.12)). To this end, note that by (1.8)-(1.9),

$$\Theta \Pi \cdot P \cdot \Pi' \Theta = \Theta \Pi \cdot (B \cdot \text{diag}(\lambda_1, \ldots, \lambda_K) \cdot B') \cdot \Pi' \Theta;$$

recall $b_1$ is the first column of $B$. It follows that $P = B \cdot \text{diag}(\lambda_1, \ldots, \lambda_K) \cdot B'$. As $P$ has unit diagonals, the diagonals of $B \cdot \text{diag}(\lambda_1, \ldots, \lambda_K) \cdot B'$ are all 1. Combining this with the relation between $B$ and $v_1, v_2, \ldots, v_K$ (e.g., (1.11)),

$$b_1(k) = [\lambda_1 + v'_k \text{diag}(\lambda_2, \ldots, \lambda_K) v_k]^{-1/2}, \quad 1 \leq k \leq K.$$  

\[\text{we call a vector a weight vector if all its entries are nonnegative with a sum of 1.}\]

\[A k\text{-simplex is the } k\text{-dimensional polytope that is the convex hull of its } (k+1)\text{ vertices.}\]

\[\text{We call a weight vector degenerate if one of its entry is 1 and all other entries are 0.}\]
Therefore, $b_1$ can be conveniently retrieved using $\lambda_1, \ldots, \lambda_K$ and $v_1, \ldots, v_K$.

The above gives rise to the following three-stage algorithm which we call Ideal Mixed-SCORE. Input: $\Omega$. Output: $\pi_i$, $1 \leq i \leq n$.

- **SCORE step.** Obtain $(\lambda_1, \xi_1), \ldots, (\lambda_K, \xi_K)$ and obtain the matrix $R$.
- **Vertices Hunting (VH) step.** Determine $v_1, v_2, \ldots, v_K$ using the matrix $R$ and some convex hull algorithm.
- **Membership Reconstruction (MR) step.** Compute $b_1$ by (1.13) and compute $\pi_i$ by (1.12).

The following theorem is proved in Section 6.

**Theorem 1.1 (Ideal Mixed-SCORE).** Fix $K > 1$ and $n > 1$. Consider a DCMM $(n, P, \Theta, \Pi)$ where each community has at least one pure node (i.e., the set $\{1 \leq i \leq n : \pi_i(k) = 1\}$ is non-empty for all $1 \leq k \leq K$). Despite that $\Xi$ may not be uniquely defined, $b_1$ and $\{w_i\}_{i=1}^n$ are uniquely defined and the Ideal Mixed-SCORE exactly recovers the membership matrix $\Pi$.

**Remark.** In obtaining the simplex structure, a key component is (1.10), a normalization step first proposed by Jin [22]. Jin [22] has proposed more than one normalization approaches to removing the nuisance effects of $\Theta$: for example, an alternative is to normalize each row of the matrix $\Xi$ by its Euclidean $\ell_q$-norm, $q > 0$. But we prefer to normalize $\Xi$ as in (1.10) since the resultant geometric structure is much more simpler: we do not have the simplex structure if we normalize $\Xi$ row-wise using the $\ell_q$-norm.

**Remark.** Another popular normalization idea is to directly normalize the adjacency matrix $A$ using either the (diagonal) degree matrix or an estimate of $\Theta$. Such an idea won’t work for (see definitions of $(\hat{\lambda}_k, \hat{\xi}_k)$ shortly below) $A = (I) + (II)$, where $(I) = \sum_{k=1}^K \hat{\lambda}_k \hat{\xi}_k \hat{\xi}_k'$ and $(II) = \sum_{k=K+1}^n \hat{\lambda}_k \hat{\xi}_k \hat{\xi}_k'$ correspond to the “signal” part $\Omega$ and the “noise” part $A - \Omega$, respectively. While such a normalization approach may be right for (I), it is clearly not right for (II), so normalizing $A$ directly won’t work. Our idea is to first obtain $(I)$ from $A$ using PCA and then normalize $(I)$ with the SCORE.

1.3. **Mixed-SCORE and a two stage Vertices Hunting algorithm.** We now extend the previous idea to the real case where $(A, K)$ are given but $\Omega$ is unknown. Let $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_K$ be the $K$ largest (in magnitude) eigenvalues of $A$, and let $\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_K$ be the corresponding eigenvectors.\footnote{When the network is connected, by Perron’s theorem [19], all entries of $\hat{\xi}_i$ are strictly positive. When the network is not connected, we take the giant component of it.} Fixing a threshold $T > 0$, let $\hat{R} = [\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n]'$ be the empirical counterpart of $R$.
such that for $1 \leq i \leq n$ and $1 \leq k \leq K - 1$,

$$
(1.14) \quad \hat{R}(i, k) = \text{sign}(\hat{\xi}_{k+1}(i)/\hat{\xi}_1(i)) \cdot \min\{|\hat{\xi}_{k+1}(i)/\hat{\xi}_1(i)|, T\}.
$$

The following algorithm, which we call Mixed-SCORE, is a natural extension of the Ideal Mixed-SCORE. Input: $A, K$. Output: $\hat{\pi}_i$, $1 \leq i \leq n$.

- **SCORE step.** Obtain $(\hat{\lambda}_1, \hat{\xi}_1), \ldots, (\hat{\lambda}_K, \hat{\xi}_K)$ and $\hat{R} = [\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n]'$.
- **Vertices Hunting (VH) step.** By an algorithm to be determined, obtain an estimate of the vertices of the Ideal Simplex: $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_K$.
- **Membership Reconstruction (MR) step.** Obtain an estimate of $b_1$ by

$$
(1.15) \quad \hat{b}_1(k) = [\hat{\lambda}_1 + \hat{v}_k' \text{diag}(\hat{\lambda}_2, \ldots, \hat{\lambda}_K) \hat{v}_k]^{-1/2}, \quad 1 \leq k \leq K.
$$

For each $1 \leq i \leq n$, let $\hat{w}_i \in \mathbb{R}^K$ be the unique vector such that $\hat{r}_i = \sum_{k=1}^{K} \hat{w}_i(k) \hat{v}_k$ and $\sum_{k=1}^{K} \hat{w}_i(k) = 1$. Define a vector $\hat{\pi}_i^* \in \mathbb{R}^{K}$ by

$$
\hat{\pi}_i^*(k) = \max\{0, \hat{w}_i(k)/\hat{b}_1(k)\}, \quad 1 \leq k \leq K.
$$

Estimate $\pi_i$ by $\hat{\pi}_i = \hat{\pi}_i^*/\|\hat{\pi}_i^*\|_1$, $1 \leq i \leq n$.

Here, Steps 1 and 3 are straightforward extensions of Steps 1 and 3 in Ideal Mixed-SCORE, respectively. The main challenge is how to extend Step 2 (i.e., Vertices Hunting) of Ideal Mixed-SCORE: in the point cloud formed by $\{\hat{r}_i\}_{i=1}^n$ in $\mathbb{R}^{K-1}$, the Ideal Simplex is blurred and is not directly observable, so we can not directly use a convex hull algorithm as before.

The point is illustrated in Figure 1, where the data is generated according to a DCMM with $(n, K) = (500, 3)$, $P \in \mathbb{R}^{3.3}$ has unit diagonals and 0.3 on all off-diagonals, $\{\theta^{-1}(i)\} \sim \text{Unif}[1, 5]$. Among all nodes, 300 are pure nodes, with 100 in each community, and 200 are mixed nodes evenly distributed in 4 groups, where the PMFs equal to $(0.8, 0.2, 0.0), (0.0, 0.2, 0.8), (0.2, 0.4, 0.4)$ and $(1/3, 1/3, 1/3)$, in each of the four groups, respectively.

We propose a two-stage Vertices Hunting algorithm. The main idea is to first apply classical $k$-means to the point cloud and identify a few (but more than $K$) “local centers”. We recognize that under mild conditions (including that each community has sufficiently many pure nodes):

- Each vertex of the Ideal Simplex is surrounded by a cluster of points, where each point represents a row of $\hat{R}$ corresponding to a pure node; as a result, each vertex falls close to one of the “local centers”.
- The remaining “local centers” lie in the interior of the Ideal Simplex.

Such a geometric structure allows us for accurate Vertices Hunting. In detail, fixing a tuning integer $L \geq K$, the Vertices Hunting step is as follows.
Fig 1. Left: rows of $\mathbb{R}$ (many rows are equal so a point may represent many rows). Middle: each point is a row of $\hat{\mathbb{R}}$. Right: same as the middle panel except that a triangle (solid blue) estimated by the Vertices Hunting algorithm is added. In all panels, dashed triangle is the Ideal Simplex, and red/green points correspond to pure/mixed nodes respectively.

- **Local clustering.** Apply the classical $k$-means algorithm to $\hat{r}_1, \cdots, \hat{r}_n$ assuming there are $L$ clusters. 8 Denote the centers of the clusters by $\hat{m}_1, \hat{m}_2, \ldots, \hat{m}_L \in \mathbb{R}^{K-1}$.

- **Combinatorial Vertices Search.** For any $K$ distinct indices $1 \leq j_1 < \ldots < j_K \leq L$, let $\mathcal{H}(\hat{m}_{j_1}, \ldots, \hat{m}_{j_K})$ be the convex hull of $\hat{m}_{j_1}, \ldots, \hat{m}_{j_K}$, and denote the maximal Euclidean distance between the convex hull and those cluster centers outside the convex hull by

$$d_L(j_1, \cdots, j_K) = \max_{1 \leq j \leq L} \text{dist}(\hat{m}_j, \mathcal{H}(\hat{m}_{j_1}, \cdots, \hat{m}_{j_K})).$$

Let $\hat{j}_1 < \ldots < \hat{j}_{K}$ be the indices such that

$$\hat{(j_1, j_2, \ldots, j_K)} = \arg\min_{1 \leq j_1 < j_2 < \ldots < j_K \leq L} d_L(j_1, j_2, \cdots, j_K).$$

We estimate the Ideal Simplex (IS) by $\mathcal{S}(\hat{v}_1, \ldots, \hat{v}_K)$—the simplex with vertices $\hat{v}_1, \ldots, \hat{v}_K$, where

$$\hat{v}_k = \hat{m}_{\hat{j}_k}, \quad 1 \leq k \leq K.$$

In the unlikely event where $\mathcal{S}(\hat{v}_1, \ldots, \hat{v}_K)$ is degenerate, replace it by the standard simplex in $\mathbb{R}^{K-1}$.

A challenging problem is how to set the tuning parameter $L$. We suggest two approaches, where the first one is for theoretical study, and the second one is for practical use and works well for the data sets we study in the

8In a rare event, classical $k$-means may output less than $L$ clusters; in this case, we estimate the Ideal Simplex by the standard simplex in $\mathbb{R}^{K-1}$.

9The distance between the convex hull and any point in its interior is thought of as 0.
paper. Consider the first one. For each \( L \), suppose we apply the classical \( k \)-means to all rows of \( \hat{R} \) assuming \( \leq L \) clusters, and let \( \epsilon_L(\hat{R}) \) be the sum of squared residuals. We set \( L \) as

\[
(1.17) \quad \hat{L}_n(A) = \min\{ L \geq K + 1 : \epsilon_{L+1}(\hat{R}) < \epsilon_L(\hat{R})/\log(\log(n)) \}.
\]

Consider the second approach. For each \( L \), we compute the quantity \( d_L(\hat{R}) = d_L(\hat{j}_1, \cdots, \hat{j}_K) \) as in (1.16) and the quantity \( \delta_L(\hat{R}) \) defined as

\[
\min_{\{j_1, \cdots, j_K\} : \text{a permutation of } \{1, \cdots, K\}} \max_{1 \leq k \leq K} \{ \| \hat{\nu}_k^{(L)} - \hat{\nu}_k^{(L-1)} \| \}.
\]

We choose

\[
(1.18) \quad \hat{L}^*_n(A) = \arg\min_{K+1 \leq L \leq 3K} \{ \delta_L(\hat{R})/(1 + d_L(\hat{R})) \};
\]

pick the largest index if there is a tie. Figure 1 (right panel) displays the estimated vertices from the algorithm, where \( L \) is chosen from (1.18).

How to set \( L \) in a data driven fashion that works well both in theory and in practice is a hard problem and we leave this to future study.

1.4. Main results. Consider a sequence of models \( DCMM(n, P, \Theta, \Pi) \) indexed by \( n \) where \( (\Theta, \Pi) \) change with \( n \) but \( K, P \in \mathbb{R}^{K,K} \) are fixed.

We impose three mild regularity conditions as follows. Recall that \( \{ \theta(i) \}_{i=1}^n \) are the degree heterogeneity parameters. Let \( \theta_{\max} = \max_{1 \leq i \leq n} \{ \theta(i) \}, \theta_{\min} = \min_{1 \leq i \leq n} \{ \theta(i) \} \), and let \( \mathcal{N}_k = \mathcal{N}_k(\Pi) = \{ 1 \leq i \leq n : \pi_i(k) = 1 \} \) be the set of pure nodes of community \( k, 1 \leq k \leq K \). First, we assume that there are constants \( c_1, c_2 \in (0, 1) \) and \( c_3 > 0 \) such that

\[
(1.19) \quad \min_{1 \leq k \leq K} \# \mathcal{N}_k \geq c_1 n, \quad \min_{1 \leq k \leq K} \sum_{i \in \mathcal{N}_k} \theta^2(i) \geq c_2 \| \theta \|_2^2, \quad \theta_{\max} \leq c_3.
\]

Second, we assume that as \( n \to \infty \),

\[
(1.20) \quad \log(n)\text{err}_n \to 0, \quad \text{where } \text{err}_n = \text{err}_n(\Theta) \equiv \frac{\log(n)\sqrt{\theta_{\max}\| \theta \|_1}}{\sqrt{n\theta_{\min}\| \theta \|}}.
\]

Note that in the special case where \( \theta_{\max}/\theta_{\min} \leq C, \text{err}_n \sim n^{-1/2}\theta_{\max}^{-1} \log(n) \). Last, let \( G = G(\Theta, \Pi) = \| \theta \|^{-2} \Pi^T \Theta^2 \Pi \in \mathbb{R}^{K,K} \). We assume that there is a constant \( c_4 > 0 \) such that (\( \lambda_k \) is the \( k \)-th largest eigenvalue in magnitude)

\[
(1.21) \quad |\lambda_1(PG)| \geq c_4 + \max_{2 \leq k \leq K} \lambda_k(PG);
\]

(1.21) is only mild for the matrix \( G \) is properly scaled and \( K \) is small.
We discuss two cases: (A) there are many mixed nodes, and (B) there are relatively few mixed nodes, separately.

Consider Case (A). Let \( M = M(\Pi) = \{1 \leq i \leq n : \max_{1 \leq k \leq K} \pi_i(k) < 1\} \) be the set of all mixed nodes. Fixing an integer \( L_0 \), we assume there is a partition of \( M \), \( M = M_1 \cup \cdots \cup M_{L_0} \), a set of PMF’s \( \gamma_1, \cdots, \gamma_{L_0} \), and constants \( c_5, c_6 > 0 \) such that (1.22)

\[
\{ \min_{1 \leq j \neq \ell \leq L_0} \| \gamma_j - \gamma_\ell \|, \min_{1 \leq \ell \leq L_0, 1 \leq k \leq K} \| \gamma_\ell - e_k \| \} \geq c_5,
\]

and for each \( 1 \leq \ell \leq L_0 \),

(1.23) \[
|M_\ell| \geq c_0 |M| \geq n \log(n) \text{err}_2 n, \quad \max_{i \in M_\ell} \| \pi_i - \gamma_\ell \| \leq 1/ \log(n).
\]

Note that the first item in (1.23) requires that there are moderately many mixed nodes. The following theorem is proved in Section 3.

**Theorem 1.2.** Fix \( K \geq 2 \) and \( P \in \mathbb{R}^{K,K} \) that satisfies (1.3). Consider a sequence of DCM \( M(n, P, \Theta, \Pi) \) where \( (\Theta, \Pi) \) change with \( n \). Suppose (1.19)-(1.23) hold. In the Mixed-SCORE, let \( T = \sqrt{\log(n)} \) in (1.14) and \( L = \hat{L}_n(A) \) be defined as in (1.17). As \( n \to \infty \), with probability \( 1 - o(n^{-3}) \),

- \( \hat{L}_n(A) = L_0 + K \) and \( (1/n) \sum_{i=1}^n \| \hat{\pi}_i - \pi_i \|^2 \leq C \cdot \text{err}_2 n \),
- if additionally \( \theta_{\max} \leq C \theta_{\min} \), then \( (1/n) \sum_{i=1}^n \| \hat{\pi}_i - \pi_i \|^2 \leq C \log(n)/n \theta_{\max}^{-2} \).

**Remark.** We use the conditions (1.22)-(1.23) to ensure the success of the Vertices Hunting step. More specifically, such conditions ensure that among all cluster centers identified by the first sub-step of Vertices Hunting, (a) there are \( K \) of the cluster centers each of which is reasonably close to one of the vertices of the Ideal Simplex, and (b) all other cluster centers fall within the \((K - 1)\)-simplex formed by the \( K \)-centers in (a). Theorem 1.2 continues to hold if conditions (1.22)-(1.23) are relaxed or replaced by a different set of conditions, as long as (a)-(b) hold with overwhelming probabilities.

Consider Case (B). We assume

(1.24) \[
|M| \leq C n \cdot \text{err}_2 n, \quad \min_{i \in M, 1 \leq k \leq K} \| \pi_i - e_k \| \geq c_5,
\]

so there are relatively few mixed nodes. In such a case, since most nodes are pure nodes, we can simply choose the tuning parameter \( L \) in the Vertices Hunting step as \( K \). The following theorem is proved in Section 3.
**Theorem 1.3.** Fix $K \geq 2$ and $P \in \mathbb{R}^{K,K}$ that satisfies (1.3). Consider a sequence of DCMM$(n, P, \Theta, \Pi)$ where $(\Theta, \Pi)$ change with $n$. Suppose (1.19)-(1.21), (1.24) hold. In the Mixed-SCORE, let $T = \sqrt{\log(n)}$ in (1.14) and $L = K$. As $n \to \infty$, with probability $1 - o(n^{-3})$, $(1/n)\sum_{i=1}^{n} \|\hat{\pi}_i - \pi_i\|_2^2 \leq C \cdot \text{err}_n^2$. If additionally $\theta_{\max} \leq C \theta_{\min}$, then $(1/n)\sum_{i=1}^{n} \|\hat{\pi}_i - \pi_i\|_2^2 \leq C \cdot \log(n) / n \theta_{\max}^{-2}$.

1.5. **Summary.** We propose Mixed-SCORE as a new approach to estimating network mixed memberships. The method contains four ingredients: (a) dimension reduction by PCA, (b) eigenvector normalization by SCORE, (c) a surprising connection between normalized eigenvectors and the Ideal Simplex, and (d) an easy-to-use Vertices Hunting algorithm that allows for a convenient reconstruction of the memberships.

We analyze Mixed-SCORE carefully under the DCMM model. However, Mixed-SCORE is not tied to DCMM and may be successful in much broader settings. For example, for the data sets in Table 1, where the DCMM only holds approximately (at most), yet the simplex structure is clearly visible, and Mixed-SCORE performs quite satisfactorily; see Section 2 for details.

DCMM is closely related to Mixed Membership Stochastic Block (MMSB) model by Airoldi et al. [2], but MMSB does not model degree heterogeneity. It is also different from the Latent Position Cluster (LPC) model by Handcock et al. [18]. DCMM is similar to the Overlapping Continuous Community Assignment (OCCAM) model by Zhang et al. [39], but their models and interpretation on $\Pi$ are very different.

DCMM is closely related to Newman’s DCBM [27], and is related to the recent literature on DCBM [10, 22, 35, 36, 40]. However, these works are mostly focused on community detection, not on membership estimation.

Mixed-SCORE consists of several ideas, each of which can be extended in different ways. For example, the SCORE step can be useful for directed or bipartite networks, and the Vertices Hunting step can be extended to address Topic Modeling in text mining. See Section 5 for more discussion.

1.6. **Content and notations.** In Section 2, we apply Mixed-SCORE to all data sets in Table 1 and interpret the results. In Section 3, we prove the main results Theorems 1.2-1.3. Section 4 contains simulations and Section 5 contains discussions. Proofs of secondary results are relegated to Section 6.

For any vector $x$, $\|x\|_q$ denotes the $\ell_q$-norm, $q > 0$. The subscript is dropped for simplicity if $q = 2$. For any matrix $M$, $\|M\|$ denotes the spectral norm, $\|M\|_F$ denotes the Frobenius norm, and $\|M\|_1$ denotes the matrix $\ell_1$-norm. We use $C$ to denote a generic positive constant that may vary from occurrence to occurrence. For two positive sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$,
we say $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$, and we say $a_n \asymp b_n$ if there is a constant $C > 1$ such that $b_n/C \leq a_n \leq Cb_n$ for sufficiently large $n$.

**2. Application to all network data sets in Table 1.** Let $\hat{\pi}_i$ be the estimated PMF for node $i$, $1 \leq i \leq n$. We need the following definition.

**Definition 2.1.** Fix $1 \leq i \leq n$. We call $\max_{1 \leq k \leq K}\{\hat{\pi}_i(k)\}$ the (estimated) purity of node $i$ and call community $k$ the (estimated) home base of node $i$ if $k = \arg\max_{1 \leq \ell \leq K}\{\hat{\pi}_i(\ell)\}$.

When applying Mixed-SCORE to all data sets, we set $T = \log(n)$ in obtaining $\hat{R}$ and use the data-driven choice of $L$ in (1.18).

**2.1. The two networks for statisticians.** In a recent paper, Ji and Jin [21] has collected a network data set for statisticians, based on all published papers in Annals of Statistics, Biometrika, JASA, and JRSS-B, 2003 to the first half of 2012. The data set allows us to construct many networks. For reasons of space, we focus our study on a coauthorship network and a citee network, where each node is an author, and edges are defined as follows.

- **Coauthorship network.** There is an edge between two authors if they have coauthored at least two papers in the range of the data set. Our study focuses on the giant component of the network (236 nodes).
- **Citee network.** There is an edge between two authors if they have been cited at least once by the same author (other than themselves). We also focus on the giant component (1790 nodes) for our study.

Consider the Coauthorship network first. The network was suggested by [21] as the “High Dimensional Data Analysis” group which has a “Carroll-Hall” sub-group (including researchers in nonparametric and semi-parametric statistics, functional estimation, etc.) and a “North Carolina” sub-group (including researchers from Duke, North Carolina, and NCSU, etc.). In light of this, we consider a DCMM model assuming (a) there are two communities called “Carroll-Hall” and “North Carolina” respectively, and (b) some of the nodes have mixed memberships in two communities. We have applied Mixed-SCORE to the network, and the results are in Table 2.

In particular, it was found in [21] that the “Fan” group (Jianqing Fan and collaborators) has strong ties to both communities. Our results confirm such a finding but shed new light on the “Fan” group: many of the nodes (e.g., Yingying Fan, Rui Song, Yichao Wu, Chunming Zhang, Wenyang Zhang) have highly mixed memberships, and for each mixed node, we are able to quantify its weights in two communities. For example, both Runze Li (former graduate of UNC-Chapel Hill) and Jiancheng Jiang (former post-doc
Table 2
Left and Middle: high-degree pure nodes in the “Carroll-Hall” community and the “North Carolina” community. Right: highly mixed nodes (data: Coauthorship network).

<table>
<thead>
<tr>
<th>Name</th>
<th>Deg.</th>
<th>Name</th>
<th>Deg.</th>
<th>Name</th>
<th>Deg.</th>
<th>Estimated PMF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peter Hall</td>
<td>21</td>
<td>Joseph G Ibrahim</td>
<td>14</td>
<td>Jianqing Fan</td>
<td>16</td>
<td>54% of Carroll-Hall</td>
</tr>
<tr>
<td>T Tony Cai</td>
<td>18</td>
<td>David Dunson</td>
<td>8</td>
<td>Jason P Fine</td>
<td>5</td>
<td>54% of Carroll-Hall</td>
</tr>
<tr>
<td>Hans-Georg Muller</td>
<td>10</td>
<td>Donglin Zeng</td>
<td>7</td>
<td>Michael R Kosorok</td>
<td>5</td>
<td>57% of Carroll-Hall</td>
</tr>
<tr>
<td>Enno Mammen</td>
<td>6</td>
<td>Alan E Gelfand</td>
<td>5</td>
<td>J S Marron</td>
<td>4</td>
<td>55% of North Carolina</td>
</tr>
<tr>
<td>Jian Huang</td>
<td>6</td>
<td>Ming-Hui Chen</td>
<td>5</td>
<td>Yufeng Liu</td>
<td>4</td>
<td>52% of North Carolina</td>
</tr>
<tr>
<td>Yanjun Ma</td>
<td>5</td>
<td>Bing-Yi Jing</td>
<td>4</td>
<td>Xiaotong Shen</td>
<td>4</td>
<td>55% of North Carolina</td>
</tr>
<tr>
<td>Bani Mallick</td>
<td>4</td>
<td>Dan Yu Lin</td>
<td>4</td>
<td>Kung-Sik Chan</td>
<td>4</td>
<td>55% of North Carolina</td>
</tr>
<tr>
<td>Jens Perch Nielsen</td>
<td>4</td>
<td>Guosheng Yin</td>
<td>4</td>
<td>Yichao Wu</td>
<td>3</td>
<td>51% of Carroll-Hall</td>
</tr>
<tr>
<td>Marc G Genton</td>
<td>4</td>
<td>Heiping Zhang</td>
<td>4</td>
<td>Yacine Ait-Sahalia</td>
<td>3</td>
<td>54% of Carroll-Hall</td>
</tr>
<tr>
<td>Xihong Lin</td>
<td>4</td>
<td>Qi-Man Shao</td>
<td>4</td>
<td>Wenyang Zhang</td>
<td>3</td>
<td>51% of Carroll-Hall</td>
</tr>
<tr>
<td>Aurore Delaigle</td>
<td>3</td>
<td>Sudipto Banerjee</td>
<td>4</td>
<td>Howell Tong</td>
<td>2</td>
<td>52% of North Carolina</td>
</tr>
<tr>
<td>Bin Nan</td>
<td>3</td>
<td>Amy H Herring</td>
<td>3</td>
<td>Chuming Zhang</td>
<td>2</td>
<td>51% of Carroll-Hall</td>
</tr>
<tr>
<td>Bo Li</td>
<td>3</td>
<td>Bradley S Peterson</td>
<td>3</td>
<td>Yingying Fan</td>
<td>2</td>
<td>52% of North Carolina</td>
</tr>
<tr>
<td>Fang Yao</td>
<td>3</td>
<td>Debajyoti Sinha</td>
<td>3</td>
<td>Rui Song</td>
<td>2</td>
<td>52% of Carroll-Hall</td>
</tr>
<tr>
<td>Jane-Ling Wang</td>
<td>3</td>
<td>Kani Chen</td>
<td>3</td>
<td>Per Aslak Mykland</td>
<td>2</td>
<td>52% of North Carolina</td>
</tr>
<tr>
<td>Jiashun Jin</td>
<td>3</td>
<td>Wei Lin</td>
<td>3</td>
<td>Bee Leng Lee</td>
<td>2</td>
<td>54% of Carroll-Hall</td>
</tr>
</tbody>
</table>

at UNC-Chapel Hill and current faculty member at UNC-Charlotte) have mixed memberships, but Runze Li is more on the “Carroll-Hall” community (weight: 73%) and Jiancheng Jiang is more on the “North Carolina” community (weight: 62%).

We now move to the Citee network. Ji and Jin [21] suggested that the network has three meaningful communities: “Large Scale Multiple Testing” (MulTest), “Spatial and Nonparametric Statistics” (SpatNon) and “Variable Selection” (VarSelect). In light of this, we use a DCMM model with $K = 3$, and apply the Mixed-SCORE to the data. Figure 2 (left) presents the rows of $\hat{R} \in \mathbb{R}^{n \times 2}$, where a 2-simplex (i.e., triangle) is clearly visible in the cloud.

Tables 3-4 present the estimated PMF of high degree nodes. The results confirm those in [21] (especially on the existence of three communities aforementioned), but also shed new light on the network. First, it seems that high degree nodes in VarSelect are frequently observed to have an interest in MulTest, and this is not true the other way around (e.g., compare Jianqing Fan, Hui Zou with Yoav Benjamini, Joseph Romano). Second, the citations from SpatNon to either MulTest or VarSelect are comparably lower than those between MulTest and VarSelect. This fits well with our impression.

Conceivably, a node with higher degree tends to be more senior and so tends to be more mixed. This is confirmed by our results. Figure 2 (right) presents the plot of the node purity (see Definition 2.1) versus the estimated degree heterogeneity parameter $\hat{\theta}(i)$.\(^{10}\) The results show a clear negative

\(^{10}\) Letting $\xi_1$, $b_1$ and $\{\hat{w}_i\}_{i=1}^n$ be the same as those in Mixed-SCORE, we estimate $\theta(i)$ by $\hat{\theta}(i) = \xi_1(i) \sum_{k=1}^K [\hat{w}_i(k)/b_1(k)]$. In the oracle setting, the right hand side equals to $\theta(i)$.\(^{10}\)
Fig 2. Left: rows of $\hat{R}$; the dashed line traces the estimated 2-simplex by Mixed-SCORE. Right: node purity versus degree; x-axis represents the estimated degree parameters $\hat{\theta}(i)$ which are grouped together with an interval of .2; we plot the mean and standard deviation of $\hat{\theta}(i)$ in each group (data: Citee network).

correlation between two quantities (especially on the right end, which corresponds to nodes with high degrees), which indicates that nodes with higher degrees tend to be more mixed.

Table 3
Estimated PMF of the 100 nodes with the highest degrees in the Citee network, among which only the 12 purist nodes in each community are reported.

<table>
<thead>
<tr>
<th>Name</th>
<th>Deg</th>
<th>MulTest</th>
<th>SpatNon</th>
<th>VarSelect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Felix Abramovich</td>
<td>366</td>
<td>0.943</td>
<td>0.057</td>
<td></td>
</tr>
<tr>
<td>Joseph Romano</td>
<td>377</td>
<td>0.966</td>
<td>0.032</td>
<td></td>
</tr>
<tr>
<td>Sara van de Geer</td>
<td>372</td>
<td>0.034</td>
<td>0.166</td>
<td></td>
</tr>
<tr>
<td>Nave Benjamins</td>
<td>478</td>
<td>0.021</td>
<td>0.179</td>
<td></td>
</tr>
<tr>
<td>David Donoho</td>
<td>404</td>
<td>0.019</td>
<td>0.181</td>
<td></td>
</tr>
<tr>
<td>Christopher Genovese</td>
<td>521</td>
<td>0.010</td>
<td>0.190</td>
<td></td>
</tr>
<tr>
<td>Larry Wasserman</td>
<td>535</td>
<td>0.000</td>
<td>0.200</td>
<td></td>
</tr>
<tr>
<td>Jan Willer</td>
<td>387</td>
<td>0.798</td>
<td>0.05</td>
<td>0.152</td>
</tr>
<tr>
<td>Alexandre Tsybakov</td>
<td>521</td>
<td>0.794</td>
<td>0.236</td>
<td></td>
</tr>
<tr>
<td>Jihun Jin</td>
<td>441</td>
<td>0.790</td>
<td>0.040</td>
<td></td>
</tr>
<tr>
<td>Ningting Fan</td>
<td>410</td>
<td>0.740</td>
<td>0.259</td>
<td></td>
</tr>
<tr>
<td>John Storey</td>
<td>544</td>
<td>0.737</td>
<td>0.263</td>
<td></td>
</tr>
</tbody>
</table>

Table 4
Estimated PMF of the 12 nodes with the highest degrees in the Citee network.

<table>
<thead>
<tr>
<th>Name</th>
<th>Deg</th>
<th>MulTest</th>
<th>SpatNon</th>
<th>VarSelect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jianqiu Fan</td>
<td>797</td>
<td>0.305</td>
<td>0.220</td>
<td>0.415</td>
</tr>
<tr>
<td>Raymond Carroll</td>
<td>650</td>
<td>0.282</td>
<td>0.294</td>
<td>0.424</td>
</tr>
<tr>
<td>Hui Zou</td>
<td>324</td>
<td>0.348</td>
<td>0.025</td>
<td>0.427</td>
</tr>
<tr>
<td>Peter Hall mass</td>
<td>780</td>
<td>0.051</td>
<td>0.032</td>
<td>0.467</td>
</tr>
<tr>
<td>Runze Li</td>
<td>778</td>
<td>0.282</td>
<td>0.226</td>
<td>0.491</td>
</tr>
<tr>
<td>Ming Yuan</td>
<td>748</td>
<td>0.391</td>
<td>0.166</td>
<td>0.444</td>
</tr>
</tbody>
</table>

2.2. The Polbook network. The network has 105 nodes, each represents a book on US politics published around the time of the 2004 presidential election and sold by the online bookseller Amazon.com. The edges are assigned by Amazon, where two books have an edge if they are frequently co-
purchased by the same buyers, as indicated by the “customers who bought this book also bought these other books” feature on Amazon. By reading the descriptions and reviews of the books posted on Amazon, Mark Newman (see [29]) labeled each book as liberal, neutral, or conservative. Such labels are not exactly accurate but can be used as a reference.

We view the network as having two communities (liberal and conservative) and view neutral nodes as having mixed memberships in two communities, and so a DCMM model with $K = 2$ is appropriate. We applied MixedSCORE to the data with $K = 2$ and Figure 3 presents the estimated PMF for all nodes; note that for each node, the two entries of the estimated PMF are the estimated weights in liberal and conservative respectively. Since two weights sum to 1, Figure 3 only reports the weights in liberal.

<table>
<thead>
<tr>
<th>Title</th>
<th>Author</th>
<th>Estimated PMF</th>
<th>Newman’s label</th>
<th>Reasons for discrepancy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empire</td>
<td>Michael Hardt</td>
<td>91.1% liberal</td>
<td>neutral</td>
<td>liberal book</td>
</tr>
<tr>
<td>The Future of Freedom</td>
<td>Fareed Zakaria</td>
<td>98.1% liberal</td>
<td>neutral</td>
<td>liberal book</td>
</tr>
<tr>
<td>Rise of the Vulcans</td>
<td>Michael Hardt</td>
<td>95.6% liberal</td>
<td>conservative</td>
<td>liberal book</td>
</tr>
<tr>
<td>All the Shah’s Men</td>
<td>Stephen Kinzer</td>
<td>98.2% liberal</td>
<td>neutral</td>
<td>liberal author</td>
</tr>
<tr>
<td>Bush at War</td>
<td>Bob Woodward</td>
<td>93.2% liberal</td>
<td>conservative</td>
<td>liberal author</td>
</tr>
<tr>
<td>Plan of Attack</td>
<td>Bob Woodward</td>
<td>96.8% liberal</td>
<td>neutral</td>
<td>liberal author</td>
</tr>
<tr>
<td>Power Plays</td>
<td>Dick Morris</td>
<td>98.6% conservative</td>
<td>neutral</td>
<td>conservative author</td>
</tr>
<tr>
<td>Meant To Be</td>
<td>Lauren Morrill</td>
<td>98.7% conservative</td>
<td>neutral</td>
<td>not a political book</td>
</tr>
<tr>
<td>The Bushes</td>
<td>Peter Schweizer</td>
<td>60.3% liberal</td>
<td>conservative</td>
<td>our estimation is inaccurate</td>
</tr>
</tbody>
</table>

For all except 9 books listed in Table 5, our results are nicely consistent with the community labels assigned by Newman: for a book that is labeled
as liberal or conservative by Newman, our estimated PMF has a weight of approximately 1 in liberal or conservative, respectively; for a book that is labeled as neutral by Newman, our estimated PMF has significant weights in both liberal and conservative.

For the 9 books in Table 5, our results do not agree well with the labels assigned by Newman, and we have checked the background information of these books using multiple online resources (e.g., reader’s comments, news pages). For books #1-#6, we find either the book or the author is liberal. Note that we estimate these books as highly liberal, while Newman labeled them as either neutral or conservative. Similarly, for book #7, we find the author is conservative. Note that we estimate this book as conservative while Newman labeled it as neutral. For these reasons, we believe our estimates for these 7 books are more accurate. See [20, 33] where the authors also found Newman’s labels could be incorrect for some of the nodes.

Book #8 is not a political book; this may explain the discrepancy between our result and Newman’s label. For book #9, Newman’s label seems to be right and our estimate may not be accurate enough.

2.3. The Football network. This is a network for American football games between Division I-A college teams during the regular football season of Fall 2000 (Girvan and Newman [14]). Each node represents a team and there is an edge between two teams if they have played one or more games. There are a total of 115 nodes, where 5 of them are called “Independents”. For administration purpose, the remaining 110 nodes are manually divided into 11 conferences, each with a size from 7 to 13; see Table 6.

We note that a conference is not necessarily a community, and vice versa. We hypothesize a DCMM model holds with fewer than 11 communities; such a viewpoint is different from [14] which assumes a non-mixing non-overlapping network model where each conference is interpreted as a community. In this spirit, we have applied Mixed-SCORE to the network assuming there are $K$ communities for $2 \leq K \leq 6$, and it seems $K = 4$ gives the most interpretable results. Below, we report the result for $K = 4$.

In particular, for $K = 4$, if we let $\hat{R} \in \mathbb{R}^{n,3}$ be the matrix of entry-wise ratios and view each row of $\hat{R}$ as a point in $\mathbb{R}^3$ as before, then in the cloud of points, a nice 3-simplex is clearly visible; see Figure 4 (left panel).

In Table 6, for each of the 11 conferences, we tabulate the average of the estimated PMF (i.e., $\hat{\pi}_i$) across different teams. The results suggest that geographical locations play an important role in the community structures:

- The four communities can be interpreted as “North East”\footnote{Most of the teams are located in north east or in middle west.}, “South

...
The average of estimated PMF across different teams in each conference (the 4 entries of the PMF are in Columns 3-6, respectively; numbers in the brackets: standard deviations).

<table>
<thead>
<tr>
<th>Conference (abbreviation)</th>
<th>size</th>
<th>“North East”</th>
<th>“South East”</th>
<th>“South Central”</th>
<th>“West Coast”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mid-American (MAC)</td>
<td>13</td>
<td>.93 (.06)</td>
<td>.03 (.05)</td>
<td>.03 (.04)</td>
<td>.01 (.03)</td>
</tr>
<tr>
<td>Southeastern (SEC)</td>
<td>12</td>
<td>.03 (.04)</td>
<td>.94 (.04)</td>
<td>.01 (.02)</td>
<td>.02 (.03)</td>
</tr>
<tr>
<td>Big Twelve (Big 12)</td>
<td>12</td>
<td>.03 (.04)</td>
<td>.02 (.02)</td>
<td>.92 (.06)</td>
<td>.03 (.06)</td>
</tr>
<tr>
<td>Pacific Ten (PAC 10)</td>
<td>10</td>
<td>.02 (.02)</td>
<td>0 (.0)</td>
<td>.02 (.03)</td>
<td>.96 (.05)</td>
</tr>
<tr>
<td>Atlantic Coast (ACC)</td>
<td>9</td>
<td>.24 (.04)</td>
<td>.73 (.04)</td>
<td>0 (.0)</td>
<td>.03 (.02)</td>
</tr>
<tr>
<td>Big Ten (Big 10)</td>
<td>11</td>
<td>.56 (.05)</td>
<td>0 (.0)</td>
<td>.25 (.06)</td>
<td>.19 (.06)</td>
</tr>
<tr>
<td>Conference USA (CUSA)</td>
<td>10</td>
<td>.10 (.11)</td>
<td>.61 (.18)</td>
<td>.26 (.15)</td>
<td>.03 (.08)</td>
</tr>
<tr>
<td>Mountain West (MWC)</td>
<td>8</td>
<td>0 (.0)</td>
<td>.23 (0.10)</td>
<td>.12 (.09)</td>
<td>.65 (.12)</td>
</tr>
<tr>
<td>Sun Belt (Sun Belt)</td>
<td>7</td>
<td>.06 (.11)</td>
<td>.40 (.16)</td>
<td>.33 (.20)</td>
<td>.21 (.25)</td>
</tr>
<tr>
<td>Western Athletic (WAC)</td>
<td>10</td>
<td>.02 (.07)</td>
<td>.16 (.09)</td>
<td>.53 (.15)</td>
<td>.29 (.13)</td>
</tr>
</tbody>
</table>

East”, “South Central”, and “West Coast”, respectively.

- The four conferences MAC, SEC, Big 12, and PAC 10 consist of most of the pure nodes in “North East”, “South East”, “South Central”, and “West Coast”, respectively.\(^\text{12}\)
- The other seven conferences contain mostly mixed nodes (the 5 independent teams are also mixed nodes).

Figure 4 presents the geographical locations for all 115 teams (teams in the

\(^{12}\)Due to estimation errors, we rarely see an estimated PMF has exactly 1 nonzero entry; we think a node as pure if the estimated purity (Definition 2.1) is very close to 1.
same conference are in the same color). For illustration, we have grouped the teams in MAC, SEC, Big 12, and PAC 10 with a contour in orange, green, blue, and purple, respectively, to highlight the connection between the community partition and the geographical locations (e.g., the purple contour circumvents all teams in PAC 10; note that some other teams also fall within the contour).

For most of the mixed nodes, our estimated PMF is consistent with the geographical distance of the node to each of the four communities. One example is MWC (i.e., Mountain West Conference), where for most teams in this conference, the estimated PMF has a high weight in "West Coast". This is consistent with the fact that these teams are close to West Coast geographically. Another example is WAC (i.e., Western Athletic Conference), where a similar claim can be drawn. Especially, for each team in WAC, the estimated PMF has very little weight in "North East".

Compared to Girvan and Newman [14], our results (especially that on the connection between geographical locations and community structures) shed new light on the data set and provide very different perspectives.

3. Proof of Theorems 1.2-1.3. The proofs have two main parts, where we study the entry-wise ratio matrix \( \hat{R} = [\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n]' \) with spectral analysis (in Section 3.1) and analyze the Vertices Hunting step (in Section 3.2), respectively. The two theorems are proved in Section 3.3. Through out this section, \( C \) denotes a generic constant the value of which may vary from occurrence to occurrence.

3.1. Spectral analysis. First, we study the leading eigenvalues of \( \Omega \) and \( A \). Let \( G = \|\theta\|^{-2}\Pi\Theta^2\Pi \in \mathbb{R}^{K,K} \) be as in Section 1.4 and let \( P \in \mathbb{R}^{K,K} \) be as in (1.3). Let \( a_1, \ldots, a_K \) be all the eigenvalues of \( PG \), let \( \lambda_1, \ldots, \lambda_K \) be all the nonzero eigenvalues of \( \Omega \), and let \( \hat{\lambda}_1, \ldots, \hat{\lambda}_K \) be the \( K \) largest eigenvalues of \( A \) (in magnitude), all sorted descendingly in magnitude.

**Lemma 3.1.** Suppose conditions of either Theorem 1.2 or Theorem 1.3 hold. For all \( 1 \leq k \leq K \), \( \lambda_k = \|\theta\|^2 a_k \) and \( |\lambda_k| \geq C\|\theta\|^2 \). Moreover, \( \lambda_1 \geq \max_{2 \leq k \leq K} |\lambda_k| + C\|\theta\|^2 \).

**Lemma 3.2.** Under conditions of either Theorem 1.2 or Theorem 1.3, with probability \( 1 - o(n^{-3}) \), \( \max_{1 \leq k \leq K} \{ |\hat{\lambda}_k - \lambda_k| / |\lambda_k| \} \leq C \sqrt{\log(n)} \theta_{\max} \|\theta\|_1 / \|\theta\|^2 \).

Next, we characterize the leading eigenvectors of \( \Omega \) and \( A \). Let \( \xi_1, \ldots, \xi_K \) be the eigenvectors of \( \Omega \) associated with \( \lambda_1, \ldots, \lambda_K \), respectively, and let

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\(^{13}\)The figure was downloaded from [SouthernCollegeSports.com](http://www.SouthernCollegeSports.com). For very few teams, the figure does not match the data set because conferences change occasionally.
\( \hat{\xi}_1, \ldots, \hat{\xi}_K \) be the eigenvectors of \( A \) associated with \( \hat{\lambda}_1, \ldots, \hat{\lambda}_K \), respectively. Write \( \Xi_0 = [\xi_1, \ldots, \xi_{K-1}] \in \mathbb{R}^{n \times K-1} \) and \( \hat{\Xi}_0 = [\hat{\xi}_1, \ldots, \hat{\xi}_K] \in \mathbb{R}^{n \times K-1} \). For any vector \( v \in \mathbb{R}^n \), define \( \text{OSC}(v) = (\max_{1 \leq i \leq n} |v(i)|) / (\min_{1 \leq i \leq n} |v(i)|) \), where \( \text{OSC}(v) = \infty \) if \( v(i) = 0 \) for some \( i \).

**Lemma 3.3.** Under conditions of either Theorem 1.2 or Theorem 1.3, all the entries of \( \xi_1 \) are strictly positive, and \( \text{OSC}(\Theta^{-1} \xi_1) \leq C \).

**Lemma 3.4.** Under conditions of either Theorem 1.2 or Theorem 1.3, with probability \( 1 - o(n^{-3}) \), there exists an orthogonal matrix \( H \in \mathbb{R}^{K^{-1}, K-1} \) (which depends on \( A \) and is stochastic) such that \( \max \{ \| \hat{\xi}_1 - \xi_1 \|, \| \hat{\Xi}_0 - \Xi_0 H \|_F \} \leq C \sqrt{\log(n)} \theta_{\max} \theta_1 / \theta_1^4 \).

Last, we study the entry-wise ratio matrices \( R \) and \( \hat{R} \).

**Lemma 3.5.** Suppose conditions of either Theorem 1.2 or Theorem 1.3 hold.

- The vertices of the IS satisfy that \( \max_{1 \leq k \leq K} \| v_k \| \leq C \) and \( \min_{k \neq \ell} \| v_k - v_\ell \| \geq C \). Moreover, the volume of the IS is at least \( C \).

- There exist \( L_0 \) points \( m_1, \ldots, m_{L_0} \in \mathbb{R}^{K^{-1}} \) such that \( \max_{i \leq j} \| r_i - m_j \| \leq C / \log(n) \) for \( 1 \leq j \leq L_0 \). Moreover, \( m_1, \ldots, m_{L_0} \) satisfy that \( \min_{1 \leq j \neq \ell \leq L_0} \| m_j - m_\ell \| \geq C \) and \( \min_{1 \leq \ell \leq L_0} \min_{1 \leq k \leq K} \| m_\ell - v_k \| \geq C \).

**Lemma 3.6.** Under conditions of either Theorem 1.2 or Theorem 1.3, with probability \( 1 - o(n^{-3}) \), there exists an orthogonal matrix \( H \in \mathbb{R}^{K^{-1}, K-1} \) (which depends on \( A \) and is stochastic) such that \( \sum_{i=1}^n \| \hat{r}_i - H r_i \|^2 \leq C n \cdot \text{err}_n^2 \).

Compared to the spectral analysis in [22], the settings considered here are more complicated, so most parts of the proofs here are new. For example, the proof of Lemma 3.4 uses the sin-theta theorem (which [22] does not) so it does not require a strong condition on the gaps between nonzero eigenvalues of \( \Omega \) as in [22]. The proof of Lemma 3.5 is also much more difficult than those in [22] due to the presence of mixed memberships; in particular, it is more complicated to quantify the connection between \( \| \pi_i - \pi_j \| \) and \( \| r_i - r_j \| \) for all pairs \( (i, j) \).

### 3.2. Analysis of the Vertices Hunting algorithm

We focus our discussion on Theorem 1.2. Theorem 1.3 is a simpler case and it uses a different choice of \( L \), but the idea is similar, so we discuss it in Section 6.12.

First, we study \( L_n(A) \) in (1.17). Recall that \( \epsilon_L(R) \) is the sum of squared residuals after applying \( k \)-means to rows of \( R \), assuming \( \leq L \) clusters.
Lemma 3.7. Under conditions of Theorem 1.2, with probability $1-o(n^{-3})$,

$$
\epsilon_L(\hat{R}) \left\{ \begin{array}{ll}
\geq Cn(K-L), & L < K, \\
\geq C|\mathcal{M}|(L_0 + K - L), & K \leq L < K + L_0, \\
\leq C \sum_{\ell=1}^{L_0} \sum_{i \in \mathcal{M}_\ell} \|\pi_i - \gamma_\ell\|^2 + Cn \cdot err_n^2, & L = L_0 + K.
\end{array} \right.
$$

As a result, $\hat{L}_n(A) = L_0 + K$ with probability at least $1 - o(n^{-3})$.

Since $\hat{L}_n(A) = L_0 + K$ with overwhelming probability, we can consider the Vertices Hunting with a (non-stochastic) tuning parameter $L = L_0 + K$, without loss of generality.

Lemma 3.8. Suppose the conditions of Theorem 1.2 hold, and we apply Vertices Hunting to $\hat{R}$ with $L = L_0 + K$. With probability $1 - o(n^{-3})$,

- The local clustering step identifies $(L_0+K)$ cluster centers, where there is a unique $(K-1)$-simplex such that $K$ of these centers (denoted by $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_K$) are its vertices, and all other centers fall within the simplex. The $K$ vertices can be easily identified by a convex hull algorithm.
- There is a permutation $\kappa$ of $\{1, \cdots, K\}$ such that $\max_{1 \leq k \leq K} \|\hat{v}_{\kappa(k)} - v_k\| \leq Cerr_n$.

We write $\hat{v}_{\kappa(k)} = \hat{v}_k$ for simplicity.

3.3. Proof of Theorems 1.2-1.3. Let $H$ be the same as in Lemma 3.6. We aim to show that, with probability $1 - o(n^{-3})$, for all $1 \leq i \leq n$,

$$
\|\hat{\pi}_i - \pi_i\|^2 \leq C\|\hat{r}_i - Hr_i\|^2 + C\left( \max_{1 \leq k \leq K} \{ |\hat{\lambda}_k - \lambda_k| / |\lambda_k| \} \right)^2 + C\left( \max_{1 \leq k \leq K} \|\hat{v}_k - v_k\| \right)^2.
$$

(3.25)

This says that the estimation errors for $\pi_i$ attribute to three sources: difference between $\hat{R}$ and $R$, difference between the eigenvalues of $A$ and those of $\Omega$, and estimation errors in the Vertices Hunting step. Once (3.25) is proved, the claim follows. To see the point, note that by Lemma 3.2 and the assumption (1.20), the second term on the Right Hand Side (RHS) is $O(err_n^2)$. Also, by Lemma 3.8 and Lemma 6.4, in the settings of both theorems, the third term on the RHS is $O(err_n^2)$. Inserting these into (3.25) gives

$$
\sum_{i=1}^{n} \|\hat{\pi}_i - \pi_i\|^2 \leq C\sum_{i=1}^{n} \|\hat{r}_i - Hr_i\|^2 + Cn \cdot err_n^2 \leq Cn \cdot err_n^2,
$$

where the last inequality is due to Lemma 3.6. This gives the claim.
We now show (3.25). The goal is to show that the inequality holds provided that the last two terms on the RHS of (3.25) are $o(1)$ (ensured by the conditions of the theorems). In the Membership Reconstruction (MR) step, we compute $\hat{\omega}_i$ and $\hat{b}_1$ first, and then use them to construct $\hat{\pi}_i$. As preparation, we first analyze $\hat{\omega}_i$ and $\hat{b}_1$, respectively.

Consider $\hat{\omega}_i$. By definition, it satisfies both that $\hat{\omega}_i = \sum_{k=1}^K \hat{\omega}_i(k) \hat{\nu}_k$ and that $\sum_{k=1}^K \hat{\omega}_i(k) = 1$. So if we let $\hat{V} = [\hat{v}_1,\ldots,\hat{v}_K]$ and $Q = [1, V']$, then

$$\hat{\omega}_i = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \hat{\nu}_1 & \cdots & \hat{\nu}_K \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \hat{Q}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

where $1_K \in \mathbb{R}^K$ is the vector of 1. Recall that $V = [v_1,v_2,\ldots,v_K]$, where $v_1,\ldots,v_K$ are the vertices of the Ideal Simplex. So if we let $Q = [1, V']$, then by definitions, $w_i = Q^{-1}(1, r')$. It follows that

$$\|\hat{\omega}_i - w_i\| \leq \|\hat{\omega}_i - \hat{\nu}_i\| + \|\hat{\nu}_i - w_i\| \leq \hat{Q}^{-1}\|\hat{\nu}_i - r_i\| + \|r_i\|\hat{Q}^{-1} - Q^{-1} = \|\hat{Q}^{-1}\|\hat{\nu}_i - r_i\| + \|r_i\|\hat{Q}^{-1} - Q^{-1}\|\hat{Q} - Q\|.$$

We first bound $\|\hat{Q} - Q\|$. Using the connection between $Q$ and $V$, $\|\hat{Q} - Q\| \leq \|\hat{Q} - V\|_1 = \|\hat{V} - V\|_1 = \max_{1 \leq k \leq K} \|\hat{\nu}_k - v_k\|_1 \leq \sqrt{K} \max_{1 \leq k \leq K} \|\hat{\nu}_k - v_k\|_1$. We then bound $\|Q^{-1}\|$. Write $Q = B' \text{diag}(b_1,\ldots,b_1(K))^{-1}$, where in the proof of Lemma 3.3, we have seen that $BB' = \|\theta\|^{-2}G^{-1}$ by (6.38) and $C^{-1}\|\theta\|^{-1} \leq b_1(1) \leq C\|\theta\|^{-1}$ by (6.37). So $\lambda_{\min}(Q'Q) \geq C^{-2}\|G\|^{-1}$.

Furthermore, since $G$ is a non-negative matrix, $\|G\| \leq \sum_{k,\ell=1}^K G(k,\ell) = \|\theta\|^{-2} \sum_{k,\ell=1}^K \sum_{i=1}^n \pi_i(k) \pi_i(\ell) \theta^2(i) = 1$. So $\lambda_{\min}(Q'Q) \geq C^{-2}$. It follows that $\|Q^{-1}\| \leq C$. Also, since $\|\hat{Q} - Q\| \leq C \max_{1 \leq k \leq K} \|\hat{\nu}_k - v_k\| = o(1)$, it also holds that $\|Q^{-1}\| \approx C$. Last, by Lemma 3.5, $\max_{1 \leq i \leq n} \|r_i\| \leq C$. Combing the above, we obtain

$$\|\hat{\omega}_i - w_i\| \leq C\|\hat{\nu}_i - r_i\| + C \max_{1 \leq k \leq K} \|\hat{\nu}_k - v_k\|.$$

Consider $\hat{b}_1$, where we recall that it is defined by

$$\hat{b}_1(k) = [\hat{\lambda}_1 + \hat{\nu}_k \text{diag}(\hat{\lambda}_2,\ldots,\hat{\lambda}_K)\hat{\nu}_k]^{-1/2}.$$

First, by Lemma 3.1, $\lambda_1 \leq C\|\theta\|^2$ and $|\lambda_K| \geq C^{-1}\|\theta\|^2$. Second, $C^{-1}\|\theta\|^{-1} \leq b_1(k) \leq C\|\theta\|^{-1}$ for $1 \leq k \leq K$. Last, we have proved $\lambda_{\min}(Q'Q) \geq C$ above; noting that the $k$-th diagonal of $Q'Q$ is $1 + \|v_k\|^2$, we have $\|v_k\| \geq C$ for all $1 \leq k \leq K$. It is then easy to see that

$$\|\hat{b}_1(k) - b_1(k)\|/b_1(k) \leq \max_{1 \leq \ell \leq K} \{\|\hat{\lambda}_\ell - \lambda_\ell\|/|\lambda_\ell|\} + C \max_{1 \leq \ell \leq K} \|\hat{\nu}_\ell - v_\ell\|.$$


We are now ready to show (3.25). It suffices to show

\[ (3.28) \quad \| \hat{\pi}_i - \pi_i \| \leq C \left( \| \hat{w}_i - w_i \| + \max_{1 \leq k \leq K} \{ \| \hat{b}_1(k) - b_1(k) \| / b_1(k) \} \right), \]

for once this is proved, plugging (3.26)-(3.27) into it gives (3.25) directly.

We now show (3.28). Recall that once \( \hat{w}_i \) and \( \hat{b}_1 \) are ready, Mixed-SCORE first derives

\[ \hat{\pi}_i^* (k) = \max \{ 0, \hat{w}_i(k)/\hat{b}_1(k) \}, \quad 1 \leq k \leq K, \]

and then estimates \( \pi_i \) by \( \hat{\pi}_i = \hat{\pi}_i^*/\| \hat{\pi}_i^* \|_1 \). Let \( \pi_i^* (k) = w_i(k)/b_1(k) \) be the non-stochastic counterpart of \( \hat{\pi}_i^* (k) \), \( 1 \leq k \leq K \). Since \( \pi_i^* (k) \geq 0 \), we have \( |\hat{\pi}_i^* (k) - \pi_i^* (k) | \leq |\hat{w}_i(k)/\hat{b}_1(k) - w_i(k)/b_1(k) | \). It follows from the triangular inequality that

\[ |\hat{\pi}_i^* (k) - \pi_i^* (k) | \leq \frac{1}{b_1(k)} |\hat{w}_i(k) - w_i(k) | + \hat{w}_i(k) \left( \frac{1}{b_1(k)} - \frac{1}{\hat{b}_1(k)} \right), \]

where we have used \( \hat{w}_i(k) \leq 1 \) in the last inequality. First, \( b_1(k) \geq C \| \theta \|^{-1} \) by (6.37). Second, \( |\hat{b}_1(k) - b_1(k) | \) is much smaller than \( b_1(k) \), by (3.27). So \( \hat{b}_1(k) \geq b_1(k)/2 \geq C \| \theta \|^{-1} \). Combining these, the right hand side is bounded by \( C \| \theta \| \cdot (|\hat{w}_i(k) - w_i(k) | + |\hat{b}_1(k) - b_1(k) |/\hat{b}_1(k)) \). It follows that

\[ (3.29) \quad \| \hat{\pi}_i - \pi_i \| \leq C \| \theta \| (\| \hat{w}_i - w_i \| + \max_{1 \leq k \leq K} \{ |\hat{b}_1(k) - b_1(k) |/b_1(k) \} ). \]

At the same time, recall that \( \hat{\pi}_i = \hat{\pi}_i^*/\| \hat{\pi}_i^* \|_1 \). By the triangular inequality,

\[ |\hat{\pi}_i (k) - \pi_i (k) | \leq \frac{1}{\| \hat{\pi}_i^* \|_1} |\hat{\pi}_i^* (k) - \pi_i^* (k) | + \hat{\pi}_i^* (k) \left( \frac{1}{\| \hat{\pi}_i^* \|_1} - \frac{1}{\| \pi_i^* \|_1} \right), \]

First, \( \| \hat{\pi}_i^* \|_1 - \| \pi_i^* \|_1 \leq \| \hat{\pi}_i^* - \pi_i^* \|_1 \leq \sqrt{K} \| \hat{\pi}_i^* - \pi_i^* \| \), by Cauchy-Schwartz inequality. Second, since \( b_1(k) \leq C \| \theta \|^{-1} \) for all \( 1 \leq k \leq K \), we have \( \| \pi_i^* \|_1 = \sum_{k=1}^{K} w_i(k)/b_1(k) \geq C^{-1} \| \theta \| \sum_{k=1}^{K} w_i(k) = C^{-1} \| \theta \| \). Combining the above gives \( |\hat{\pi}_i (k) - \pi_i (k) | \leq C \| \theta \|^{-1} (|\hat{\pi}_i^* (k) - \pi_i^* (k) | + \| \hat{\pi}_i^* - \pi_i^* \|). \) It follows that

\[ (3.30) \quad |\hat{\pi}_i - \pi_i | \leq C \| \theta \|^{-1} \| \hat{\pi}_i - \pi_i^* \|. \]

Combining (3.29)-(3.30) gives (3.28), and completes the proof.
4. Simulations. We investigate the performance of Mixed-SCORE via a small-scale numerical study. We compare our method with OCCAM [39] (in Experiments 2-5) and LPC [18] (in Experiment 6). The reason for choosing these two competitors is that they are both model-based methods that output node “memberships”, and they both account for degree heterogeneity (explicitly or implicitly). OCCAM assigns to each node a non-negative \(\ell_2\)-norm; we renormalize these vectors by their \(\ell_1\)-norms and use them as the estimated PMF. LPC outputs a posterior PMF for each node (describing its posterior probabilities of being drawn from different components of a mixture), which we use as the estimated PMF of that node; to implement LPC, we use the R package \texttt{latentnet} and the default algorithm parameters.

For most experiments below, we set \(n = 500\) and \(K = 3\). For \(0 \leq n_0 \leq 160\), let each community have \(n_0\) number of pure nodes. Fixing \(x \in (0,1/2)\), let the mixed nodes have four different memberships \((x,x,1-2x),(x,1-2x,x), (1-2x,x,x)\) and \((1/3,1/3,1/3)\), each with \((500-3n_0)/4\) number of nodes. Fixing \(\rho \in (0,1)\), the matrix \(P\) has diagonals 1 and off-diagonals \(\rho\). Fixing \(z \geq 1\), we generate the degree parameters such that \(1/\theta(i) \sim U(1,z)\), where \(U(1,z)\) denotes the uniform distribution on \([1,z]\). The tuning parameter \(L\) is selected as in (1.18). For each parameter setting, we report \(n^{-1} \sum_{i=1}^{n} \|\hat{\pi}_i - \pi_i\|_2^2\) averaged over 100 repetitions.

Experiment 1: Tuning parameter selection. We first study the choice of the tuning parameter \(L\) in Mixed-SCORE. We aim to see (i) how the estimation errors change for a range of \(L\), and (ii) how the adaptive choice \(\hat{L}_n^*(A)\) in (1.18) performs. Fix \((x,\rho,z) = (0.4,0.2,5)\) and let \(n_0\) range in \{60,80,100\}. For each setting, we run Mixed-SCORE with \(L \in \{4,5,\cdots,9\}\) and \(\hat{L}_n^*(A)\). The results are displayed in Figure 5. First, when there are relatively few mixed nodes (e.g., \(n_0 = 100\), small values of \(L\) yield good performance; but as the number of mixed nodes going up, we favor larger values of \(L\); these match our theoretical results (Theorems 1.2-1.3). Second, under the circumstances of a moderate number of mixed nodes (e.g., \(n_0 = 60,80\), for a range of \(L\) (e.g., \(L \in \{7,8,9\}\)), the statistical errors of Mixed-SCORE are similar, and \(\hat{L}_n^*(A)\) falls in this range with high probability. Figure 6 shows the estimated 2-simplex in one repetition \((n_0 = 80)\), and the simplex changes very little when \(L\) falls in a range.

Experiment 2: Fraction of pure nodes. Fix \((x,\rho,z) = (0.4,0.1,5)\) and let \(n_0\) range in \{40,60,80,100,120,160\}. As \(n_0\) increases, the fraction of pure nodes increases from around 25% to around 95%. The results are displayed in top left panel of Figure 7. It suggests that when the fraction of pure nodes is < 70%, Mixed-SCORE significantly outperforms OCCAM; when the fraction
Fig 5. Performance of Mixed-SCORE as the tuning parameter $L$ varies (y-axis: estimation errors; $L^*_n(A)$ is plotted in red; both mean and standard deviation are displayed). From left to right, there are 60, 80, 100 pure nodes in each community, respectively.

Fig 6. Illustration of the Vertices Hunting step. From left to right, $L = 7, 8, 9$. Although the local cluster centers (blue points) are different, the estimated 2-simplex (dashed black) changes very little, and it approximates the IS (solid red) well.

of pure nodes is $> 70\%$, the two methods have similar performance.

Experiment 3: Connectivity across communities. Fix $(x, n_0, z) = (0.4, 80, 5)$ and let $\rho$ range in $\{0.05, 0.1, 0.15, \ldots, 0.5\}$. The larger $\rho$, the more edges across different communities. The results are presented in top right panel of Figure 7. We see that the performance of Mixed-SCORE improves as $\rho$ decreases. One possible reason is that, for $\rho$ large, it is relatively more difficult to identify the vertices of the Ideal Simplex. Furthermore, Mixed-SCORE is better than OCCAM in all settings.

Experiment 4: Purity of mixed nodes. Fix $(n_0, \rho, z) = (80, 0.1, 5)$ and let $x$ range in $\{0.05, 0.1, 0.15, \ldots, 0.5\}$. We recall Definition 2.1 for the “purity” of a node. In our settings, there are four types of mixed nodes, and the purity of the first three types of mixed nodes is $(1 - 2x)1\{x \leq 1/3\} + x1\{x > 1/3\}$. Therefore, as $x$ increases to $1/3$, these nodes become less pure; then, as $x$ further increases, these nodes become more pure. The results are in bottom left panel of Figure 7. It suggests that estimating the memberships becomes harder as the purity of mixed nodes decreases, and Mixed-SCORE outperforms OCCAM in almost all settings. Especially in the highly “mixing” case (say, $x$ is close to $1/3$), Mixed-SCORE is much better than OCCAM.
**Experiment 5: Degree heterogeneity.** Fix \((x, n_0, \rho) = (0.4, 80, 0.1)\) and let \(z\) range in \(\{1, 2, \ldots, 8\}\). Since \(1/\theta(i) \sim U(1, z)\), a larger \(z\) implies that the nodes have lower degrees and are more heterogeneous (hence, the problem becomes more difficult). The results are presented in bottom right panel of Figure 7. It suggests that Mixed-SCORE uniformly outperforms OCCAM. Interestingly, when \(z\) is small (so the problem is “easy”), Mixed-SCORE is very accurate, but the performance of OCCAM is unsatisfactory.

**Experiment 6: Comparison with latent space approach.** We compare Mixed-SCORE with the Bayesian method based on LPC [18] (we use the R package \textit{latentnet}). In this experiment, we fix \(n = 120, K = 3, (x, \rho, z) = (0.4, 0.3, 5)\), and let \(n_0\) range in \(\{12, 16, 20, \ldots, 32, 36\}\) (so the number of mixed nodes in each group decreases from 21 to 3). The results are displayed in Figure 8. We find that, when the fraction of mixed nodes is comparably small, LPC has a perfect performance; however, as the fraction of mixed nodes increases to more than 40%, the performance of LPC deteriorates rapidly; one reason is that, when \(n_0\) is not very large, LPC often estimates the PMF of all the nodes as the same. In contrast, the performance of Mixed-SCORE is quite stable. In terms of computing time, Mixed-SCORE takes only seconds for
one repetition while LPC takes > 20 minutes (both measured in R).

![Graph](image)

Fig 8. Estimation errors of Mixed-SCORE and LPC (y-axis: $n^{-1} \sum_{i=1}^{n} \left\| \hat{\pi}_i - \pi_i \right\|^2$).

5. Discussion. The paper is closely related to [22] on SCORE, but it is different in important ways: (i) the focus of [22] is on community detection, while the focus here is on membership estimation which is more challenging; (ii) the paper discovers an interesting connection between the Ideal Simplex and mixed memberships, which was not discovered in the literature; (iii) we propose Mixed-SCORE as a new approach to membership estimation, at the heart of which is a new Vertices Hunting algorithm; (iv) the theory here is more complicated than that in [22]. The paper is also related to the recent literature on DCBM [10, 22, 35, 36, 40]. However, these works mainly focused on community detection rather than membership estimation.

Our model, DCMM is a natural extension of the DCBM by Karrer and Newman [27]. It can also be viewed as an extension of the Mixed Membership Stochastic Block (MMSB) model [2, 3, 16]: but DCMM models degree heterogeneity, while MMSM does not. Our approach is also different from theirs: [2, 16] used a Bayesian approach and [3] used a tensor approach, and we use a spectral approach, which is computationally more efficient.

DCMM is similar to the Overlapping Continuous Community Assignment model (OCCAM) [39]. However, both DCMM and MMSB [2, 3, 16] regard $\Pi$ in (1.8) as the matrix of PMF’s (each row is a PMF so the $\ell_1$-norm is 1) which seems to be scientifically meaningful, but [39] thinks the matrix in a way so that each row has a unit $\ell_2$-norm (which seems hard to interpret). The methods and theory in [39] are also very different from here.

Our work is also related to works on overlapping communities learning in networks, including methods based upon local connectivity patterns [17, 30] and methods that are developed under overlapping community models [5, 31]. However, these works do not output a membership vector for each node.
directly as our method does. Our work is also related to [18], where they proposed a latent space model, and the Bayesian method introduced there estimated a posterior PMF for each node that can be viewed as a mixed membership. However, the model and method there are different from ours.

In a high level, the work is related to recent interest in Nonnegative Matrix Factorization (NMF) [32]. Especially, Donoho and Stodden [12] provided a geometrical interpretation of NMF as “the problem of finding a simplicial cone ... contained in the positive orthant”. The Ideal Simplex we discover here is reminiscent of the simplicial cone, but is of course very different.

Our method is a new PCA approach at the heart of which is the idea of multi-stage complexity/dimension reduction. In a high level, it is related to the recent work on IF-PCA [26], sparse PCA [6, 9], and high-dimensional clustering [4, 25, 26].

The work can be extended in many directions. In a forthcoming manuscript [24], we extend the work (on undirected networks) to the settings where the network is directed or bi-partite. In another forthcoming manuscript [23], we investigate the optimality of Mixed-SCORE using a minimax framework. There is also an interesting connection between DCMM and Topic Models in text mining [1, 7, 8], a problem that has received a lot of attention. In our forthcoming work [28], we extend methods and theory developed here to attack problems on Topic Models. Extensions to dynamic networks and to network data sets where some covariates (i.e., ages and affiliations of the authors in the statistician’s network) are available are also interesting research directions that are worthy of future work.

There are several open problems. For example, it remains unclear how to estimate the number of communities $K$, and there is plenty of room for improvement in the modeling, methods, theory, as well as (real-world) applications. For example, some conditions in our theorems can be relaxed, and a different Vertices Hunting algorithm may also be successful. Also, it is always possible that we may need a model that is more sophisticated than DCMM. For reasons of space, we leave such studies to the future.

6. Appendix.

6.1. A preliminary lemma and its proof. We state a useful lemma about the matrix $B$ and its connection to the spectrum of $\Omega$.

Lemma 6.1. Consider $DCMM(n, P, \Theta, \Pi)$ and assume there is at least one pure node for each community. The following statements are true:

- There is a non-singular matrix $B \in \mathbb{R}^{K \times K}$ such that $\Theta \Pi B = \Xi$, and $B$ is unique once $\Xi$ is chosen.
For $1 \leq k \leq K$, denote by $a_k$ the $k$-th largest (in magnitude) eigenvalue of $PG$. All the nonzero eigenvalues of $\Omega$ are $a_1\|\theta\|^2, \cdots, a_K\|\theta\|^2$, i.e., $\lambda_k = a_k\|\theta\|^2$, for $1 \leq k \leq K$.

For $1 \leq k \leq K$, $b_k$ is a (right) eigenvector of $PG$ associated with $a_k$.

$\lambda_1 > 0$ and it has a multiplicity 1 (so $\xi_1$ is uniquely determined up to a factor of $\pm 1$).

$\xi_1$ can be chosen such that all of its entries are positive. For this choice of $\xi_1$, all the entries of the associated $b_1$ are also positive.

**Proof of Lemma 6.1:** Consider the first claim. It suffices to show that the column space of $\Theta\Pi$ is the same as the column space of $\Xi$. Then, since $\xi_1, \cdots, \xi_K$ form an orthonormal basis of this subspace, there is a unique, non-singular matrix $\tilde{B}$ such that $\Theta\Pi = \Xi\tilde{B}$. We then take $B = \tilde{B}^{-1}$.

By the assumption that there is at least one pure node in each community, we can find $K$ rows of $\Pi$ such that they form a $K \times K$ identity matrix. So $\Pi$ has a rank $K$. Since $\Theta$ and $P$ are both non-singular matrices, $\Omega = \Theta\Pi\Pi^\prime\Theta$ also has a rank $K$. This shows that $\Omega$ indeed has $K$ nonzero eigenvalues. By definition, $\Omega \xi_k = \lambda_k \xi_k$, for $1 \leq k \leq K$. It follows that $\Theta\Pi(P\Pi^\prime\Theta\xi_k) = \lambda_k \xi_k$.

So $\xi_k$ is in the column space of $\Theta\Pi$ for $1 \leq k \leq K$. This means the column space of $\Xi$ is contained in the column space of $\Theta\Pi$. Since both matrices have a rank $K$, the two column spaces are the same.

Consider the second claim. For any matrices $A \in \mathbb{R}^{m,n}$ and $B \in \mathbb{R}^{n,m}$, if $m \geq n$, then the nonzero eigenvalues of $AB$ are the same as the nonzero eigenvalues of $BA$. It follows that the nonzero eigenvalues of $\Omega = (\Theta\Pi)(\Pi^\prime\Theta)$ are the same as the nonzero eigenvalues of $(\Pi^\prime\Theta)(\Theta\Pi) = \|\theta\|^2PG$.

Consider the third claim. Write $\tilde{G} = \|\theta\|^2G = \Pi^\prime\Theta^2\Pi^\prime$. Note that $\Omega \xi_k = \lambda_k \xi_k$ and $\xi_k = \Theta\Pi b_k$. Therefore, $(\Theta\Pi\Pi^\prime\Theta)(\Theta\Pi b_k) = \lambda_k(\Theta\Pi b_k)$. Multiplying both sides by $\Pi^\prime\Theta$ from the left, we have

$$\tilde{G}PGb_k = \lambda_k \tilde{G}b_k$$

Since $\tilde{G}$ is non-singular, $P\tilde{G}b_k = \lambda_k b_k$. Plugging in $\tilde{G} = \|\theta\|^2G$ and $\lambda_k = a_k\|\theta\|^2$, we obtain $PGb_k = a_k b_k$. This shows that $b_k$ is an eigenvector of $PG$ associated with $a_k$.

Consider the fourth claim. Since $\lambda_1 = a_1\|\theta\|^2$, it suffices to show that $a_1 > 0$ and that it has a multiplicity 1. Since $P$ is nonnegative and irreducible, and all the diagonal entries of $P$ are strictly positive, $P$ is a primitive matrix [19, Chapter 8.5]. By definition of primitive matrices, there exists an integer
\[ m > 0 \text{ such that } P^m > 0. \] Let \( \alpha = \min_{1 \leq k \leq K} G(k,k) \). Note that \( G \geq \alpha I \) because the off-diagonal entries of \( G \) are all nonnegative. Then, \( PG \geq \alpha P \geq 0 \) and \((PG)^m \geq \alpha^m P^m > 0 \) \cite[Page 520]{19}. So

\[(6.31) \quad PG \text{ is primitive.} \]

By Perron-Frobenius theorem \cite[Theorem 8.4.4]{19}, \( a_1 > 0 \) and it has a multiplicity 1.

Consider the last claim. Note that \( b_1 \) is the eigenvalue of \( PG \) associated with \( a_1 \). Since \( a_1 \) has a multiplicity 1, \( b_1/\|b_1\| \) is unique up to a factor of \( \pm 1 \) (depending on the choice of \( \xi_1 \)). By Perron-Frobenius theorem again, \( b_1/\|b_1\| \) can be chosen such that all the entries are positive. The associated \( \xi_1 = \Theta \Pi b_1 \), where \( \Theta \Pi \) is a nonnegative matrix with positive row sums. So all the entries of \( \xi_1 \) are also positive.

6.2. Proof of Theorem 1.1. By Lemma 6.1, all the entries of \( \xi_1 \) and \( b_1 \) are positive, so \( \{r_i\}_{i=1}^n \) and \( \{v_k\}_{k=1}^K \) are well-defined. We now show that \( b_1 \) and \( \{w_i\}_{i=1}^n \) are unique even though \( \Xi \) may not be. Note that by Lemma 6.1, \( \xi_1 \) is uniquely determined. For each \( 1 \leq k \leq K \), there is at least one pure node \( i \) in community \( k \). Since \( \xi_1 = \Theta \Pi b_1 \), for this node \( i \), \( \xi_1(i) = \theta(i) b_1(k) \). So the uniqueness of \( \xi_1 \) implies the uniqueness of \( b_1 \). Furthermore, \( w_i \) is defined through \( b_1 \) and \( \pi_i \), so it is also uniquely determined.

To show that the Ideal Mixed-SCORE exactly recovers all the \( \pi_i \), we first show that the simplex structure exists, i.e., \((1.12)\) holds. Since \( \Xi = \Theta \Pi B \), \( \xi_1(i) = \theta(i) \sum_{k=1}^K \pi_i(k) b_1(k) = \theta(i) b_1 \odot \pi_1 \) for \( 1 \leq \ell \leq K \). It follows from \( R(i, \ell) = \xi_{\ell+1}(i)/\xi_1(i) \) that

\[
R(i, \ell) = \frac{\theta(i) \sum_{k=1}^K \pi_i(k) b_{\ell+1}(k)}{\theta(i) \|b_1 \odot \pi_1\|_1} = \sum_{k=1}^K \frac{b_1(k) \pi_i(k)}{\|b_1 \odot \pi_i\|_1} \cdot \frac{b_{\ell+1}(k)}{b_1(k)} = \sum_{k=1}^K w_i(k) v_k(\ell).
\]

This yields that \( r_i = \sum_{k=1}^K w_i(k) v_k = V w_i \).

Once the simplex structure holds, by applying any convex hull algorithm to rows of \( R \), we can exactly identify \( v_1, \ldots, v_K \). Furthermore, for each \( i \), we can recover \( w_i \) from \( r_i \) and \( v_1, \ldots, v_K \) by solving the linear equations (1 is the vector of 1's)

\[
V w_i = r_i, \quad 1' w_i = 1, \quad \text{where } V \in \mathbb{R}^{K-1,K}, w_i \in \mathbb{R}^K.
\]

\footnote{For any matrix \( A \), we write \( A \geq 0 \) if all the entries of \( A \) are nonnegative, and \( A > 0 \) if all the entries are positive; for any two matrices \( A, B \) of the same dimension, we write \( A \geq B \) if \( A - B \geq 0 \), and \( A > B \) if \( A - B > 0 \).}
Last, if $b_1$ is known, since $w_1 \propto (b_1 \circ \pi_i)$ and $\pi_i$ is a PMF, we can exactly recover $\pi_i$ by $\pi_i = \pi_i^* / \|\pi_i^*\|_1$ where $\pi_i^*(k) = w_i(k)/b_1(k)$, $1 \leq k \leq K$.

It remains to show that we can recover $b_1$, i.e., (1.13) holds. Write $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_K)$. Then, $\Omega = \Xi \Lambda \Xi'$. First, plugging in $\Xi = \Theta \Pi B$, we find that $\Omega = \Theta \Pi (B \Lambda B') \Pi' \Theta$. Multiplying both sides by $\Pi'$ from the left and $\Pi$ from the right, we have $\Pi' \Omega \Pi = \tilde{G}(B \Lambda B') \tilde{G}$, where $\tilde{G} = \Pi' \Theta^2 \Pi$ is a non-singular matrix. Second, since $\Omega = \Theta \Pi \Pi' \Theta'$, we have $\Pi' \Omega \Pi = GP \tilde{G}$. Combining the above gives

$$\tilde{G} P \tilde{G} = \tilde{G}(B \Lambda B') \tilde{G} \iff P = B \Lambda B'. $$

As a result, $1 = P(k, k) = \sum_{\ell=1}^{K} \lambda_\ell b_\ell^2(k) = b_1^2(k) [\lambda_1 + \sum_{\ell=2}^{K} \lambda_\ell^2 v_\ell (\ell - 1)]$. This gives (1.13).

6.3. Proof of Lemma 3.1. We have seen that $\lambda_k = a_k \|\theta\|^2$ for $1 \leq k \leq K$ and $\lambda_1 > 0$ in Lemma 6.1. Combining them with the assumption (1.21) gives $\lambda_1 - \max_{2 \leq k \leq K} |\lambda_k| \geq c_4 \|\theta\|^2$. It remains to prove $\min_{1 \leq k \leq K} |\lambda_k| \geq C \|\theta\|^2$, and we only need to show that

$$\min_{1 \leq k \leq K} |a_k| \geq C. \tag{6.32}$$

We note that $a_k^2$ is an eigenvalue of $G'P'PG$, where $P'P$ is a positive definite matrix that does not depend on $n$. So $a_k^2 \geq C \lambda_{\min}(G'G)$. Here $G$ is a positive semi-definite matrix. Therefore, to show (6.32), it suffices to show that

$$\lambda_{\min}(G) \geq C. \tag{6.33}$$

Recall that $G = ||\theta||^2 \Pi' \Theta^2 \Pi$. Let $\Theta_1 \in \mathbb{R}^n$ be the diagonal matrix whose $i$-th diagonal is $\theta(i)$ if $i$ is a pure node and 0 otherwise. Let $G_1 = ||\theta||^{-2} \Pi' \Theta_1 \Pi$. Then, $\lambda_{\min}(G) \geq \lambda_{\min}(G_1)$ because $\Theta - \Theta_1$ is positive semi-definite. Moreover, $G_1$ is a diagonal matrix, whose $k$-th diagonal is equal to $||\theta||^{-2} \sum_{i \in N_k} \theta^2(i)$. By (1.19), $\lambda_{\min}(G_1) \geq c_2$. This proves (6.33).

6.4. Proof of Lemma 3.2. First, by Weyl’s inequality, $\max_{1 \leq k \leq K} |\hat{\lambda}_k - \lambda_k| \leq \|A - \Omega\|$. Second, by Lemma 3.1, $\min_{1 \leq k \leq K} |\lambda_k| \geq C \|\theta\|^2$. Combining the above, it suffices to show that with probability $1 - o(n^{-3})$,

$$\|A - \Omega\| \leq C \sqrt{\log(n) \theta_{\max}^2 \|\theta\|_1}. \tag{6.34}$$

We now prove (6.34). Write $A - \Omega = W + \text{diag}(\Omega)$, where $W \equiv A - E[A]$. 

First, consider $\text{diag}(\Omega)$. Note that $\Omega(i, i) = \theta^2(i) \sum_{k, \ell=1}^{K} \pi_i(k) \pi_i(\ell) P(k, \ell) \leq \theta^2(i) \max_{k, \ell} P(k, \ell) \leq C\theta^2(i)$.

(6.35) \[ \|\text{diag}(\Omega)\| \leq C\theta_{\max}^2 \leq C\sqrt{\theta_{\max}\|\theta\|_1}, \]

where the last inequality comes from that $\theta_{\max}^2 \leq c_3\theta_{\max}$ by (1.19) and that $\theta_{\max} \leq \sqrt{\theta_{\max}\|\theta\|_1}$.

Second, consider $W$. Our strategy is to write $W$ as the sum of independent matrices and then apply the matrix Bernstein inequality [37, Theorem 6.2]. For $i \neq j$, $W(i, j)$ follows a Bernoulli distribution with parameter $\Omega(i, j)$. Let $e_i$ be the $n \times 1$ vector such that $e_i(j) = 1$ for $i = j$ and 0 otherwise. For $1 \leq i < j \leq n$, define the $n \times n$ matrix $W^{(i,j)} = W(i, j)(e_i e'_j + e_j e'_i)$. Then,

\[ W = \sum_{1 \leq i < j \leq n} W^{(i,j)}, \quad \text{where } E[W^{(i,j)}] = 0, \|W^{(i,j)}\| \leq |W(i, j)| \leq 1. \]

Note that $E[W^2(i, j)] = \Omega(i, j) \leq C\theta(i)\theta(j)$, and $[W^{(i,j)}]^2 = W^2(i, j)(e_i e'_j + e_j e'_i)$. So $\sum_{1 \leq i < j \leq n} E[W^{(i,j)}]^2$ is a diagonal matrix, where the $i$-th diagonal is $\sum_{j: j \neq i} E[W^2(i, j)] \leq C\theta(i)\|\theta\|_1$. Therefore,

\[ \sigma^2 \equiv \| \sum_{1 \leq i < j \leq n} E[(W^{(i,j)})^2] \| \leq C\theta_{\max}\|\theta\|_1. \]

We apply the matrix Bernstein inequality [37, Theorem 6.2], and find that for any $t > 0$,

\[ P(\|W\| > t) \leq 2n \exp \left( -\frac{-t^2/2}{C\theta_{\max}\|\theta\|_1 + t/3} \right). \]

Let $t = C_0\sqrt{\log(n)\theta_{\max}\|\theta\|_1}$ for a constant $C_0 > 0$ to be determined. Since $\theta_{\max}\|\theta\|_1 \geq \|\theta\|^2 \gg \log(n)$ by (1.20), we have $t \ll \theta_{\max}\|\theta\|_1 \ll t^2$. Therefore, when $C_0$ is large enough, the probability on the right hand side is $o(n^{-3})$, i.e., with probability $1 - o(n^{-3})$,

(6.36) \[ \|W\| \leq C\sqrt{\log(n)\theta_{\max}\|\theta\|_1} \]

Combining (6.35)-(6.36) gives (6.34).

6.5. Proof of Lemma 3.3. Since $\Xi = \Theta P B$, $\xi_1(i) = \theta(i) \sum_{k=1}^{K} \pi_i(k) b_1(k)$. So for all $1 \leq i \leq n$, \[
\min_{1 \leq k \leq K} \{b_1(k)\} \leq \xi_1(i)/\theta(i) \leq \max_{1 \leq k \leq K} \{b_1(k)\}.
\]
To show the claim, it suffices to show that
\begin{equation}
(6.37) \quad C^{-1} \|\theta\|^{-1} \leq b_1(k) \leq C\|\theta\|^{-1}, \quad \text{for all } 1 \leq k \leq K.
\end{equation}

We now show (6.37). First, consider the upper bound. Write $\hat{G} = \|\theta\|^2 G = \Pi'\Theta^2\Pi$. Since $\Xi = \Theta\Pi B$, we have $B'\Pi'\Theta^2\Pi B = I_K$, or equivalently, $B'\hat{G}B = I_K$. Multiplying both sides by $B$ from the left and $B'$ from the right, we obtain $BB'\hat{G}BB' = BB'$. Since $BB'$ is non-singular, it implies
\begin{equation}
(6.38) \quad BB' = \hat{G}^{-1}.
\end{equation}
We note that $b_1^2(k)$ is upper bounded by the $k$-th diagonal of $BB'$. By (6.33) and (6.38),
\begin{equation}
(6.37) \quad b_1^2(k) \leq \lambda_{\max}(BB') \leq C\|\theta\|^{-2}.
\end{equation}
By Lemma 6.1, $b_1(k) > 0$. This gives the upper bound in (6.37).

Next, consider the lower bound. Write $b_1 = b_1^{(n)}$, $\theta = \theta^{(n)}$ and $G = G^{(n)}$ to emphasize the dependence on $n$. Suppose there is a subsequence $\{n_\ell\}_{\ell=1}^\infty$ such that $\lim_{\ell \to \infty} \min_{1 \leq k \leq K} \{b_1^{(n_\ell)}(k)\} \to 0$. Note that all the entries of $G^{(n_\ell)}$ are bounded. So there exists a subsequence of $\{n_\ell\}_{\ell=1}^\infty$, which we still denote by $\{n_\ell\}_{\ell=1}^\infty$ for notation convenience, such that $G^{(n_\ell)} \to G_0$ for a fixed matrix $G_0 \in \mathbb{R}^{K,K}$ and that $b_1^{(n_\ell)}(k) \to 0$ for some $1 \leq k \leq K$. First, it is easy to see that $G_0$ is a nonnegative symmetric matrix whose diagonals are positive. Using similar argument as that in (6.31), $PG_0$ is primitive. Second, by perturbation theory (e.g., Lemma 6.2) and the assumption (1.21), $b_1^{(n_\ell)}/\|b_1^{(n_\ell)}\|$, up to a factor of $\pm 1$, tends to the leading (unit-norm) eigenvector of $PG_0$. This implies that the $k$-th entry of the leading eigenvector of $PG_0$ is zero. However, since $PG_0$ is primitive, by Perron’s theorem [19], the entries of the leading eigenvector of $PG_0$ are either all positive or all negative. This yields a contradiction. So the lower bound in (6.37) holds.

6.6. Proof of Lemma 3.4. We need the sin-theta theorem [11]. Below we state a simpler version of it (adapted from [9, Theorem 10]).

**Lemma 6.2.** Let $G$ and $\hat{G}$ be two $p \times p$ symmetric matrices. For $1 \leq k \leq p$, let $\lambda_k$ be the $k$-th largest eigenvalue of $G$, $\xi_k$ and $\hat{\xi}_k$ be the eigenvector associated with the $k$-th largest eigenvalue of $G$ and $\hat{G}$, respectively. Suppose for some $\delta > 0$ and $1 \leq k_1 \leq k_2 \leq p$, we have $\lambda_{k_1-1} > \lambda_{k_1} + \delta$, $\lambda_{k_2+1} < \lambda_{k_2} - \delta$ and $\|\hat{G} - G\| \leq \delta/2$. Write $U = [\xi_{k_1}, \ldots, \xi_{k_2}]$ and $\hat{U} = [\hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_2}]$. Then, $\|U\hat{U}' - UU'\| \leq 2\delta^{-1}\|\hat{G} - G\|$. 


We first consider $\|\hat{\xi}_1 - \xi_1\|$. By Lemma 3.1, $\lambda_1 > 0$ and it is at least $C\|\theta\|^2$ larger than all the other eigenvalues of $\Omega$; moreover, by (6.34) and (1.20), $\|A - \Omega\| = o(\|\theta\|^2)$. So Lemma 6.2 yields that $\|\hat{\xi}_1 - \xi_1\| \leq C\|\theta\|^{-2}\|A - \Omega\|$. By elementary linear algebra, $\xi_1\xi'_1 - \xi_1\xi'_1$ has two nonzero eigenvalues $\pm (\xi'_1\xi_1)^{1/2} = \pm \frac{1}{\sqrt{2}} \|\xi_1 - \xi_1\|$. It follows that $\|\hat{\xi}_1 - \xi_1\| \leq C\sqrt{2}\|\theta\|^{-2}\|A - \Omega\|$. Combining it with (6.34), with probability $1 - o(n^{-3})$, \begin{equation} (6.39) \quad \|\hat{\xi}_1 - \xi_1\| \leq C\sqrt{\log(n)\theta_{\max}}\|\theta\|_1/\|\theta\|^4. \end{equation}

We then consider $\|\hat{\Xi}_0 - \Xi_0\|$. For any integer $m \geq 1$, let $\mathcal{O}_m$ be the set of all orthogonal matrices of dimension $m$. Introduce $S_+ = \{2 \leq k \leq K : \lambda_k > 0\}$ and $S_- = \{2 \leq k \leq K : \lambda_k < 0\}$. Define $\Xi^+_0$ and $\hat{\Xi}^+_0$ as the respective submatrices of $\Xi_0$ and $\hat{\Xi}_0$ by restricting to columns with indices in $S_+$, and define $\Xi^-_0$ and $\hat{\Xi}^-_0$ similarly. It is easy to see that $$\min_{H \in \mathcal{O}_{K-1}} \|\hat{\Xi} - \Xi\|_F \leq \min_{H \in \mathcal{O}_{S_+}} \|\hat{\Xi}^+_0 - \Xi^+_0\|_F + \min_{H \in \mathcal{O}_{S_-}} \|\hat{\Xi}^-_0 - \Xi^-_0\|_F.$$

We only consider the first term on the right hand side, and the second term is similar. By Lemma 3.1, $C\|\theta\|^2 \leq |\lambda_k| \leq \lambda_1 - C\|\theta\|^2$ for all $2 \leq k \leq K$, so there is a gap of at least $C\|\theta\|^2$ between the eigenvalues $\lambda_j : j \in S_+$ and the other eigenvalues. So Lemma 6.2 implies $\|\hat{\Xi}^+_0 (\hat{\Xi}^+_0)' - \Xi^+_0 (\Xi^+_0)'\| \leq C\|\theta\|^{-2}\|A - \Omega\|$. Since the rank of $\hat{\Xi}^+_0 (\hat{\Xi}^+_0)' - \Xi^+_0 (\Xi^+_0)'$ is no larger than $2K$, $$\|\hat{\Xi}^+_0 (\hat{\Xi}^+_0)' - \Xi^+_0 (\Xi^+_0)'\|_F \leq \sqrt{2K} C\|\theta\|^{-2}\|A - \Omega\|.$$

By [26, Lemma 2.4], there exists an orthogonal matrix $H$ such that $\|\hat{\Xi}^+_0 - \Xi^+_0\|_F \leq \|\hat{\Xi}^+_0 (\hat{\Xi}^+_0)' - \Xi^+_0 (\Xi^+_0)'\|$. We plug it into the above inequality and apply (6.34), it follows that with probability at least $1 - o(n^{-3})$, \begin{equation} (6.40) \quad \min_{H \in \mathcal{O}_{S_+}} \|\hat{\Xi}^+_0 - \Xi^+_0\|_F \leq C\sqrt{K\log(n)\theta_{\max}}\|\theta\|_1/\|\theta\|^4. \end{equation}

The claim then follows.

6.7. Proof of Lemma 3.5. The key of the proof is the following lemma, which we prove in Section 6.8.

Lemmma 6.3. Under conditions of Theorem 1.2, there is a constant $C \geq 1$ such that $C^{-1}\|\pi_i - \pi_j\| \leq \|r_i - r_j\| \leq C\|\pi_i - \pi_j\|$ for all $1 \leq i, j \leq n$.

We now consider the claims in the second bulletin. For $1 \leq \ell \leq L_0$, let $\tilde{w}_\ell = (\gamma_\ell \circ b_1)/\|\gamma_\ell \circ b_1\|_1$ and $m_\ell = V\tilde{w}_\ell$. Then, we can view $m_\ell$ as the $r_i$
associated with \( \pi_i = \gamma_{\ell} \). So the claims follow immediately from Lemma 6.3 and the assumptions (1.22)-(1.23).

We then consider the claims in the first bulletin. First, for any \( 1 \leq k \neq \ell \leq K \), we take a pure node \( i \) of community \( k \) and a pure node \( j \) of community \( \ell \). Then, \( \|v_k - v_{\ell}\| = \|r_i - r_j\| \geq C^{-1}\|\pi_i - \pi_j\| = C^{-1}\|e_k - e_{\ell}\| = C^{-1}\sqrt{2} \), where the inequality is due to Lemma 6.3. Second, write \( B' = [u_1, \cdots, u_K] \) such that \( u_k \) denotes the \( k \)-th row of \( B \). By definition, \( (1, v'_{\ell}) = b_{1}^{-1}(k)u_k \), so \( \|v_k\| \leq \|u_k\|/b_1(k) \). Here \( \|u_k\| \leq [\lambda_{\max}(BB')]^{1/2} \), and by (6.33) and (6.38) \( \lambda_{\max}(BB') \leq C\|\theta\|^{-2} \); moreover, \( b_1(k) \geq C\|\theta\|^{-1} \) by (6.37). It follows that \( \|v_k\| \leq C \). Third, to show that the volume of IS is non-diminishing, we define \( \vec{\pi} = (1/K, \cdots, 1/K)' \), \( \vec{w} = (b_1(1)/\|b_1(1)\|_1 \text{ and } \vec{v} = V \vec{\pi} \). Then \( \vec{v} \) is in the interior of IS. We aim to show that

\[
(6.41) \quad \|\vec{v} - u\| \geq C, \text{ for any point } u \text{ on the boundary of the IS.}
\]

Once (6.41) is true, the IS contains a ball centered at \( \vec{v} \) with a radius \( C \), so its volume is non-diminishing. We now show (6.41). By definition,\[ u = V\vec{w} \text{ for a weight vector } \vec{w} \text{ at least one coordinate of which is zero.} \]

Suppose \( \vec{w}(1) = 0 \) without loss of generality. Define \( \vec{\gamma} = R^K \) by \( \vec{\gamma}(k) = [\vec{w}(k)/b_1(k)/\sum_{\ell=1}^{K} \vec{w}(\ell)/b_1(\ell)] \). Then, we can view \( \vec{v} \) and \( u \) as the \( r_i \) associated with \( \pi_i = \vec{\pi} \) and \( \pi_i = \vec{\gamma} \), respectively. By Lemma 6.3, \( \|\vec{v} - u\| \geq C^{-1}\|\vec{\pi} - \vec{\gamma}\| \), where \( \|\vec{\pi} - \vec{\gamma}\| \geq \|\vec{\pi}(1) - \vec{\gamma}(1)\| = 1/K \). This proves (6.41).

6.8. Proof of Lemma 6.3. By (1.12), \( r_i = Vw_i \), where \( \sum_{k=1}^{K} w_i(k) = 1 \).

Let \( \vec{r}_i = [1, r'_i]' \), \( \vec{v}_k = [1, v'_k]' \) and \( \vec{V} = [\vec{r}_1, \cdots, \vec{r}_K] \). Then, \( \vec{r}_i = \vec{V}w_i \) and

\[
\|r_i - r_j\| = \|\vec{r}_i - \vec{r}_j\| = \|\vec{V}(w_i - w_j)\|, \quad \text{for any } 1 \leq i, j \leq n.
\]

By definition of \( v_k \), \( \vec{V} = B'[\text{diag}(b_1(1), \cdots, b_1(K))]^{-1} \). Combining this with (6.37) and (6.38), the eigenvalues of \( \vec{V}'\vec{V} \) are sandwiched by \( C^{-2}\lambda_{\max}^{-1}(G) \) and \( C^2\lambda_{\min}^{-1}(G) \). In (6.33), we have seen that \( \lambda_{\min}(G) \geq C \). Moreover, \( \lambda_{\max}(G) \leq \sum_{k, \ell=1}^{K} G(k, \ell) = \|\theta\|^{-2} \sum_{k, \ell} \sum_{i} \theta^2(i)\pi_i(k)\pi_i(\ell) = 1 \). Together, we have

\[
C^{-1} \leq \lambda_{\min}(\vec{V}'\vec{V}) \leq \lambda_{\max}(\vec{V}'\vec{V}) \leq C.
\]

Combining the above gives

\[
(6.42) \quad C^{-1}\|w_i - w_j\| \leq \|r_i - r_j\| \leq C\|w_i - w_j\|.
\]

Therefore, to show the claim, it suffices to show that

\[
(6.43) \quad C^{-1}\|\pi_i - \pi_j\| \leq \|w_i - w_j\| \leq C\|\pi_i - \pi_j\|.
\]
We now show (6.43). Note that \( w_i = (b_1 \circ \pi_i)/\|b_1 \circ \pi_i\|_1 \), where \( C^{-1}\|\theta\|^{-1} \leq b_1(k) \leq C\|\theta\|^{-1} \) for all \( k \) by (6.37). Define \( \tilde b_1 \in \mathbb{R}^K \) by \( \tilde b_1(k) = 1/b_1(k) \), \( 1 \leq k \leq K \). It follows that \( \pi_i = (\tilde b_1 \circ w_i)/\|\tilde b_1 \circ \pi_i\|_1 \), where \( C^{-1}\|\theta\|^{-1} \leq \tilde b_1(k) \leq C\|\theta\|^{-1} \) for all \( k \). Hence, as long as we have proved \( \|w_i - w_j\| \leq C\|\pi_i - \pi_j\| \), we can use exactly the same proof to obtain \( \|\pi_i - \pi_j\| \leq C\|w_i - w_j\| \), i.e., \( \|w_i - w_j\| \geq C^{-1}\|\pi_i - \pi_j\| \). It suffices to show the right inequality of (6.43).

Since \( w_i(k) = \pi_i(k) b_1(k)/\|\pi_i \circ b_1\|_1 \), we write
\[
w_i(k) - w_j(k) = \frac{\pi_i(k) - \pi_j(k)}{\|\pi_i \circ b_1\|_1} b_1(k) + \pi_j(k) b_1(k) \left[ \frac{1}{\|\pi_i \circ b_1\|_1} - \frac{1}{\|\pi_j \circ b_1\|_1} \right]
\]
First, since \( C^{-1}\|\theta\|^{-1} \leq b_1(k) \leq C\|\theta\|^{-1} \) and \( \pi_i \circ b_1 \geq C^{-1}\|\theta\|^{-1} \), we have \( \|b_1\| \leq C\|\theta\|^{-1} \) and \( \pi_i \circ b_1 \geq C^{-1}\|\theta\|^{-1} \). It follows that \( b_1(k)/\|\pi_i \circ b_1\|_1 \leq C \). Second, \( |w_j(k)| \leq 1 \).

Last, \( \|\pi_j \circ b_1 \| - \|\pi_i \circ b_1\|_1 \) is \( |\sum_{k=1}^{K} \pi_i(k) - \pi_j(k)| b_1(k) \leq \|b_1\| \|\pi_i - \pi_j\| \leq C\|\theta\|^{-1} \|\pi_i - \pi_j\| \), where we have used the Cauchy-Schwartz inequality and (6.37). Combining the above gives
\[
|w_i(k) - w_j(k)| \leq C|\pi_i(k) - \pi_j(k)| + C\|\pi_i - \pi_j\|.
\]
Using the inequality \((a + b)^2 \leq 2a^2 + 2b^2\) and summing over \( k \), we find that \( \|w_i - w_j\| \leq C\|\pi_i - \pi_j\|^2 \). This proves the right inequality of (6.43).

6.9. Proof of Lemma 3.6. Let \( g_n = \sqrt{\log(n)}\theta_{\max}\|\theta\|_1/\|\theta\|^2 \) and \( H \) be the orthogonal matrix in Lemma 3.4. Let \( u_i \) and \( \hat u_i \) be the \( i \)-th row of \( [\xi_1, \Xi_0 H] \) and \( \hat \Xi, \) respectively. Then, \((1, H r_i')' = [\xi_1(i)]^{-1} u_i \) and \((1, \hat r_i')' \) is a thresholded version of \( [\xi_1(i)]^{-1} \hat u_i \), for \( 1 \leq i \leq n \).

By Lemma 3.4, with probability \( 1 - o(n^{-3}) \), \( \sum_{i=1}^{n} \|\hat u_i - u_i\|^2 \leq C\|\theta\|^{-2} g_n^2 \). By Lemma 3.3 and that \( \|\xi\| = 1 \), we have \( \xi_1(i) \geq C\theta(i)/\|\theta\| \). Combining them gives
\[
\sum_{i=1}^{n} \frac{\|\hat u_i - u_i\|^2}{\xi_1^2(i)} \leq C g_n^2/\theta_{\min}^2 = C_0 g_n^2/\theta_{\min}^2 / \log(n),
\]
for a constant \( C_0 > 0 \). Let \( \mathcal{J} = \{1 \leq i \leq n : \|\hat u_i - u_i\| \leq \xi_1(i)/2\} \). It follows from (6.44) that
\[
|\mathcal{J}| \leq 4C_0 n \cdot err_n^2 / \log(n).
\]
For \( i \in \mathcal{J} \), \( \|\hat r_i - H r_i\| \leq \|\hat r_i\| + \|H r_i\| \), where \( \|\hat r_i\| \leq C\sqrt{\log(n)} \) due to the definition (1.14), and \( \|H r_i\| = \|r_i\| \leq C \) by Lemma 3.5. Together, we have
\[
\sum_{i \in \mathcal{J}} \|\hat r_i - H r_i\|^2 \leq |\mathcal{J}| \cdot C \log(n) \leq C n \cdot err_n^2.
\]
We then consider $i \in \mathcal{J}$. Define $\hat{r}_i^*(k) = \hat{\xi}_{k+1}(i)/\hat{\xi}_1(i)$, $1 \leq k \leq K - 1$. Then, $\hat{r}_i(k) = \hat{r}_i^*(k) \cdot 1\{|\hat{r}_i^*(k)| \leq \sqrt{\log(n)}\}$. So

$$
\|\hat{r}_i^* - Hr_i\| = \|\hat{\xi}_1(i)^{-1}\hat{u}_i - [\xi_1(i)]^{-1}u_i\|.
$$

For $i \in \mathcal{J}$, using the triangle inequality, $\|\hat{r}_i^* - Hr_i\| \leq [\hat{\xi}_1(i)]^{-1}\|\hat{u}_i - u_i\| + [\xi_1(i)]^{-1}\|\hat{u}_i - \hat{\xi}_1(i)\| - [\xi_1(i)]^{-1}\|\hat{\xi}_1(i) - \xi_1(i)\|$. Since $|\hat{\xi}_1(i) - \xi_1(i)| \leq \|\hat{u}_i - u_i\| \leq \xi_1(i)/2$, we have that $[\xi_1(i)]^{-1} \leq 2[\xi_1(i)]^{-1}$. Moreover, $[\xi_1(i)]^{-1}\|u_i\| \leq 1 + \|Hr_i\| \leq C$ due to Lemma 3.5. Combining the above gives

$$
(6.46) \quad \|\hat{r}_i^* - Hr_i\| \leq C[\xi_1(i)]^{-1}\|\hat{u}_i - u_i\|, \quad \text{for } i \in \mathcal{J}.
$$

By (6.46) and the definition of $\mathcal{J}$, $\|\hat{r}_i^* - Hr_i\| \leq C$ for $i \in \mathcal{J}$. It follows that $\|\hat{r}_i^*\| \leq C$ for $i \in \mathcal{J}$. As a result,

$$
(6.47) \quad \hat{r}_i = \hat{r}_i^*, \quad \text{for } i \in \mathcal{J}.
$$

Combining (6.46)-(6.47), we find that

$$
(6.48) \quad \sum_{i \in \mathcal{J}} \|\hat{r}_i - Hr_i\|^2 = \sum_{i \in \mathcal{J}} \|\hat{r}_i^* - Hr_i\|^2 \leq \sum_{i \in \mathcal{J}} \frac{\|\hat{u}_i - u_i\|^2}{[\hat{\xi}_1(i)]} \leq Cn \cdot \text{err}_n^2,
$$

where the last inequality is because of (6.44). The claim follows from (6.45) and (6.48).

6.10. **Proof of Lemma 3.7.** Let $H$ be the same as that in Lemma 3.6, and $m_1, \ldots, m_{L_0}$ be the same as those in Lemma 3.5. Since $H$ is an orthogonal matrix, the claims of Lemma 3.5 still hold if we replace all the $(r_i, v_k, m_\ell)$ by $(Hr_i, Hv_k, Hm_\ell)$. For notation simplicity, we still use $(r_i, v_k, m_\ell)$ below, but note that they actually mean $(Hr_i, Hv_k, Hm_\ell)$.

Write $\alpha_n^2 = n^{-1}\sum_{j=1}^{L_0} \sum_{i \in \mathcal{M}_j} \|r_i - m_j\|^2$. We introduce the notation $m_{J_i}$ for all $1 \leq i \leq n$. For pure nodes, let $m_{J_i} = v_k$ if $i \in \mathcal{N}_k$; for mixed nodes, let $m_{J_i} = m_\ell$ if $i \in \mathcal{M}_\ell$. By Lemmas 3.5-3.6, $\alpha_n \leq C/\log^2(n)$ and

$$
(6.49) \quad \sum_{i=1}^n \|\hat{r}_i - r_i\|^2 \leq Cn \cdot \text{err}_n^2, \quad \sum_{i=1}^n \|r_i - m_{J_i}\|^2 = n\alpha_n^2.
$$

We then introduce the notation $\hat{m}_{J_i} = \hat{m}_{J_i}^{(L)}$ for all $1 \leq i \leq n$ and $L \geq 1$. Let $\hat{m}_1, \ldots, \hat{m}_L$ be the cluster centers of $k$-means (assuming $\leq L$ clusters). For a node $i$, let $J_i$ be the unique $j$ such that $\hat{r}_i$ belongs to the cluster associated with $\hat{m}_j$ (so that $\hat{m}_{J_i} = \hat{m}_j$). We often omit the superscript $L$ when there is no confusion. Then, $\epsilon_L(\hat{R}) = \sum_{i=1}^n \|\hat{r}_i - \hat{m}_{J_i}\|^2$. 


First, consider \( L = L_0 + K \). In this case, \( \{m_{j_i}, 1 \leq i \leq n\} \) contains \( \leq L \) distinct points. Therefore,

\[
\epsilon_{L_0 + K}(\hat{R}) = \sum_{i=1}^{n} \left\| \hat{r}_i - \hat{m}_{j_i} \right\|^2 \leq \sum_{i=1}^{n} \left\| \hat{r}_i - m_{J_i} \right\|^2 
\]

(6.50)

\[
\leq 2 \sum_{i=1}^{n} \left\| r_i - m_{J_i} \right\|^2 + 2 \sum_{i=1}^{n} \left\| r_i - r_i \right\|^2 \leq Cn(\alpha_n^2 + \text{err}_n^2).
\]

Here, the first inequality comes from the definition of \( k \)-means, the second inequality is because of the triangle inequality and that \((a + b)^2 \leq 2a^2 + 2b^2\), and the last inequality follows from (6.49). Furthermore, by Lemma 6.3, \( n\alpha_n^2 = \sum_{i=1}^{L_0} \sum_{i \in M_\ell} \| r_i - m_\ell \|_2^2 \leq C \sum_{i=1}^{L_0} \sum_{i \in M_\ell} \| \pi_i - \gamma_\ell \|_2^2 \). Combining it with (6.50) gives the claim for \( L = L_0 + K \).

Second, consider \( K \leq L < L_0 + K \). By Lemma 3.5, for a constant \( C_0 > 0 \), \( \min_{j \neq s} \| m_j - m_s \| \geq C_0 \), \( \min_{j \neq k} \| m_j - v_k \| \geq C_0 \) and \( \min_{k \neq \ell} \| v_k - v_\ell \| \geq C_0 \).

Since \( L < L_0 + K \), there are at least \( (L_0 + K - L) \) number of \( \ell \) such that no cluster center is located within a distance of \( C_0/2 \) to \( m_\ell \). If \( i \in M_\ell \) is such that \( \| \hat{r}_i - r_i \| \leq C_0/4 \), then its distance to the closest \( k \)-means center is at least \( C_0/2 - \| \hat{r}_i - r_i \| - \| \hat{r}_i - m_\ell \| \geq C_0/2 - C_0/4 - C/\log(n) \geq C_0/4 \).

Furthermore, due to the first inequality in (6.49), the number of \( i \) such that \( \| \hat{r}_i - r_i \| > C_0/4 \) is at most \( Cn \cdot \text{err}_n^2 \); so the number of \( i \in M_\ell \) such that \( \| \hat{r}_i - r_i \| \leq C_0/4 \) is at least \( |M_\ell| - Cn \cdot \text{err}_n^2 \geq c_0|M| - Cn \cdot \text{err}_n^2 \geq c_0|M| \), where we have used (1.23). As a result,

\[
\epsilon_L(\hat{R}) \gtrsim (L_0 + K - L) \cdot c_6|M| \cdot (C_0/4)^2 \geq C(L_0 + K - L)|M|.
\]

Last, consider \( K < L \). There are at least \( (K - L) \) number of \( k \) such that no cluster center is located within a distance \( C_0/2 \) to \( v_k \). Using a similar argument as above,

\[
\epsilon_L(\hat{R}) \gtrsim (K - L) \cdot c_1 n \cdot (C_0/4)^2 \geq C(K - L)n.
\]

Once the claims for \( \epsilon_L(\hat{R}) \) hold, it follows immediately that \( \hat{L}_n(A) = L_0 + K \).

6.11. Proof of Lemma 3.8. We use the same notations \( \{m_{j_i}, \hat{m}_{j_i}, 1 \leq i \leq n\} \) as in the proof of Lemma 3.7. By Lemma 3.5, there is a constant \( C_0 > 0 \) such that the distance between any two points of \( \{v_1, \cdots, v_K, m_1, \cdots, m_{L_0}\} \) is at least \( C_0 \). We are going to show the key argument:

- With probability \( 1 - o(n^{-3}) \), there exist \( L = L_0 + K \) non-overlapping balls each containing one \( \hat{m}_{j_i} \), where \( K \) of them center at \( v_1, \cdots, v_K \) with radius \( \leq C\text{err}_n \) and \( L_0 \) of them center at \( m_1, \cdots, m_{L_0} \) with radius \( < C_0/6 \).
Given that the above is true, we now prove the claims. First, for each \( 1 \leq k \leq K \), there is a unique \( \hat{j}_k \) such that \( \hat{m}_{\hat{j}_k} \) is contained in the ball centering at \( v_k \). Let \( \hat{v}_k = \hat{m}_{\hat{j}_k} \). Then \( \max_{1 \leq k \leq K} \| \hat{v}_k - v_k \| \leq \text{Cerr}_n \). Second, note that \( v_1, \ldots, v_K \) form a \((K - 1)\)-simplex with a non-diminishing volume (by Lemma 3.5), and each \( \hat{v}_k \) is within a distance \( \text{Cerr}_n = o(1) \) to \( v_k \). It is easy to see that \( \hat{v}_1, \ldots, \hat{v}_K \) form a non-degenerate \((K - 1)\)-simplex \( S(\hat{v}_1, \ldots, \hat{v}_K) \).

Third, we show that all the other \( \hat{m}_j \) fall within \( S(\hat{v}_1, \ldots, \hat{v}_K) \). Note that the \( L_0 \) balls centering at \( m_1, \ldots, m_{L_0} \) are contained in the Ideal Simplex (IS) \( S(v_1, \ldots, v_K) \), and are at least a distance of \( C_0 - C_0/6 \) to any vertex of IS. So they are at least a distance \( C' \) to the boundary of the IS.\(^{15}\) From IS to \( S(\hat{v}_1, \ldots, \hat{v}_K) \), the boundary shifts at most \( \text{Cerr}_n \). Hence, those \( L_0 \) balls are also contained in \( S(\hat{v}_1, \ldots, \hat{v}_K) \), so are the \( \hat{m}_j \) in them. Last, we show that minimizing (1.16) indeed gives \((\hat{m}_1, \ldots, \hat{m}_K) = 0\). Note that \( d_L(\hat{j}_1, \ldots, \hat{j}_K) = 0 \).

Also, each of the \( L_0 \) balls are at least a distance of \( C' - \text{Cerr}_n \geq C'/2 \) to the boundary of \( S(\hat{v}_1, \ldots, \hat{v}_K) \). Therefore, for any other choice of \( \{j_1, \ldots, j_K\} \), \( d_L(j_1, \ldots, j_K) \geq C'/2 \).

It remains to show the key argument. It suffices to show that, in each of the \( L = L_0 + K \) balls, there is at least one \( \hat{m}_j \). Since there are \( \leq L \) cluster centers in total, each ball contains one and only one \( \hat{m}_j \).

We first show that for each \( 1 \leq \ell \leq L_0 \), there is at least one \( \hat{m}_j \) within a distance \( C_0/6 \) to \( m_\ell \). Note that \( \{m_{\ell_1}, \ldots, m_{\ell_{L_0}}\} \) contains at most \( L \) distinct points, so the definition of \( k \)-means yields that \( \sum_{i=1}^{n} \| \hat{r}_i - \hat{m}_{\hat{j}_i} \|^2 \leq \sum_{i=1}^{n} \| \hat{r}_i - m_{\ell_i} \|^2 \).

Using this fact and the inequality \((a + b)^2 \leq 2a^2 + 2b^2\), we have

\[
\sum_{i=1}^{n} \| r_i - \hat{m}_{\hat{j}_i} \|^2 \leq 2 \sum_{i=1}^{n} \| \hat{r}_i - \hat{m}_{\hat{j}_i} \|^2 + 2 \sum_{i=1}^{n} \| \hat{r}_i - r_i \|^2 \\
\leq 2 \sum_{i=1}^{n} \| \hat{r}_i - m_{\ell_i} \|^2 + 2 \sum_{i=1}^{n} \| \hat{r}_i - r_i \|^2 \\
\leq 4 \sum_{i=1}^{n} \| r_i - m_{\ell_i} \|^2 + 6 \sum_{i=1}^{n} \| \hat{r}_i - r_i \|^2 \\
\leq C n \alpha_n^2 + C n \cdot \text{err}_n^2,
\]

(6.51)

where the last inequality follows from (6.49). Suppose there exists an \( \ell \) such that, within a distance of \( C_0/6 \) to \( m_\ell \), there is no cluster center of \( k \)-means. First, for \( i \in M_\ell \), since \( \| r_i - m_\ell \| = o(1) \), \( r_i \) is at least a distance \( \geq C_0/6 \) to the closest cluster center, so \( \| r_i - \hat{m}_{\hat{j}_i} \| \geq C_0/6 \). Second, \( |M_\ell| \geq c_0 |M| \).

\(^{15}\)The \( C' \) here depends on \( C_0 \) and the shape of the Ideal Simplex.
Combining the above, we have

$$\sum_{i \in M} ||r_i - \hat{m}_j||^2 \geq c_6 |M| \cdot (C_0/6)^2 \geq C |M|.$$ 

By (1.23), $|M|/(n \cdot err_n^2) \geq \log(n)$. Also, $n \alpha_n^2 \leq |M| \cdot \max_{1 \leq \ell \leq L_0, i \in M} ||r_i - m_\ell||^2 \leq |M| \cdot C \max_{1 \leq \ell \leq L_0, i \in M} ||\pi_i - \gamma\ell||^2$ by Lemma 6.3. It then follows from (1.23) that $n \alpha_n^2 \leq C |M|/\log^2(n)$. So the right hand side above is at least $C n \log(n)(err_n^2 + \alpha_n^2)$. This yields a contradiction to (6.51).

We then show that, for each $1 \leq k \leq K$, there is at least one $\hat{m}_j$ within a distance $C_{err}$ to $v_k$. We shall consider two cases separately: (A1) $\alpha_n \leq C_{err}$, and (A2) $\alpha_n/err_n \rightarrow \infty$.

Consider Case (A1). Now, (6.51) reduces to $\sum_{i=1}^n ||r_i - \hat{m}_j||^2 \leq C_1 n \cdot err_n^2$ for a constant $C_1 > 0$. Let $C_2 = \sqrt{2C_1/e_1}$ where $e_1$ is the same as in (1.19). Suppose there is at least one $k$ such that, within a distance of $C_2 err_n$ to $v_k$, there is no cluster center of $k$-means. Recall that $N_k$ is the set of pure nodes of community $k$. Since $|N_k| \geq c_1 n$ and $r_i = v_k$ for each $i \in N_k$, we have $\sum_{i=1}^n ||r_i - \hat{m}_j||^2 \geq \sum_{i \in N_k} ||r_i - \hat{m}_j||^2 \geq (c_1 n) \cdot (C_2^2 err_n^2) = 2C_1 n \cdot err_n^2$. This yields a contradiction.

Consider Case (A2). Now, (6.51) reduces to

$$\sum_{i=1}^n ||r_i - \hat{m}_j||^2 \leq C_3 n \alpha_n^2,$$

for a constant $C_3 > 0$. Since $\alpha_n/err_n \rightarrow \infty$, using a similar proof to that in Case (A2), we are not able to get a contradiction. However, noting that $\alpha_n = o(1)$, by constructing a contradiction to (6.52), we can prove a weaker result: for each $1 \leq k \leq K$, there is at least one $\hat{m}_j$ within a distance $C_0/6$ to $v_k$. Below, we show that each of these $\hat{m}_j$ is indeed within a distance $C_{err}$ to the corresponding $v_k$.

So far, we have shown that, there are $L$ non-overlapping balls centering at $v_1, \cdots, v_K, m_1, \cdots, m_{L_0}$ with radius $C_0/6$, where each contains one and only one $\hat{m}_j$. Fix $1 \leq k \leq K$, and let $\hat{m}_j$ be the unique cluster center in the ball centering at $v_k$. Denote by $\hat{V}$ the set of nodes that are clustered to $\hat{m}_j$ in $k$-means. For a large constant $C_4 > 0$ to be determined, we aim to show that,

$$\sum_{i \in \hat{V}} ||\hat{r}_i - v_k||^2 < \sum_{i \in \hat{V}} ||\hat{r}_i - \hat{m}_j||^2, \quad \text{if } ||\hat{m}_j - v_k|| > 2C_4 err_n.$$

In other words, whenever $||\hat{m}_j - v_k|| > 2C_4 err_n$, we can keep the current cluster assignment but alter $\hat{m}_j$ to $v_k$ to strictly decrease the $k$-means objective. This yields a contradiction. It follows that $||\hat{m}_j - v_k|| \leq 2C_4 err_n$. 
Below, we show (6.53). There are four types of nodes in $\hat{V}$:

- $\hat{V}_1$: $i$ is not a pure node of community $k$, and $\|\hat{r}_i - r_i\| \leq C_0/6$.
- $\hat{V}_2$: $i$ is not a pure node of community $k$, and $\|\hat{r}_i - r_i\| > C_0/6$.
- $\hat{V}_3$: $i$ is a pure node of community $k$, and $\|\hat{r}_i - r_i\| \leq C_4\text{err}_n$.
- $\hat{V}_4$: $i$ is a pure node of community $k$, and $\|\hat{r}_i - r_i\| > C_4\text{err}_n$.

Since $\sum_{i=1}^n \|\hat{r}_i - r_i\|^2 \leq Cn \cdot \text{err}_n^2$, we find that

$$\sum_{i=1}^n \|\hat{r}_i - r_i\|^2 \leq Cn \cdot \text{err}_n^2$$

Also, we claim that

$$|\hat{V}_3 \cup \hat{V}_4| \gtrsim c_1 n.$$  

By the assumption (1.19), the number of pure nodes in community $k$ is at least $c_1 n$. It suffices to show that $|\hat{V}_k \setminus (\hat{V}_3 \cup \hat{V}_4)| = o(n)$. If $i$ is a pure node of community $k$ but $i$ is not assigned to $\hat{m}_j$ in $k$-means, where $\hat{m}_j$ is the only cluster center within a distance $C_0/6$ to $v_k$. Then, $\|r_i - \hat{m}_j\| > C_0/6$.

By (6.52), the number of such nodes is bounded by $(36C_3/C_0^2)n\alpha_n^2 = o(n)$.

We now look at the left hand side of (6.53). From (6.55) and that $r_i = v_k$ for $i \in \hat{V}_3 \cup \hat{V}_4$, we find $\sum_{i \in \hat{V}} \|r_i - v_k\|^2 = \sum_{i \in \hat{V}_2} \|r_i - v_k\|^2$. It follows that

$$\sum_{i \in \hat{V}} \|\hat{r}_i - v_k\|^2 \leq 2 \sum_{i \in \hat{V}} \|r_i - v_k\|^2 + 2 \sum_{i=1}^n \|\hat{r}_i - r_i\|^2 \leq 2 \sum_{i \in \hat{V}_2} \|r_i - v_k\|^2 + 2Cn \cdot \text{err}_n^2$$

$$(6.57) \leq 2 \cdot 2C^2 \cdot (36C_3/C_0)n \cdot \text{err}_n^2 + 2Cn \cdot \text{err}_n^2,$$
where the second inequality is due to (6.49), and the last inequality comes from (6.54) and Lemma 3.5, which says that for a constant $\tilde{C} > 0$, $\|r_i\| \leq \tilde{C}$ for all $1 \leq i \leq n$. We then look at the right hand side of (6.53). By (6.54) and (6.56), $|\hat{V}_3| \geq (c_1 - C/C_4^2)n$. Moreover, for $i \in \hat{V}_3$, $r_i = v_k$, $\|v_k - \hat{m}_j\| > 2C_4\text{err}_n$ and $\|\hat{r}_i - r_i\| \leq C_4\text{err}_n$. It follows that $\|\hat{r}_i - \hat{m}_j\| > C_4\text{err}_n$. Hence,

$$\sum_{i \in \mathcal{V}} \|\hat{r}_i - \hat{m}_j\|^2 \geq \sum_{i \in \hat{V}_3} \|\hat{r}_i - \hat{m}_j\|^2 \geq (c_1 - C/C_4^2)n \cdot C_4^2\text{err}_n^2.$$  

Compare (6.57) and (6.58). By choosing $C_4$ large enough, we can make (6.57) to be strictly smaller than (6.58). This proves (6.53).

6.12. Vertices Hunting in the setting of Theorem 1.3. The following lemma is a counter part of Lemma 3.8.

**Lemma 6.4.** Suppose the conditions of Theorem 1.3 hold, and we apply Vertices Hunting to $\hat{R}$ with $L = K$. With probability $1 - o(n^{-3})$, the local clustering step identifies $K$ cluster centers (denoted by $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_K$), where there is a unique $(K - 1)$-simplex such that these centers are its vertices, and there is a permutation $\kappa$ in $\{1, \cdots, K\}$ such that $\max_{1 \leq k \leq K} \|\hat{v}_{\kappa(k)} - v_k\| \leq C\text{err}_n$.

**Proof of Lemma 6.4:** It suffices to show that, for each $1 \leq k \leq K$, there is one $\hat{m}_j$ within a distance $C\text{err}_n$ to $v_k$.

For $1 \leq i \leq n$, let $m_{J_i} = v_k$ if $i \in \mathcal{N}_k$, $1 \leq k \leq K$, and let $m_{J_i} = v_1$ if $i \in \mathcal{M}$. Then, $\sum_{i=1}^n \|r_i - m_{J_i}\|^2 = \sum_{i \in \mathcal{M}} \|r_i - v_1\|^2 \leq \tilde{C}|\mathcal{M}|$, because $\max_{1 \leq i \leq n} \|r_i\| \leq \tilde{C}$ by Lemma 3.5. Since $|\mathcal{M}| \leq Cn \cdot \text{err}_n^2$ in this setting,

$$\sum_{i=1}^n \|r_i - m_{J_i}\|^2 \leq Cn \cdot \text{err}_n^2.$$  

Let $\hat{m}_{J_i}$ be the same as those in Lemma 3.7. Similar to (6.51), we can prove that, for a constant $C_1 > 0$, with probability $1 - o(n^{-3})$,

$$\sum_{i=1}^n \|r_i - \hat{m}_{J_i}\|^2 \leq C_1n \cdot \text{err}_n^2.$$  

Let $C_2 = \sqrt{2C_1/c_1}$ where $c_1$ is the same as that in (1.19). If there exists a $k$ such that no cluster center is located within a distance $C_2\text{err}_n$ to $v_k$, then $\sum_{i=1}^n \|r_i - \hat{m}_{J_i}\|^2 \geq \sum_{i \in \mathcal{N}_k} \|r_i - \hat{m}_{J_i}\|^2 = \sum_{i \in \mathcal{N}_k} \|v_k - \hat{m}_{J_i}\|^2 \geq (c_1n) \cdot (C_2\text{err}_n)^2 = 2C_1n \cdot \text{err}_n^2$. This yields a contradiction.
REFERENCES


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