10.1 Properties of Brownian Motion

10.2 Hitting Time, Maximum Value, Reflection Principle

10.1 Brownian Motion as a Limit of Random Walk

The Brownian motion is in fact a limit of rescaled generalized random walk.

Let \( X_1, X_2, \ldots \) be i.i.d. random variables, \( \mathbb{E}[X_i] = 0, \Var(X_i) = \sigma^2 \).

Define

\[
X(t) = \Delta x (X_1 + \ldots + X\lfloor t/\Delta t \rfloor)
\]

where \( \lfloor t/\Delta t \rfloor \) is the integer part of \( t/\Delta t \). We’d like to find the limit of \( X(t) \) as \( \Delta t \) and \( \Delta x \) both \( \to 0 \).

Observe

\[
\mathbb{E}[X(t)] = 0, \quad \Var(X(t)) = \sigma^2 (\Delta x)^2 \left\lfloor \frac{t}{\Delta t} \right\rfloor,
\]

To have a non-trivial limit, as \( \Delta t \) and \( \Delta x \to 0 \) we must maintain

\( \Delta t = c (\Delta x)^2 \).

Let’s take \( c = 1 \). In this case, as \( \Delta t \to 0, \Delta x \to 0, \) and \( \Delta t = (\Delta x)^2 \), we have

\[
\mathbb{E}[X(t)] = 0, \quad \Var(X(t)) \to \sigma^2 t,
\]

Moreover, since \( \Delta x = \sqrt{\Delta t} \), by CLT

\[
X(t) = \Delta x (X_1 + \ldots + X\lfloor t/\Delta t \rfloor) \approx \sqrt{t} \sigma \frac{X_1 + \ldots + X\lfloor t/\Delta t \rfloor}{\sqrt{\Delta t} \sigma} \to N(0, \sigma^2 t)
\]

in distribution.

Observe that the discrete-time process

\[
\{X(t), \ t = n\Delta t, \ n = 0, 1, 2 \ldots \}
\]

has independent and stationary increments since

\[
X(s) = \Delta x (X_1 + \ldots + X\lfloor s/\Delta t \rfloor), \quad \text{and}
\]

\[
X(t) - X(s) = \Delta x (X\lfloor \frac{t}{\Delta t} \rfloor + 1 + \ldots + X\lfloor \frac{s}{\Delta t} \rfloor)
\]

are independent, and for \( t = l\Delta t > s = m\Delta t \), the distribution of \( X(t) - X(s) \) depends on the number of terms \( \lfloor \frac{t}{\Delta t} \rfloor - \lfloor \frac{s}{\Delta t} \rfloor = (l - m) = (t - s)/\Delta t \) in the sum, but not \( s \).

Thus the limit of \( X(t) \) is a process with independent and stationary increments.
Conditional Distribution

Given $B(t) = x$, what is the conditional distribution of $B(s)$?

If $t < s$, since Brownian motion has independent increments, $B(s) - B(t)$ is independent of $B(t)$, and hence given $B(t) = x$, the condition distribution of $B(s) - B(t)$ is the same as its unconditional distribution.

\[
(B(s)|B(t) = x) = B(t) + \left[ B(s) - B(t) \right] = \sim N(0, \sigma^2(s-t)) = N(x, \sigma^2(s-t)).
\]

Hitting Times (First Passage Times)

Let $T_a = \min\{t : B(t) = a\}$ be the first time the standard Brownian motion process hits $a$. For $a > 0$, consider

\[
P(B(t) \geq a) = P(B(t) \geq a | T_a \leq t)P(T_a \leq t) + P(B(t) \geq a | T_a > t)P(T_a > t)
\]

The 2nd term on the right is clearly 0, since by continuity, the process value cannot be $> a$ without having yet hit $a$.

For the 1st term, note if $T_a \leq t$, then the process hits $a$ at some point in $[0, t]$ and, by symmetry, it is just as likely to be above $a$ or below $a$ at time $t$. That is

\[
P(B(t) \geq a | T_a \leq t) = \frac{1}{2}
\]

Thus $P(T_a \leq t) = 2P(B(t) \geq a) = 2 - 2\Phi(a/\sqrt{t})$, where $\Phi(x)$ is the CDF of $N(0,1)$.

**HW:** Show that $P(T_a < \infty) = 1$ and $\mathbb{E}[T_a] = \infty$ for $a > 0$.

What if $s < t$?

If we can find a scalar $c$ such that $\text{Cov}(B(s) - cB(t), B(t)) = 0$, then $B(s) - cB(t)$ and $B(t)$ are independent.

Thus the conditional distribution of of $B(s) - cB(t)$ given $B(t)$ is the same as its unconditional distribution $N(0, \sigma^2(s - 2cs + c^2 t))$.

Given $B(t) = x$,

\[
B(s) = cB(t) + \underbrace{B(s) - cB(t)}_{\sim N(0, \sigma^2(s - 2cs + c^2 t))} \sim N(cx, \sigma^2(s - 2cs + c^2 t)).
\]

Because

\[
\text{Cov}(B(s) - cB(t), B(t)) = \text{Cov}(B(s), B(t)) - \text{Cov}(cB(t), B(t)) = \sigma^2 s - c\sigma^2 t = \sigma^2(s - ct)
\]

we know $c = s/t$. Thus the conditional distribution of $B(s)$ given $B(t) = x$ for $s < t$ is

\[
N\left(\frac{sx}{t}, \sigma^2 \frac{s(t-s)}{t}\right).
\]

Maximum

Another random variable of interest is

\[
\max_{0 \leq s \leq t} B(s).
\]

By continuity, we know

\[
\max_{0 \leq s \leq t} B(s) \geq a \iff T_a \leq t
\]

Thus the distribution of for $\max_{0 \leq s \leq t} B(s)$ can be derived via $T_a$.

For $a > 0$

\[
P\left(\max_{0 \leq s \leq t} B(s) \geq a\right) = P(T_a \leq t) = 2P(B(t) \geq a) = P(|B(t)| \geq a) = 2 - 2\Phi(a/\sqrt{t})
\]

Note this means $\max_{0 \leq s \leq t} B(s)$ have the same distribution as $|B(t)|$. 
Stopping Time & Strong Markov Property

For a continuous time stochastic process \( \{X(t), t \geq 0\} \), a stopping time \( T \) with respect to \( \{X(t), t \geq 0\} \) is a nonnegative random variable, such that the event \( \{T \leq t\} \) depends only on \( \{X(s), 0 \leq s \leq t\} \).

**Example**

The hitting time \( T_a = \min\{t : B(t) = a\} \) is a stopping time since the event \( \{T_a \leq t\} \) is identical to the event \( \{\max_{0 \leq s \leq t} B(s) \geq a\} \).

**Theorem (Strong Markov Property)**

Let \( \{B(t), t \geq 0\} \) be a standard Brownian Motion, and let \( T \) be a stopping time respective to \( \{B(t), t \geq 0\} \). Then \( \{Z(t), t \geq 0\} \) is also a standard Brownian Motion.

(a) Define \( Z(t) = B(t + T) - B(T), t \geq 0 \).

Then \( \{Z(t), t \geq 0\} \) is also a standard Brownian Motion.

(b) For each \( t > 0 \), \( \{Z(s), 0 \leq s \leq t\} \) is independent of \( \{B(u), 0 \leq u \leq T\} \).

Reflection Principle

Let \( T_a \) be the first passage time to the value \( a \) of a standard Brownian Motion \( \{B(t), t \geq 0\} \). Define a new process

\[
\mathcal{B}(t) = \begin{cases} 
B(t) & \text{for } t \leq T_a \\
2a - B(t) & \text{for } t > T_a
\end{cases}
\]

Then \( \{\mathcal{B}(t), t \geq 0\} \) is also a standard Brownian Motion.

**Reason:** For \( t > T_a \), note

\[
B(t) = a + B(t) - a = B(T_a) + B(t) - B(T_a).
\]

By Strong Markov Property,

\[
B(s + T_a) - B(T_a) = B(s + T_a) - a \text{ is also a Brownian Motion, independent of } \{B(s), 0 \leq s \leq T_a\}.
\]

Also note that if \( \{B(t), t \geq 0\} \) is a standard Brownian motion, so is \( \{-B(t), t \geq 0\} \). Hence \( \{a - B(s + T_a), s \geq 0\} \) is also a Brownian Motion.

So \( \{B(t), t > T_a\} = \{a + B(t) - a, t > T_a\} \)

~ \( \{a - B(t), t > T_a\} \) for \( a > 0 \).

Brownian Motion Absorbed at a Value

Let \( \{B(t)\} \) be a Brownian Motion.

For \( a > 0 \), a Brownian Motion absorbed at a value \( a \) is defined as

\[
B_a(t) = \begin{cases} 
B(t) & \text{if } \max_{0 \leq s \leq t} B(s) < a \\
a & \text{if } \max_{0 \leq s \leq t} B(s) \geq a
\end{cases}
\]

What is the distribution of \( B_a(t) \)? For \( x < a \),

\[
P(B_a(t) \leq x) = P(B(t) \leq x, \max_{0 \leq s \leq t} B(s) < a) = P(B(t) \leq x) - P(B(t) \leq x, \max_{0 \leq s \leq t} B(s) \geq a) = P(B(t) \leq x) - P(B(t) \leq x, T_a \leq t)
\]

where the last equality comes from the fact

\[
\{\max_{0 \leq s \leq t} B(s) \geq a\} \iff \{T_a \leq t\}.
\]

In summary, the CDF of \( B_a(t) \) is

\[
P(B_a(t) \leq x) = \Phi\left(\frac{x}{\sqrt{t}}\right) - 1 + \Phi\left(\frac{2a - x}{\sqrt{t}}\right)
\]

since \( x \leq a, B(t) \geq 2a - x \) implies \( T_a \leq t \).
More on Reflection Principle

Let \( \{B(t), t \geq 0\} \) be a standard Brownian motion. Let’s try to find the joint distribution of

\[ W(t) = \max_{0 \leq s \leq t} B(s) \quad \text{and} \quad Y(t) = W(t) - B(t) \]

First consider \( P(W(t) \geq w, B(t) \leq x) \). By Reflection Principle,

\[ P(W(t) \geq w, B(t) \leq x) = P(B(t) \geq 2w - x) = 1 - \Phi \left( \frac{2w - x}{\sqrt{t}} \right) \]

Thus the joint density of \( W(t) \) and \( B(t) \) is

\[ f(w, x) = \frac{2w - x}{t^{3/2}} \phi \left( \frac{2w - x}{\sqrt{t}} \right) \]

\[ = \left[ 1 - \Phi \left( \frac{2w - x}{\sqrt{t}} \right) \right] \frac{d}{dw} \phi \left( \frac{2w - x}{\sqrt{t}} \right) \]

\[ = \sqrt{\frac{2}{\pi t^3}} (2w - x) \exp \left( -\frac{(2w - x)^2}{2t} \right), \ w \geq 0, \ x \leq w \]

Thus the joint density of \( W(t) \) and \( B(t) \) is

\[ f(w, x) = \sqrt{\frac{2}{\pi t^3}} (2w - x) \exp \left( -\frac{(2w - x)^2}{2t} \right), \ w \geq 0, \ x \leq w \]

By a change of variable of \( W(t), Y(t) = W(t) - B(t) \), we can find the desired joint density of \( W(t) \), and \( Y(t) \)

\[ g(w, y) = f(w, w - y) \]

\[ = \sqrt{\frac{2}{\pi t^3}} (w + y) \exp \left( -\frac{(w + y)^2}{2t} \right), \ w \geq 0, \ y \geq 0 \]

Note that the density is symmetric in \( w \) and \( y \).

Thus \( Y(t) \) has the same marginal distribution as \( W(t) \), which is also same as \(|B(t)|\).