STAT 24400 Lecture 12
Section 5.2 The Law of Large Numbers (LLN)
Section 5.3 Convergence in Distribution and the Central Limit Theorem

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Markov Inequality (p.121 in Textbook)

If X is a random variable that only take nonnegative values $\mathrm{P}(X \geq 0) = 1$ and for which $\mathrm{E}(X)$ exists, then

$$\mathrm{P}(X \geq t) \leq \frac{\mathrm{E}(X)}{t}.$$

Proof. We will prove this for the discrete case; the continuous case is entirely analogous.

$$\begin{aligned}
\mathrm{E}(X) = \sum_x xp(x) &= \overbrace{\sum_{x<t} xp(x)}^{\geq 0 \text{ since } X \geq 0} + \sum_{x \geq t} xp(x) \\
&\geq \sum_{x \geq t} xp(x) \\
&\geq \sum_{x \geq t} tp(x) = t\mathrm{P}(X \geq t)
\end{aligned}$$

Chebyshev's Inequality (p.133, Textbook)

Let X be a random variable with mean μ and variance σ^2. Then, for any $t > 0$,

$$\mathrm{P}(|X - \mu| > k) \leq \frac{\sigma^2}{k^2}.$$

Proof. Since $(X - \mu)^2$ is a nonnegative random variable, we can apply Markov's inequality (with $t = k^2$) to obtain

$$\mathrm{P}(|X - \mu| \geq k) = \mathrm{P}((X - \mu)^2 \geq k^2) \leq \frac{\overbrace{\mathrm{E}[(X - \mu)^2]}^{=\mathrm{Var}(X) = \sigma^2}}{k^2} = \frac{\sigma^2}{k^2}.$$

The Weak Law of Large Numbers (WLLN)

Let $X_1, X_2, \ldots, X_n, \ldots$ be indep. r.v.s with $\mathrm{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, for any $\varepsilon > 0$,

$$\mathrm{P}(|\overline{X}_n - \mu| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

Proof. We first find $\mathrm{E}(\overline{X}_n)$ and $\mathrm{Var}(\overline{X}_n)$:

$$\mathrm{E}(\overline{X}_n) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu.$$

Since the X_i are independent,

$$\mathrm{Var}(\overline{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{\mathrm{Var}(X_i)}_{=\sigma^2} = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}.$$

The desired result now follows immediately from Chebyshev's inequality, which

$$\mathrm{P}(|\overline{X}_n - \mu| > \varepsilon) \le \frac{\mathrm{Var}(\overline{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0, \quad \text{as } n \to \infty.$$

Counterexample for WLLN — Cauchy

- The WLLN above is proved assuming the existence of $\mathrm{Var}(X_i)$. WLLN can be proved only assuming the existence of $\mathrm{E}(X_i)$.
- WLLN does not hold if $\mathrm{E}(X_i)$ doesn't exist. A counterexample is the Cauchy distribution. If $X_1, X_2, \ldots, X_n, \ldots$ are i.i.d. Cauchy, we can show using the Characteristic function[1] in the next page that

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \text{Cauchy}$$

which doesn't converge to a single value and this implies

$$\mathrm{P}(|\overline{X}_n| > \varepsilon) = 2\int_{\varepsilon}^{\infty}\frac{1}{\pi(1+x^2)}\mathrm{d}x = 2\left[\frac{\arctan(x)}{\pi}\right]_{x=\varepsilon}^{x=\infty} = 1 - \frac{2\arctan(\varepsilon)}{\pi}$$

For example, $\mathrm{P}(|\overline{X}_n| > 1) = 1 - \frac{2\arctan(1)}{\pi} = \frac{1}{2}$ for all n, which doesn't converge to 0 as $n \to \infty$.

[1] In fact, we proved a special case in L07: if X_1 and X_2 are indep. Cauchy, $\frac{1}{2}(X_1 + X_2)$ is also Cauchy.

Distribution of Sample Mean of Cauchy R.V.'s

Recall we mentioned at the end of L11 that the **characteristic function** of the Cauchy distribution is

$$\phi_X(t) = \mathrm{E}[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} \mathrm{d}x = e^{-|t|}, \quad -\infty < t < \infty.$$

If $X_1, \ldots, X_n, \ldots$ are i.i.d. Cauchy, the characteristic function for $S_n = \sum_{i=1}^n X_i$ is

$$\phi_{S_n}(t) = [\phi_X(t)]^n = e^{-n|t|}, \quad -\infty < t < \infty.$$

Thus, the characteristic function for $\overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i = \frac{1}{n} S_n$ is

$$\phi_{\overline{X}_n}(t) = \phi_{S_n}(t/n) = e^{-n|t/n|} = e^{-|t|}, \quad -\infty < t < \infty,$$

which is exactly the characteristic function for Cauchy. As the characteristic function uniquely determines the distribution, we know $\overline{X}_n$ has the Cauchy PDF below for all n:

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty \le x < \infty.$$

Definition: Convergence in Distribution (p.181, Textbook)

Let $X_1, X_2, \ldots$ be a sequence of r.v.s with CDFs $F_1, F_2, \ldots$, and let X be a r.v. with CDF F. We say that *X_n converges in distribution to X* if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

at every point at which F is continuous.

Convergence in MGF Implies Convergence in Distribution

Suppose $X_1, X_2, \dots, X_n, \dots$ is a sequence of r.v.s, each with MGF $M_{X_i}(t)$. Furthermore, suppose that

$$\lim_{i \to \infty} M_{X_i}(t) = M_X(t),$$

for all t in an open interval containing 0, and $M_X(t)$ is an MGF. Then there is a unique CDF F_X whose moments are determined by $M_X(t)$ and, for all x where $F_X(x)$ is continuous, we have

$$\lim_{n \to \infty} F_n(x) = F(x).$$

Proof. Too advance for STAT 244.

Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each having mean μ and variance σ^2 and let $S_n = X_1 + \cdots + X_n$. The distribution of

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$,

$$\mathrm{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \le a\right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} \mathrm{d}x \quad \text{as } n \to \infty.$$

Proof of CLT (Using MGF)

We will first prove CLT for the case $\mu = \mathrm{E}(X_i) = 0$.

Let $M(t)$ be the common MGF of the X_i's. Since S_n is a sum of independent r.v.'s, we know the MGF of S_n is

$$M_{S_n}(t) = [M(t)]^n$$

The MGF for $Z_n = \dfrac{S_n}{\sqrt{n}\sigma}$ is a linear transformation of S_n, so

$$M_{Z_n}(t) = M_{S_n}\left(\frac{t}{\sqrt{n}\sigma}\right) = \left[M\left(\frac{t}{\sqrt{n}\sigma}\right)\right]^n$$

Take the Taylor series expansion of $M(s)$ about zero:

$$\begin{aligned} M(s) &= \overbrace{M(0)}^{=1} + s\,\overbrace{M'(0)}^{=\mathrm{E}(X)=0} + \frac{1}{2}s^2\,\overbrace{M''(0)}^{=\mathrm{E}[X^2]=\mathrm{Var}(X)=\sigma^2} + \varepsilon \\ &= 1 + \frac{\sigma^2}{2}s^2 + \varepsilon \quad \text{where } \varepsilon/s^2 \to 0 \text{ as } s \to 0. \end{aligned}$$

As $M(s) = 1 + \frac{\sigma^2}{2} s^2 + \varepsilon$, we have

$$M\left(\frac{t}{\sqrt{n}\sigma}\right) = 1 + \frac{\sigma^2}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 + \varepsilon_n = 1 + \frac{t^2}{2n} + \varepsilon_n$$

where $\varepsilon_n/(t^2/(n\sigma^2)) \to 0$ as $n \to \infty$.

$$M_{Z_n}(t) = \left[M\left(\frac{t}{\sqrt{n}\sigma}\right)\right]^n = \left(1 + \frac{t^2}{2n} + \varepsilon_n\right)^n \to e^{t^2/2} \quad \text{as } n \to \infty.$$

The last limit comes from the fact that

$$\lim_{n\to\infty}\left(1 + \frac{a_n}{n}\right)^n = e^a \quad \text{if} \quad \lim_{n\to\infty} a_n = a.$$

Here $e^{t^2/2}$ is the MGF of the standard normal, as was to be shown.

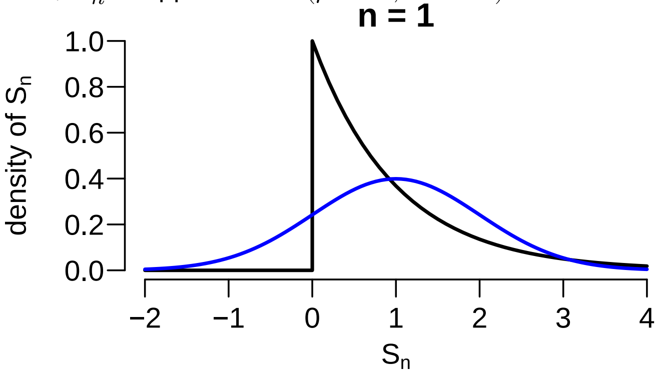
For the case $\mu = \mathrm{E}(X_i) \neq 0$, we can define $X_i' = X_i - \mu$, and let $S_n' = X_1' + \cdots + X_n'$. Then $S_n - n\mu = S_n'$ and the proof goes as the case for $\mu = 0$.

Example of CLT — Exponential

If $X_i \sim$ Exponential$(\lambda = 1)$ with the PDF

$$f(x) = e^{-x}, \quad \text{for } x > 0, \quad \mu = 1,\ \sigma^2 = 1$$

Black curve: the exact distribution of $S_n = \sum_{i=1}^{n} X_i$ is Gamma$(\alpha = n, \lambda = 1)$.
Blue curve: By CLT, S_n is approx. $\sim N(\mu = n, \sigma^2 = n)$.



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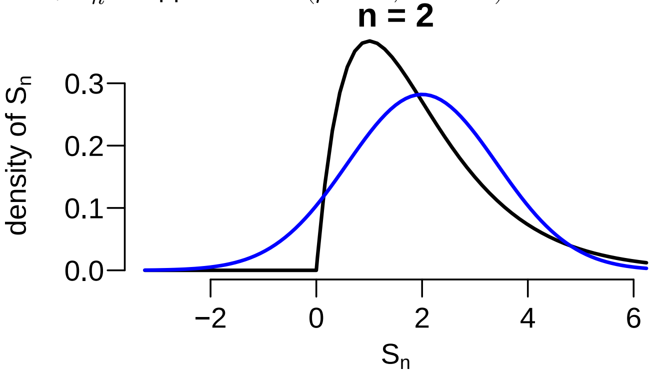

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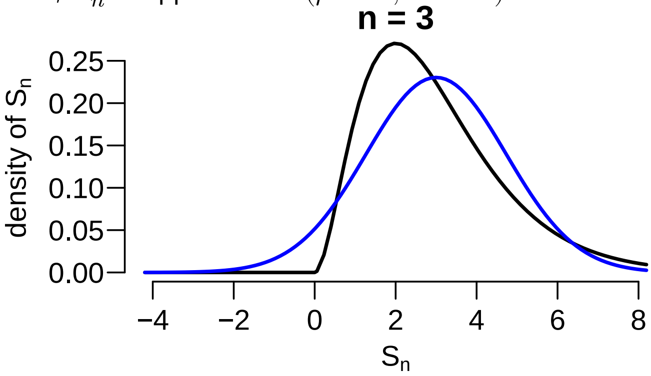

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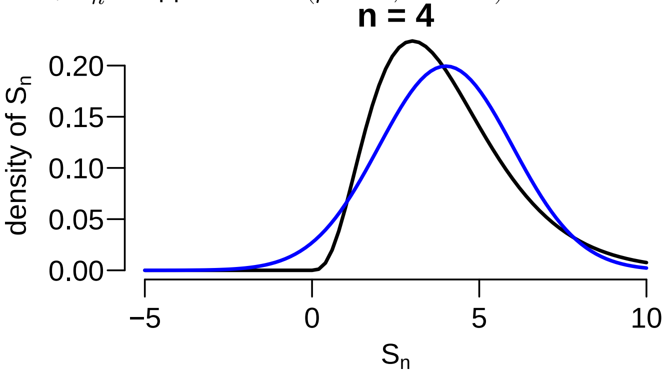

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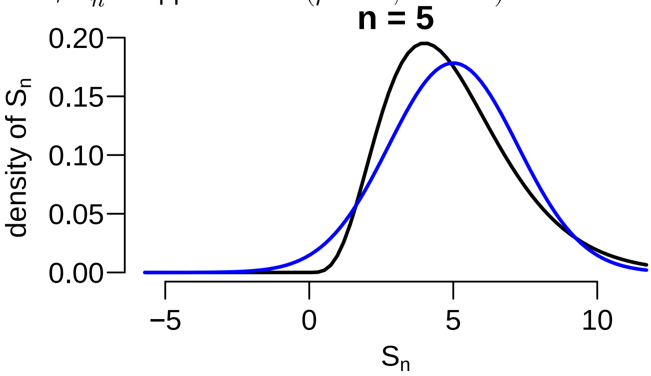

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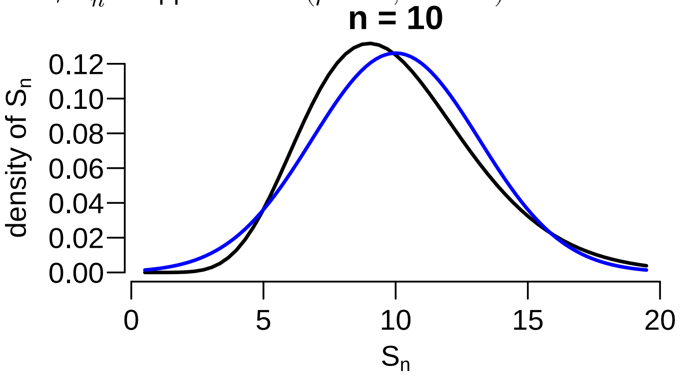

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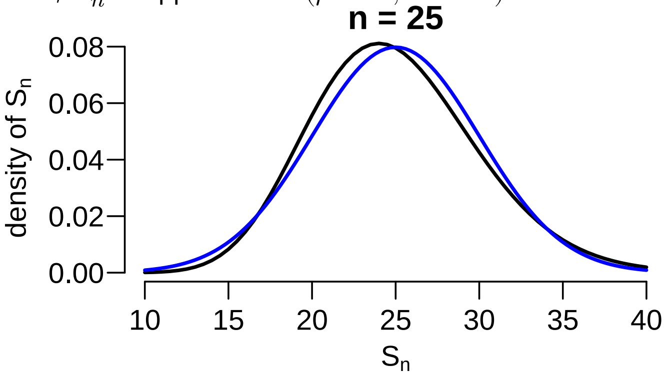

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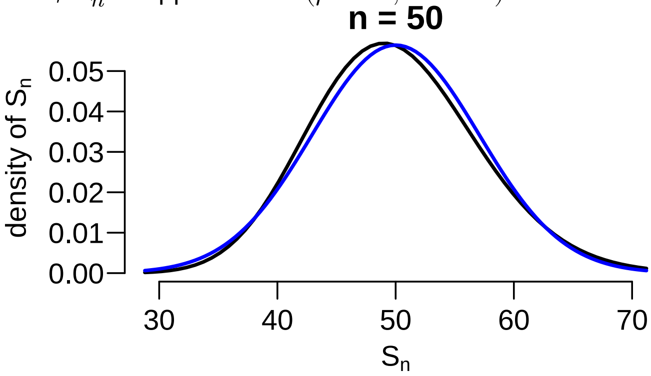

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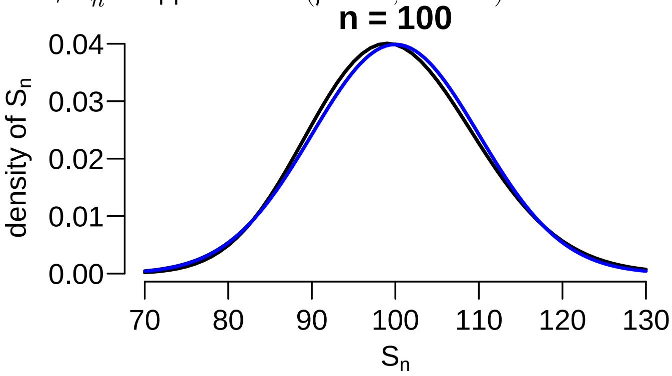

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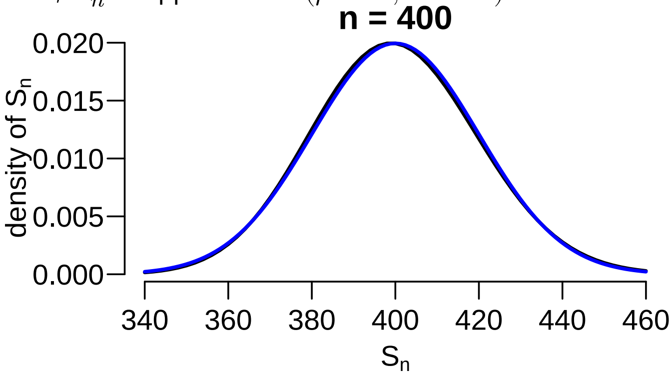

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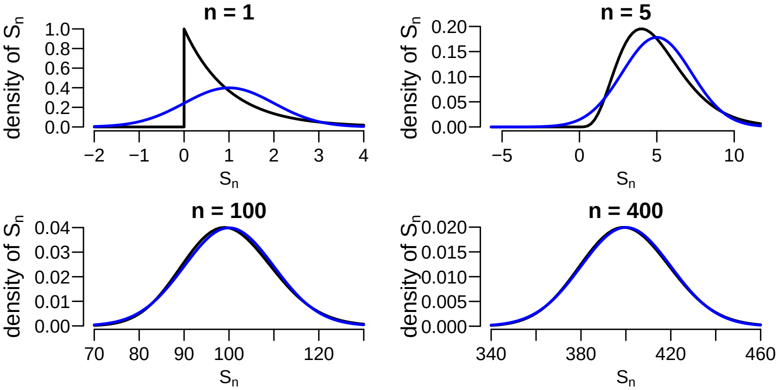

Example of CLT — Poisson

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Histogram: exact distn. of $S_n = \sum_{i=1}^{n} X_i$ is Poisson$(\lambda = n)$
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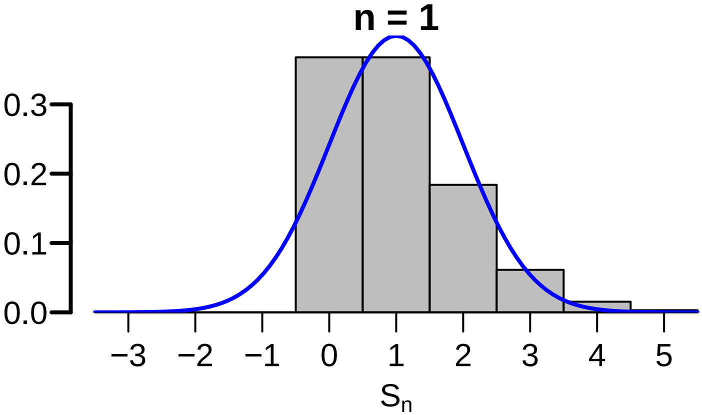

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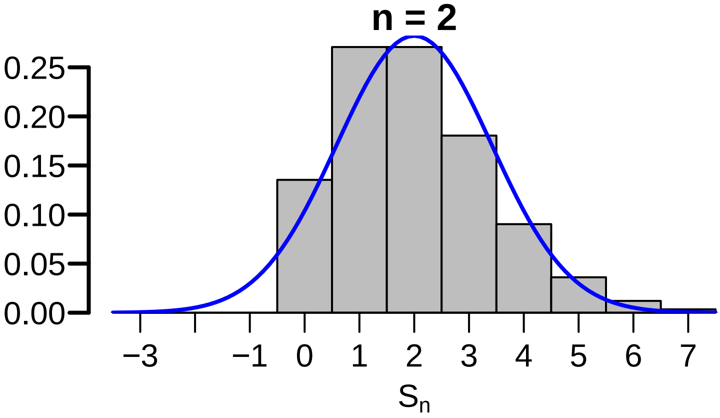

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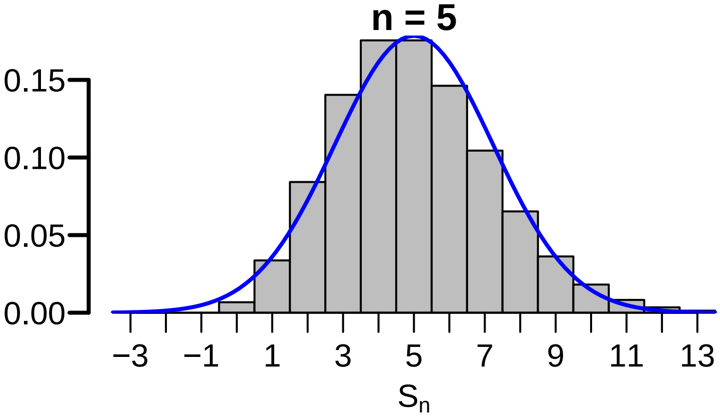

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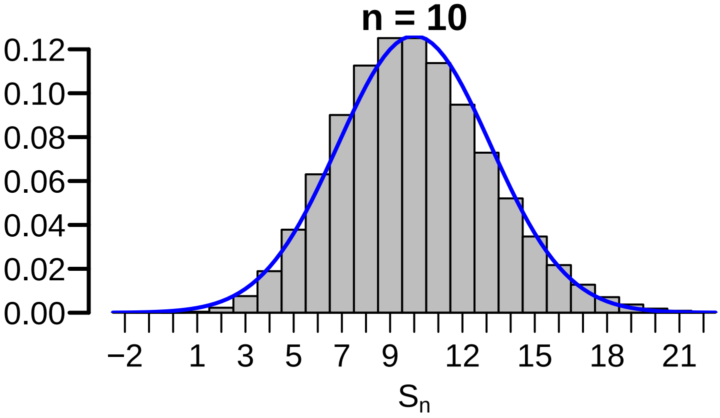

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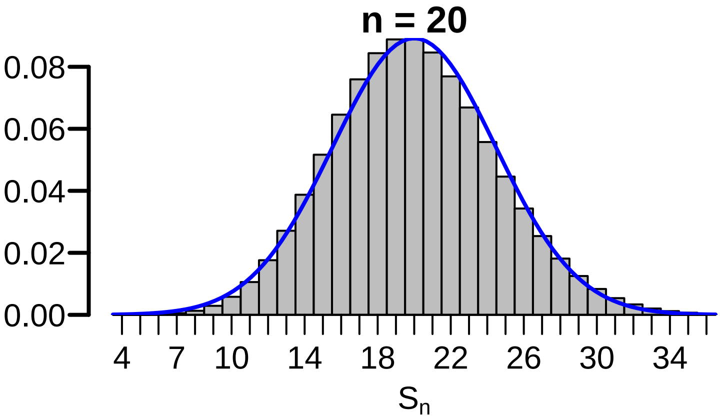

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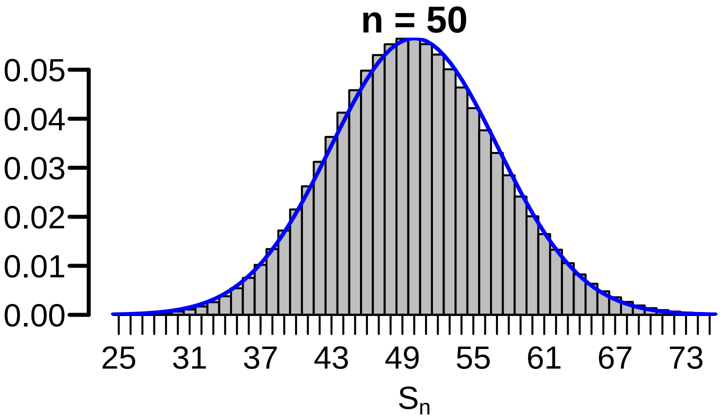

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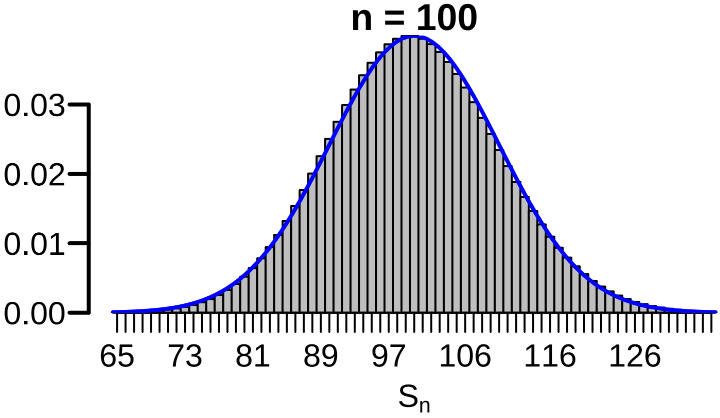

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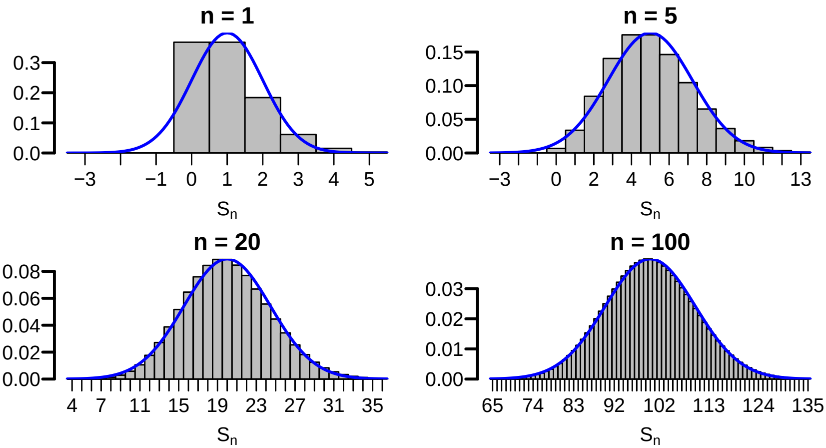
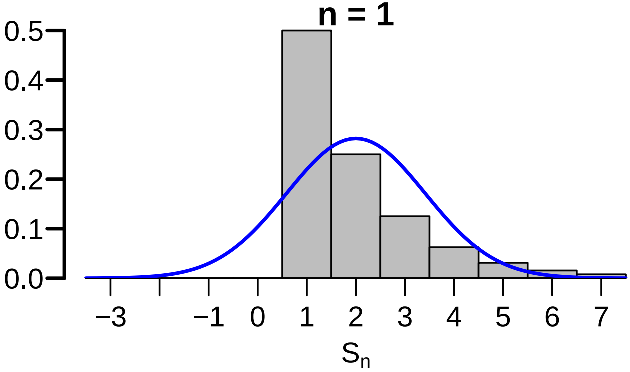


Example of CLT — Geometric

If X_i's are i.i.d. $\sim$ Geometric$(p = 0.5)$, with

$$\mathrm{P}(X_i = x) = (0.5)^x,\ x = 1, 2, 3, \ldots \Rightarrow \mu = \frac{1}{p} = 2,\ \sigma^2 = \frac{1-p}{p^2} = 2.$$

Histogram: exact distn. of $S_n = \sum_{i=1}^{n} X_i$ is NegBin$(n, p = 0.5)$.
Blue curve: By CLT, S_n is approx. $\sim N(\mu = 2n, \sigma^2 = 2n)$.



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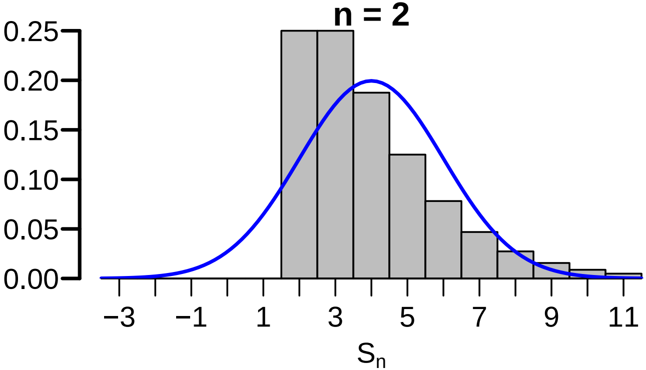

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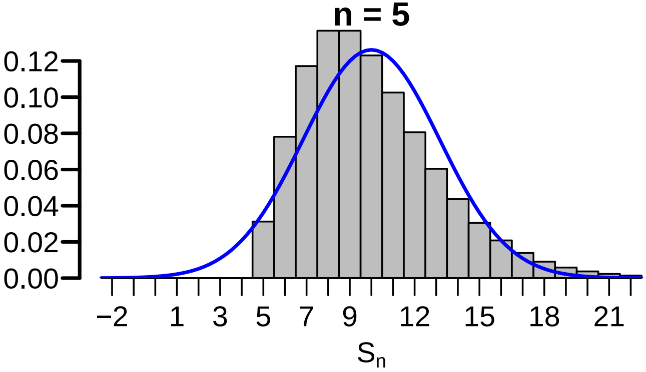

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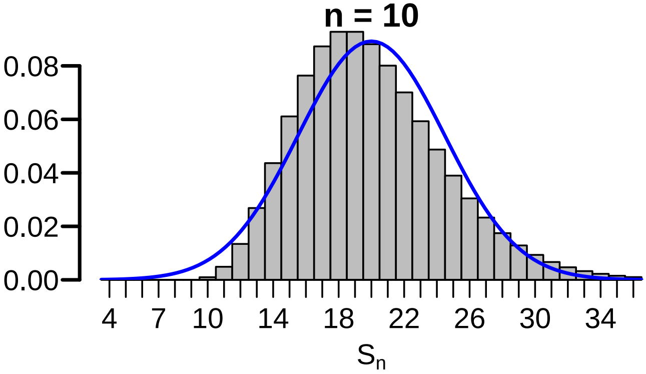
Histogram: exact distn. of $S_n = \sum_{i=1}^{n} X_i$ is NegBin$(n, p = 0.5)$.
Blue curve: By CLT, S_n is approx. $\sim N(\mu = 2n, \sigma^2 = 2n)$.



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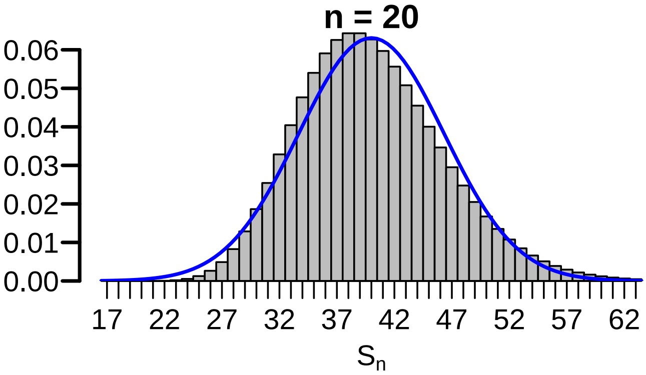

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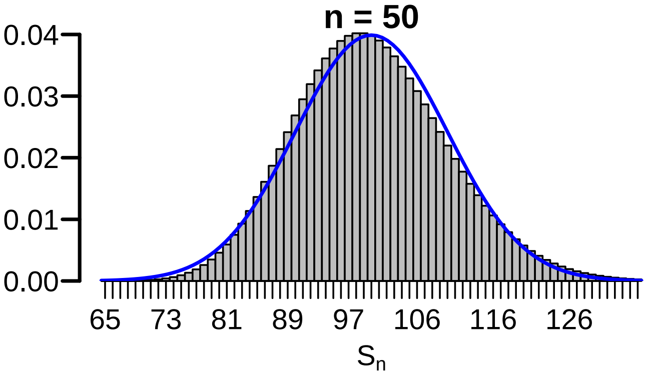

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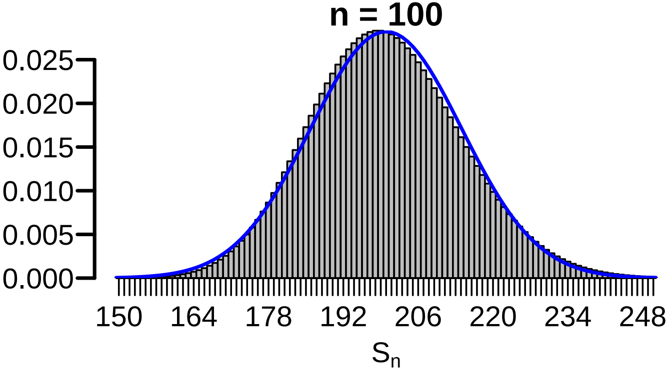

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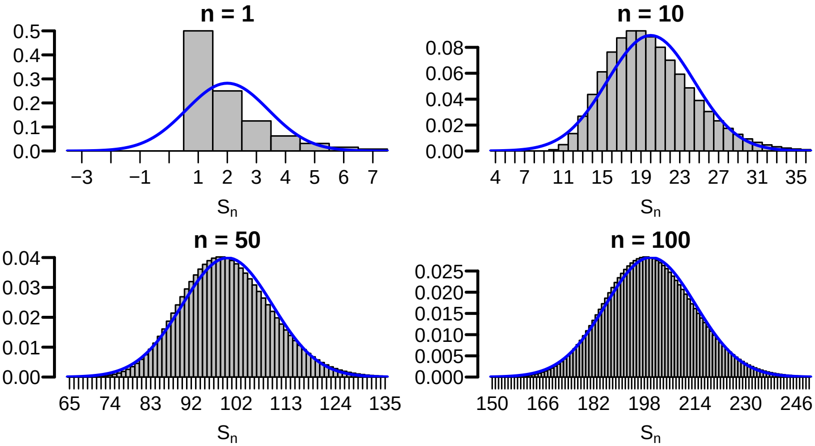


Normal Approximation to Binomial Distribution

Normal approximation to the Binomial distributions is a special case of CLT:

$$X = \sum_{i=1}^{n} X_i \sim Bin(n, p),$$

where $X_1, X_2, ..., X_n$ are n independent Bernoulli random variables with success probability p.

Therefore,

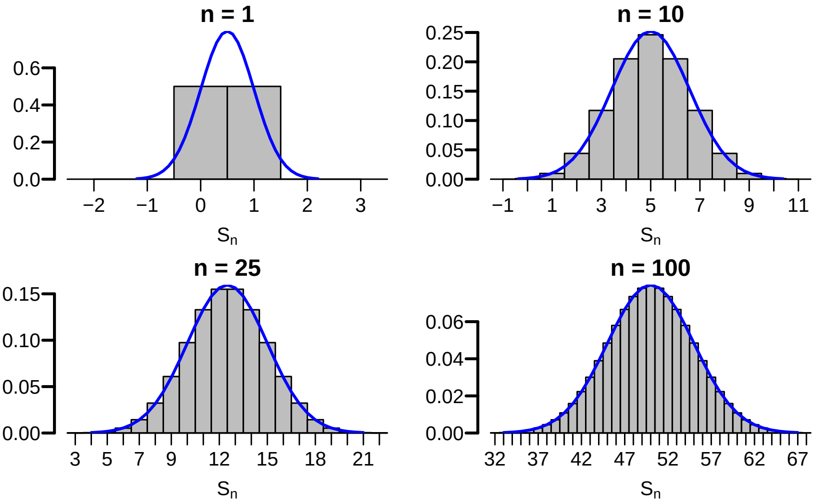
$$\mathrm{E}(X_i) = p, \quad \mathrm{Var}(X_i) = p(1-p).$$

By CLT, for large n, $Y \sim \text{Bin}(n, p)$ is approximately distributed as

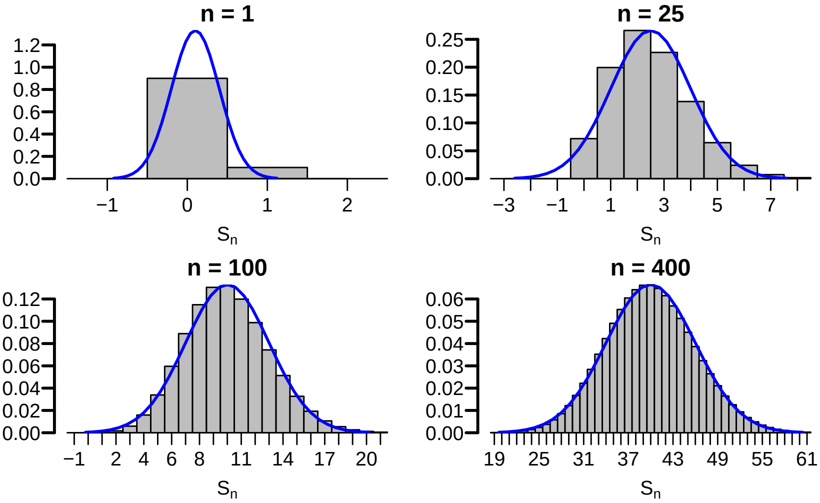
$$N(\mu_Y = np,\ \sigma_Y^2 = np(1-p)).$$

Normal Approximation to Bin$(n, p = 0.5)$

When $X_1, \ldots, X_n \sim$ Bernoulli$(p = 0.5)$, the exact distribution of S_n is $Bin(n, p = 0.5)$



For $X_1, \ldots, X_n \sim$ Bernoulli$(p = 0.1)$, the exact distribution of S_n is $Bin(n, p = 0.1)$



Example 3: Roulette Calibration

With a perfectly balanced roulette wheel, red numbers should turn up 18 in 38 of the time. To test its wheel, one casino records the results of 3800 plays. Let X be the number of reds the casino got.

Q1: If the roulette wheel is perfectly balanced, what is the chance that $X \geq 1890$?

Q2 If the casino gets 1890 reds, do you think the roulette wheel should be calibrated?

Example 3: Roulette Calibration

Q1: If the roulette wheel is perfectly balanced, find $\mathrm{P}(X \geq 1890)$.

Sol.: We know $X \sim \text{Bin}(n = 3800, p = \frac{18}{38})$.

Thus

$$\begin{aligned} \mathrm{E}(X) &= np = 3800(18/38) = 1800 \\ \mathrm{Var}(X) &= np(1-p) = 3800(18/38)(20/38) \approx 947.37 \end{aligned}$$

By CLT, X is approx. $\sim N(\mu = 1800, \sigma^2 = 947.37)$, or $Z = \frac{X - 1800}{\sqrt{947.37}} \sim N(0, 1)$ Thus,

$$P(X \geq 1890) \approx P\left(Z \geq \frac{1890 - 1800}{\sqrt{947.37}} \approx 2.92\right) \approx 1 - \Phi(2.92) \approx 0.00173.$$

```
1-pnorm(1890, m = 1800, s = sqrt(3800*(18/38)*(20/38)))
[1] 0.001728
```

Example 3: Roulette Calibration

As $X \sim \text{Bin}(n = 3800, \, p = \frac{18}{38})$, the exact probability of $X \geq 1890$ is

$$P(X \geq 1890) = \sum_{k=1890}^{3800} \binom{3800}{k} \left(\frac{18}{38}\right)^{k} \left(\frac{20}{38}\right)^{3800-k} \approx 0.00183$$

We can see normal approx. to Binomial gives fairly good approx to the exact Binomial probability.

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Q2 If the casino gets 1890 reds, do you think the roulette wheel should be calibrated?
Yes. $X \geq 1890$ is very unlikely to happen.

How Large n Has to Be to Use CLT?

- If the population is normal, then any n will do.
- If the population distribution is symmetric, then n should be at least 30 or so.
- The more skew or irregular the population, the larger n has to be
- For the Binomial distribution, a rule of thumb is that n should be such that

$$np \geq 10 \quad \text{and} \quad n(1-p) \geq 10.$$