# STAT 24400 Lecture 9
# 4.3 Covariance & Correlation

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# Expected Values for Functions of Several R.V.'s

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For **discrete** random variables $X_1, \dots, X_n$ with joint PMF $p(x_1, \dots, x_n)$, the expected value for $g(X_1, \dots, X_n)$ is

$$\mathrm{E}(g(X_1, \dots, X_n)) = \sum_{x_1, \dots, x_n} g(x_1, \dots, x_n) p(x_1, \dots, x_n),$$

provided that $\sum_{x_1, \dots, x_n} |g(x_1, \dots, x_n)| p(x_1, \dots, x_n) < \infty$.

For **continuous** random variables $X_1, \dots, X_n$ with joint PDF $f(x_1, \dots, x_n)$, the expected value for $g(X_1, \dots, X_n)$ is

$$\mathrm{E}(g(X_1, \dots, X_n)) = \int \cdots \int g(x_1, \dots, x_n) f(x_1, \dots, x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n,$$

provided that $\int \cdots \int |g(x_1, \dots, x_n)| f(x_1, \dots, x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n < \infty$.

# $\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$

If $g(X, Y) = aX + bY$ for two random variables (discrete or continuous) $X$ and $Y$ and two constants $a$ and $b$, we have

$$\mathrm{E}[g(X,Y)] = \mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y).$$

*Proof.* We will prove it for continuous $X$ and $Y$ with joint PDF $f(x, y)$. The proof for the discrete case is similar.

$$\begin{aligned}
\mathrm{E}(aX + bY) &= \iint (ax + by) f(x,y)\,\mathrm{d}x\mathrm{d}y, \quad \text{(by definition)} \\
&= \underbrace{\iint ax f(x,y)\,\mathrm{d}x\mathrm{d}y}_{\text{Part I}} + \underbrace{\iint by f(x,y)\,\mathrm{d}x\mathrm{d}y}_{\text{Part II}}
\end{aligned}$$

For Part I, we first integrate over $y$, and then over $x$.

$$\text{Part I} = \iint axf(x,y)\mathrm{d}x\mathrm{d}y = a\int\left(\int xf(x,y)dy\right)\mathrm{d}x$$

$$= a\int x\underbrace{\int f(x,y)dy}_{f_X(x)}\mathrm{d}x = a\underbrace{\int xf_X(x)\mathrm{d}x}_{\mathrm{E}(X)} = a\,\mathrm{E}(X)$$

For Part II, we first integrate over $x$, and then over $y$.

$$\text{Part II} = \iint byf(x,y)\mathrm{d}x\mathrm{d}y = b\int\left(\int yf(x,y)dx\right)\mathrm{d}y$$

$$= b\int y\underbrace{\int f(x,y)dx}_{f_Y(y)}\mathrm{d}y = b\underbrace{\int yf_Y(y)\mathrm{d}y}_{\mathrm{E}(Y)} = b\,\mathrm{E}(Y)$$

Putting Parts I & II together, we get

$$\mathrm{E}(aX+bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y).$$

# Expected Value for Linear Combination of R.V.'s

The result $\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$ can be generalized to linear combinations of several random variables

$$\mathrm{E}(a_1 X_1 + a_2 X_2 + \cdots + a_n X_n) = a_1\,\mathrm{E}(X_1) + a_2\,\mathrm{E}(X_2) + \cdots + a_n\,\mathrm{E}(X_n),$$

no matter the rv's are discrete or continuous, independent or not.

# $\mathrm{E}[g(X)h(Y)] = \mathrm{E}[g(X)]\,\mathrm{E}[h(Y)]$ if $X, Y$ are independent

When $X$ and $Y$ are **independent**, for any functions $g$ and $h$,

$$\mathrm{E}[g(X)h(Y)] = \mathrm{E}[g(X)]\,\mathrm{E}[h(Y)].$$

In particular, $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$.

*Proof.* We prove the discrete case. The continuous case is similar. Using that $p(x,y) = p_X(x)p_Y(y)$ when $X$, $Y$ are indep, one has

$$\begin{aligned}
\mathrm{E}[g(X)h(Y)] &= \sum_{xy} g(x)h(y)p(x,y) \\
&= \sum_x \sum_y g(x)h(y)p_X(x)p_Y(y) \quad \text{(by independence)} \\
&= \underbrace{\sum_x g(x)p_X(x)}_{\mathrm{E}[g(X)]} \underbrace{\sum_y h(y)p_Y(y)}_{\mathrm{E}[h(Y)]} \\
&= \mathrm{E}[g(X)]\,\mathrm{E}[h(Y)]
\end{aligned}$$

# Covariance

# Covariance

The **covariance** of $X$ and $Y$, denoted as $\mathrm{Cov}(X,Y)$ or $\sigma_{XY}$, is defined as

$$\mathrm{Cov}(X,Y) = \sigma_{XY} = \mathrm{E}[(X-\mu_X)(Y-\mu_Y)],$$

in which $\mu_X = \mathrm{E}(X)$, $\mu_Y = \mathrm{E}(Y)$

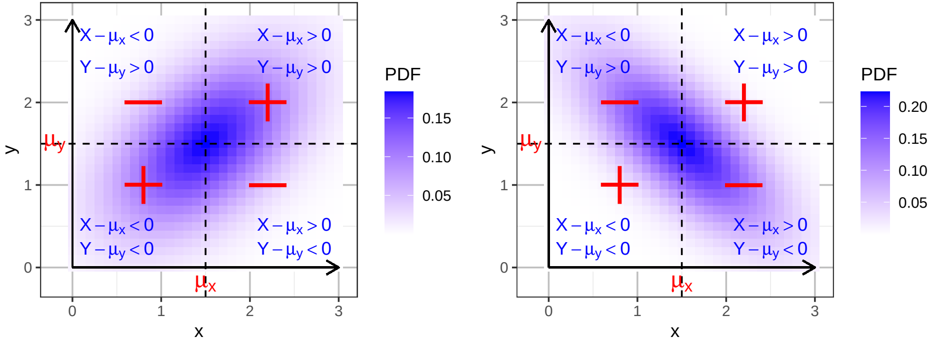
- Covariance is a generalization of variance as the variance of a random variable $X$ is just the covariance of $X$ with itself.

$$\mathrm{Var}(X) = \mathrm{Cov}(X,X) = \mathrm{E}[(X-\mu_X)^2]$$

# Sign of Covariance Reflects the Direction of $(X, Y)$ Relation

- $\mathrm{Cov}(X, Y) > 0$ means a positive relation between $X$, $Y$
  - When $X$ increases, $Y$ tends to increase
- $\mathrm{Cov}(X, Y) < 0$ means a negative relation between $X$, $Y$
  - When $X$ increases, $Y$ tends to decrease

Sign of $(X - \mu_X)(Y - \mu_Y)$



# Shortcut Formula for Covariance

$$\boxed{\operatorname{Cov}(X, Y) = \operatorname{E}(XY) - \operatorname{E}(X)\operatorname{E}(Y)}$$

- Similar to the Shortcut Formula for Variance

$$\operatorname{Var}(X) = \operatorname{E}(X^2) - [\operatorname{E}(X)]^2.$$

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- If $X$, $Y$ are independent, $\Rightarrow \mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y) \Rightarrow \mathrm{Cov}(X, Y) = 0.$

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- If $X$, $Y$ are independent, $\Rightarrow \mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y) \Rightarrow \mathrm{Cov}(X,Y) = 0$.
- However, $\mathrm{Cov}(X,Y) = 0$ does **not** imply the independence of $X$ and $Y$. In this case, we say $X$ and $Y$ are *uncorrelated.*

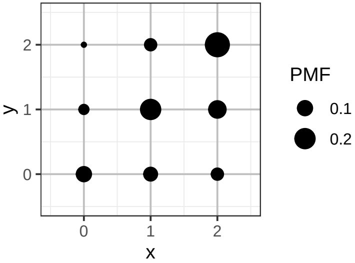
# Proof of the Shortcut Formula for Covariance

$$\begin{aligned}
\operatorname{Cov}(X, Y) &= \mathrm{E}[(X - \mu_X)(Y - \mu_Y)] \\
&= \mathrm{E}(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y) \\
&= \mathrm{E}(XY) - \mu_X \underbrace{\mathrm{E}(Y)}_{=\mu_Y} - \mu_Y \underbrace{\mathrm{E}(X)}_{=\mu_X} + \mu_X \mu_Y \\
&= \mathrm{E}(XY) - \mu_X \mu_Y.
\end{aligned}$$

# Example (Gas Station) — $\mathrm{E}(XY)$

Recall the joint PMF for the Gas Station Example in L05 is

| | $p(x,y)$ | $Y$ (full-service) 0 | 1 | 2 |
|---|---|---|---|---|
| $X$ | 0 | 0.10 | 0.04 | 0.02 |
| self- | 1 | 0.08 | 0.20 | 0.06 |
| service | 2 | 0.06 | 0.14 | 0.30 |




Guess $\mathrm{Cov}(X,Y) > 0$ or $< 0$?

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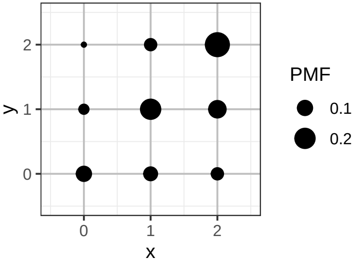



Guess $\mathrm{Cov}(X,Y) > 0$ or $< 0$?

$$
\begin{aligned}
\mathrm{E}(XY) = \sum\nolimits_{xy} xyp(x,y) &= 0\cdot 0\cdot 0.10 + 0\cdot 1\cdot 0.04 + 0\cdot 2\cdot 0.02 \\
&\quad + 1\cdot 0\cdot 0.08 + 1\cdot 1\cdot 0.20 + 1\cdot 2\cdot 0.06 \\
&\quad + 2\cdot 0\cdot 0.06 + 2\cdot 1\cdot 0.14 + 2\cdot 2\cdot 0.30 \\
&= 1.8
\end{aligned}
$$

# Example (Gas Station) — Covariance

Recall in L05, we obtained the marginal PMFs for $X$ and for $Y$:

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_X(x)$ | 0.16 | 0.34 | 0.50 |

$$\mathrm{E}(X) = 0 \cdot 0.16 + 1 \cdot 0.34 + 2 \cdot 0.5 = 1.34$$

| $y$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_Y(y)$ | 0.24 | 0.38 | 0.38 |

$$\mathrm{E}(Y) = 0 \cdot 0.24 + 1 \cdot 0.38 + 2 \cdot 0.38 = 1.14$$

By the shortcut formula, the covariance is

$$\begin{aligned} \mathrm{Cov}(X, Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) \\ &= 1.8 - 1.34 \times 1.14 = 0.2724 > 0. \end{aligned}$$

When one service island is busy, the other also tends to be busy.

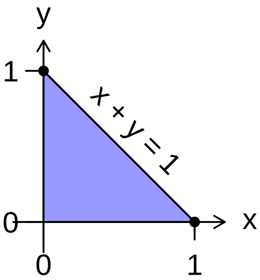
# Example (Mixed Nuts)

Recall in L05, the joint PDF for

$$X = \text{ the weight of almonds, and}$$
$$Y = \text{ the weight of cashews}$$

in a can of mixed nuts is

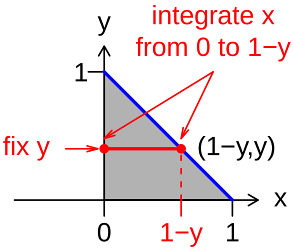
$$f(x,y) = \begin{cases} 24xy & \text{if } 0 \le x, y \le 1, x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$




Before we calculate it, guess $\mathrm{Cov}(X,Y) > 0$ or $< 0$?

# Example (Mixed Nuts) — $\mathrm{E}(XY)$

$$\mathrm{E}(XY) = \iint xyf(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \underbrace{\int_0^{1-y} 24x^2y^2\mathrm{d}x}_{\text{see below}}\,\mathrm{d}y$$




where

$$\int_0^{1-y} 24x^2y^2\,\mathrm{d}x = \left[8x^3y^2\right]_{x=0}^{x=1-y} = 8(1-y)^3y^2.$$

Putting it back to the double integral, we get

$$\mathrm{E}(XY) = \int_0^1 \int_0^{1-y} 24x^2y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 8(1-y)^3y^2\mathrm{d}y = \frac{2}{15}.$$

# Example (Mixed Nuts) — Covariance

Recall in L05, we calculated the marginal PDF's for $X$ and for $Y$:

$$f_X(x) = 12x(1-x)^2, \quad f_Y(y) = 12y(1-y)^2, \text{ for } 0 \le x, y \le 1.$$

using which we can calculate

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^1 x f_X(x)\mathrm{d}x = \int_0^1 12x^2(1-x)^2\mathrm{d}x \\
&= \int_0^1 12x^2 - 24x^3 + 12x^4\mathrm{d}x = \left[4x^3 - 6x^4 + \frac{12}{5}x^5\right]_0^1 = \frac{2}{5}.
\end{aligned}$$

Likewise, $\mathrm{E}(Y) = 2/5$. The covariance by the shortcut formula is

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = \frac{2}{15} - \frac{2}{5} \times \frac{2}{5} = -\frac{2}{75}$$

When a can of mixed nuts has more almond, it likely has less cashew, and vice versa.

# More Properties of Covariance

$$\mathrm{Cov}(X, Y) = \mathrm{E}[(X - \mathrm{E}(X))(Y - \mathrm{E}(Y))] = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y)$$

In the following, $a$, $b$ are constants. $X$, $Y$, $Z$ are random variables

- Symmetry: $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
- Scaling: $\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X, Y)$
- Right-linearity: $\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$
- Left-linearity: $\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$
- $\mathrm{Cov}(a, X) = 0$.

# Proofs for Properties of Covariance

The proofs for these propertie are all straightforward from definition. We just prove the Right-linearity as an example.

$$\begin{aligned}
\mathrm{Cov}(X+Y,Z) &= \mathrm{E}((X+Y)Z) - \mathrm{E}(X+Y)\,\mathrm{E}(Z) \\
&= \mathrm{E}(XZ) + \mathrm{E}(YZ) - [\mathrm{E}(X) + \mathrm{E}(Y)]\,\mathrm{E}(Z) \\
&= \underbrace{\mathrm{E}(XZ) - \mathrm{E}(X)\,\mathrm{E}(Z)}_{\mathrm{Cov}(X,Z)} + \underbrace{\mathrm{E}(YZ) - \mathrm{E}(Y)\,\mathrm{E}(Z)}_{\mathrm{Cov}(Y,Z)} \\
&= \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)
\end{aligned}$$

Note in the proof above, we used the property of expected value that

$$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$
$$\mathrm{E}(XZ+YZ) = \mathrm{E}(XZ) + \mathrm{E}(YZ)$$

# Variance of Linear Combinations of Two Random Variables

Recall that expectation has the following linear property:

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y).$$

We also have shown that $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.

How about $\mathrm{Var}(aX + bY)$?

$$\boxed{\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + 2ab\,\mathrm{Cov}(X, Y) + b^2\,\mathrm{Var}(Y)}$$

- If $X$ is independent of $Y$, $\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

# Proof of $\mathrm{Var}(aX + bY)$

$$
\begin{aligned}
\mathrm{Var}(aX + bY) &= \mathrm{Cov}({\color{red}aX} + {\color{blue}bY}, aX + bY) \\
&= \underbrace{\mathrm{Cov}({\color{red}aX}, aX + bY)} \quad + \quad \underbrace{\mathrm{Cov}({\color{blue}bY}, aX + bY)} \quad \text{(right-linearity)} \\
&\qquad\qquad\quad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \\
&= \overbrace{\mathrm{Cov}(aX, aX) + \mathrm{Cov}(aX, bY)} + \overbrace{\mathrm{Cov}(bY, aX) + \mathrm{Cov}(bY, bY)} \quad \text{(left-linearity)} \\
&= \mathrm{Var}(aX) + 2\,\mathrm{Cov}(aX, bY) + \mathrm{Var}(bY) \quad \text{(symmetry)} \\
&= a^2\,\mathrm{Var}(X) + 2ab\,\mathrm{Cov}(X, Y) + b^2\,\mathrm{Var}(Y) \quad \text{(scaling)}
\end{aligned}
$$

# Linear Combinations of Random Variables

For any random variables $X_1, X_2, \ldots, X_n$., a *linear combination* of $X_1, X_2, \ldots, X_n$ is

$$a_1X_1 + a_2X_2 + \cdots + a_nX_n,$$

where $a_1, a_2, \ldots, a_n$ are constant numbers. For example,

- The sum $X_1 + X_2 + \cdots + X_n$ is a linear combination of $X_1, \ldots, X_n$ with all $a_i$'s $= 1$.
- The average
$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$
is a linear combination of $X_1, X_2, \ldots, X_n$ with all $a_i$'s $= 1/n$.
- The difference $X - Y$ is a linear combination of $X$ and $Y$ with $a_1 = 1$, $a_2 = -1$

# Variance of a Linear Combination of RV's

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \,\mathrm{Var}(X_i) + 2\sum_{i<j} a_i a_j \,\mathrm{Cov}(X_i, X_j)$$

- There is a covariance term for every pair of $X_i$ and $X_j$
- When $X_1, \dots, X_n$ are independent, then

$$\mathrm{Var}(X_1 + X_2 + \dots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \dots + \mathrm{Var}(X_n).$$

- When $\mathrm{Var}(X_i) = \sigma^2$ for $i = 1, \dots, n$,
  and $\mathrm{Cov}(X_i, X_j) = \rho$ for $1 \le i \ne j \le n$, then

$$\mathrm{Var}(X_1 + \dots + X_n) = n\sigma^2 + n(n-1)\rho.$$

# Example: Variance of the Binomial Distribution

In L08, we computed the expected value for the Binomial distribution Bin$(n, p)$ is $\mathrm{E}(X) = np$.

Today, we find its variance using linear combinations to be

$$\mathrm{Var}(X) = np(1-p).$$

First for the special case $n = 1$, $X \sim$ Bin$(n = 1, p)$, $X$ only takes value 0 and 1 with the PMF below

| $x$ | 0 | 1 |
|---|---|---|
| $p(x)$ | $1-p$ | $p$ |

Hence

$$\mathrm{E}(X) = \sum\nolimits_{x=0,1} xp(x) = 0 \cdot (1-p) + 1 \cdot p = \boxed{p},$$

$$\mathrm{E}(X^2) = \sum\nolimits_{x=0,1} x^2 p(x) = 0^2 \cdot (1-p) + 1^2 \cdot p = p$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = p - p^2 = \boxed{p(1-p)}$$

For general $n$, recall a Binomial random variable $X \sim \text{Bin}(n, p)$ is the number of successes obtained in $n$ independent Bernoulli trials. For each of the $n$ trials, define

$$X_i = \begin{cases} 1 & \text{if success in the } i\text{th trial} \\ 0 & \text{if failure in the } i\text{th trial} \end{cases} \quad \Rightarrow X_i \sim \text{Bin}(n = 1, p).$$

$$\begin{aligned} \text{Then } X &= \text{the number of successes obtained in the } n \text{ trials} \\ &= X_1 + X_2 + \ldots + X_n, \end{aligned}$$

The expected value and variance of $X$ are thus

$$\mathrm{E}(X) = \underbrace{\mathrm{E}(X_1)}_{=p} + \cdots + \underbrace{\mathrm{E}(X_n)}_{=p} = np$$

$$\mathrm{Var}(X) = \underbrace{\mathrm{Var}(X_1)}_{=p(1-p)} + \cdots + \underbrace{\mathrm{Var}(X_n)}_{=p(1-p)} = np(1-p)$$

since $X_i$'s are indep. and each with mean $p$ and variance $p(1-p)$ as $X_i \sim \text{Bin}(n = 1, p)$.

# Example (Sample Mean)

Suppose $X_1, \ldots, X_n$ are i.i.d. rv's with mean $\mu$ and variance $\sigma^2$.

- i.i.d. = "independent and have an identical distribution"

Consider the *sample mean*

$$\overline{X} = \frac{1}{n}(X_1 + \cdots + X_n)$$

Then

$$\mathrm{E}(\overline{X}) = \frac{1}{n}[\mathrm{E}(X_1) + \ldots + \mathrm{E}(X_n)] = \frac{1}{n}(\underbrace{\mu + \ldots + \mu}_{n \text{ copies}}) = \mu.$$

$$\begin{aligned}\mathrm{Var}(\overline{X}) &= \frac{1}{n^2}\mathrm{Var}(X_1 + X_2 + \ldots + X_n) \quad \text{since } \mathrm{Var}(aX) = a^2 V(X) \\ &= \frac{1}{n^2}[\mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_n)] \quad \text{as all } X_i\text{'s are indep.} \\ &= \frac{1}{n^2}(\underbrace{\sigma^2 + \ldots + \sigma^2}_{n \text{ copies}}) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}\end{aligned}$$

# Example (Coupon Collector's Problem, p.127, textbook)

Suppose each box of breakfast cereals contains a coupon.

- There are $n$ different types of coupons,
- The coupon in any box is equally likely to be any of the $n$ types.

Let $N$ be the number of boxes required to collect all $n$ types of coupons.
Find $\mathrm{E}(N)$ and $\mathrm{Var}(N)$.

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Find $\mathrm{E}(N)$ and $\mathrm{Var}(N)$.

*Sol.* Let

- $X_1 =$ the number of boxes required to get the first coupon (any type). Clearly $X_1 = 1$.
- $X_2 =$ the number of additional boxes required to collect a new type of coupons after collecting first type.
- $X_i =$ the number of additional boxes required to collect $i$ types of coupons after collecting $i-1$ types, for $i = 1, 2, \dots, n$.

Observe that $N = X_1 + X_2 + \cdots + X_n$.

# Coupon Collector's Problem — Expected Value

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$$X_2 \sim \text{Geometric}(p = \frac{n-1}{n}).$$

- $X_3 \sim \text{Geometric}(p = \frac{n-2}{n})$

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- $X_3 \sim \text{Geometric}(p = \frac{n-2}{n})$
- In general, $X_i \sim \text{Geometric}(p = \frac{n-i+1}{n})$, $i = 1, 2, \dots, n$.

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- $X_3 \sim \text{Geometric}(p = \frac{n-2}{n})$
- In general, $X_i \sim \text{Geometric}(p = \frac{n-i+1}{n})$, $i = 1, 2, \ldots, n$.

Recall the expected value for Geometric$(p)$ is $1/p$. We know

$$\mathrm{E}(N) = \sum_{i=1}^{n} \mathrm{E}(X_i) = \sum_{i=1}^{n} \frac{n}{n-i+1} = n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{1}\right)$$

$$= n \sum_{r=1}^{n} \frac{1}{r} \approx n \log(n).$$

# Coupon Collector's Problem — Variance

Recall the variance for Geometric$(p)$ is $\dfrac{1-p}{p^2}$. We get

$$\begin{aligned}
\mathrm{Var}(N) &= \sum_{i=1}^{n} \mathrm{Var}(X_i) \quad \text{(since } X_1, \cdots, X_n \text{ are indep.)} \\
&= \sum_{i=1}^{n} \frac{1 - \frac{n-i+1}{n}}{(n-i+1)^2/n^2} \\
&= n \sum_{i=1}^{n} \frac{i-1}{(n-i+1)^2} \\
&= n \left( 0 + \frac{1}{(n-1)^2} + \frac{2}{(n-2)^2} + \frac{3}{(n-3)^2} + \cdots + \frac{n-1}{1^2} \right)
\end{aligned}$$

# Correlation

# How Large the Covariance Indicates a Strong Relation?

One can prove the Cauchy Inequality for covariance

$$[\mathrm{Cov}(X, Y)]^2 \leq \mathrm{Var}(X)\,\mathrm{Var}(Y)$$

# How Large the Covariance Indicates a Strong Relation?

One can prove the Cauchy Inequality for covariance

$$[\mathrm{Cov}(X,Y)]^2 \leq \mathrm{Var}(X)\,\mathrm{Var}(Y)$$

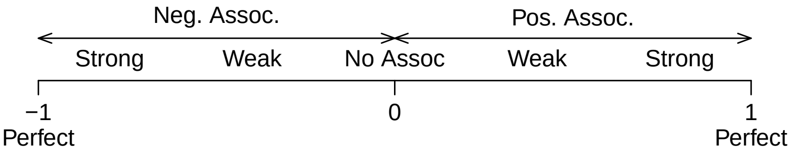
Moreover, the covariance reaches its maximum possible magnitude if and only if $X$ and $Y$ has a perfect linear relation $Y = aX + b$, $a \neq 0$.

Thus, one can assess the strength of linear relation between $X, Y$ by comparing $\mathrm{Cov}(X,Y)$ with $\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}$.

# Correlation

$$\text{Correlation } = \rho_{XY} = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

- $-1 \le \rho_{XY} \le 1$ since $\mathrm{Cov}(X, Y) \le \sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}$
- The closer $\rho_{XY}$ is to 1 or to $-1$, the stronger the linear relation between $X$ and $Y$




- $\rho_{XY} = 1$ or $-1$ if and only if $Y = aX + b$ and $a \neq 0$,
  i.e., $X$ and $Y$ has an perfect linear relation

# Covariance Is NOT Scale Invariant but Correlation Is!

Example. Let

- $X$ = amount of time studying STAT 244 per week, and
- $Y$ = final grade in STAT 244

If $X$ is measured in minutes rather than in hours, $\mathrm{Cov}(X, Y)$ would be 60 times as large.

The *strength* of $XY$ relation should be the same no matter $X$ is measured in minutes or in hours.

Correlation $\rho_{XY}$ is scale invariant and has no unit.

$$\begin{aligned}
\mathrm{Corr}(aX + c, bY + d) &= \frac{\mathrm{Cov}(aX + c, bY + d)}{\sqrt{\mathrm{Var}(aX + c)\,\mathrm{Var}(bY + d)}} \\
&= \frac{ab\,\mathrm{Cov}(X, Y)}{\sqrt{a^2\,\mathrm{Var}(X) b^2\,\mathrm{Var}(Y)}} = (\text{sign of } ab)\,\mathrm{Corr}(X, Y)
\end{aligned}$$

# Example (Gas Station) — Correlation

Recall in L05, we obtained the marginal PMFs for $X$ and $Y$:

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_X(x)$ | 0.16 | 0.34 | 0.50 |

,

| $y$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_Y(y)$ | 0.24 | 0.38 | 0.38 |

$$\begin{aligned}
\mathrm{E}(X^2) &= 0^2 \cdot 0.16 + 1^2 \cdot 0.34 + 2^2 \cdot 0.5 = 2.34 \\
\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 2.34 - 1.34^2 = 0.5444 \\
\mathrm{E}(Y^2) &= 0^2 \cdot 0.24 + 1^2 \cdot 0.38 + 2^2 \cdot 0.38 = 1.9 \\
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = 1.9 - 1.14^2 = 0.6004
\end{aligned}$$

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{0.2724}{\sqrt{0.5444 \times 0.6004}} \approx 0.476.$$

# Example (Mixed Nuts) — Correlation

Recall in L05, we calculated the marginal pdf's for $X$ and for $Y$:

$$f_X(x) = 12x(1-x)^2, \quad f_Y(y) = 12y(1-y)^2, \text{ for } 0 \le x, y \le 1.$$

using which we can calculate

$$\begin{aligned} \mathrm{E}(X^2) &= \int_0^1 x^2 f_X(x)\mathrm{d}x = \int_0^1 12x^3(1-x)^2\mathrm{d}x \\ &= \int_0^1 12x^3 - 24x^4 + 12x^5\mathrm{d}x = 3x^4 - \frac{24x^5}{5} + 2x^6\Big|_0^1 = \frac{1}{5} \end{aligned}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{1}{5} - (\frac{2}{5})^2 = \frac{1}{25}$$

Similar, one can calculate $\mathrm{Var}(Y) = 1/25$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{-2/75}{\sqrt{(1/25)(1/25)}} = -\frac{2}{3} \approx -0.667.$$

# Proof of $[\mathrm{Cov}(X,Y)]^2 \le \mathrm{Var}(X)\,\mathrm{Var}(Y)$

Since the variance of a random variable is always nonnegative,

$$
\begin{aligned}
0 \le \mathrm{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) &= \mathrm{Var}\left(\frac{X}{\sigma_X}\right) + \mathrm{Var}\left(\frac{Y}{\sigma_Y}\right) + 2\,\mathrm{Cov}\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right) \\
&= \frac{\mathrm{Var}(X)}{\sigma_X^2} + \frac{\mathrm{Var}(Y)}{\sigma_Y^2} + 2\underbrace{\frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}}_{=\rho} \\
&= \frac{\sigma_X^2}{\sigma_X^2} + \frac{\sigma_Y^2}{\sigma_Y^2} + 2\rho \\
&= 2(1+\rho), \quad \text{which implies } \rho \ge -1.
\end{aligned}
$$

Similarly, one can show that

$$
0 \le \mathrm{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 2(1-\rho), \quad \text{which implies } \rho \le 1.
$$

This proves that $-1 \le \rho \le 1 \iff 1 \ge \rho^2 = \dfrac{[\mathrm{Cov}(X,Y)]^2}{\mathrm{Var}(X)\,\mathrm{Var}(Y)}$.

# Proof that $\rho^2 = 1 \iff \mathrm{P}(Y = aX + b) = 1$

From

$$\mathrm{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 2(1-\rho)$$

we see that $\rho = 1 \iff \mathrm{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 0.$

The variance of a random variable $W$ is 0 only if

$$\mathrm{P}(W = c) = 1, \quad \text{for some constant c.}$$

Thus $\rho = 1$ if and only if

$$\mathrm{P}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} = c\right) = 1, \quad \text{for some constant } c.$$

Similarly, we can show $\rho = -1$ if and only if

$$\mathrm{P}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y} = c\right) = 1, \quad \text{for some constant } c.$$