STAT 24400 Lecture 4
Continuous Random Variables (Section 2.2)
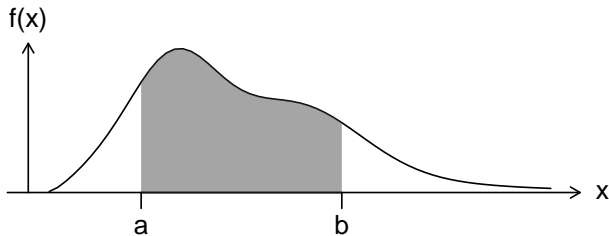## Functions of a Random Variable (Section 2.3)

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University of Chicago

Continuous Random Variables

Continuous Random Variables

A random variable X is said to have a *continuous distribution* if there exists a non-negative function f such that

$$\mathrm{P}(a < X \le b) = \int_a^b f(x)\,dx, \quad \text{for all } -\infty \le a < b \le \infty.$$

Here f is called the probability density function (PDF), the density curve, or the density of X.

Conditions of PDF

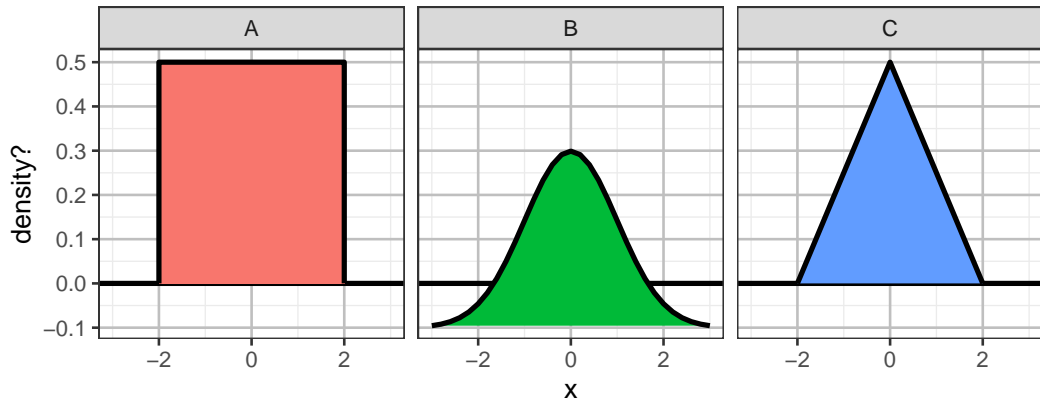A PDF $f(x)$ can be of any imaginable shape but must satisfy the following:

▶ It must be *nonnegative*
$$f(x) \geq 0 \text{ for all } x$$

▶ The total area under the PDF must be 1
$$\int_{-\infty}^{\infty} f(x)\, dx = \mathrm{P}(-\infty < X \leq \infty) = 1$$

Which of the 3 functions below is a valid probability density function (PDF)?

PDF is NOT a Probability

Suppose f is the PDF of X. If f is continuous at a point x, then for small δ

$$P\left(x - \frac{\delta}{2} < X \le x + \frac{\delta}{2}\right) = \int_{x-\delta/2}^{x+\delta/2} f(u)\, du = \delta f(x).$$

▶ Is the PDF f of a random variable always ≤ 1?

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▶ Is the PDF f of a random variable always ≤ 1?
No, the PDF $f(x)$ itself is not a probability.
It's the ⬚area⬚ underneath $f(x)$ that represents the probability.

$P(X = x) = 0$ If X Is Continuous

For any continuous random variable X

$$P(X = x) = \int_x^x f(u)du = 0$$

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$$P(X = x) = \int_x^x f(u)du = 0$$

▶ What percentage of men are 6-feet tall exactly?
Those that are 6.00001 or 5.99999 feet tall don't count.

$P(X = x) = 0$ If X Is Continuous

For any continuous random variable X

$$P(X = x) = \int_{x}^{x} f(u)du = 0$$

▶ What percentage of men are 6-feet tall exactly?
  Those that are 6.00001 or 5.99999 feet tall don't count.
▶ It doesn't matter whether the end point(s) of an interval is included when
  calculating the probability of X falling the interval if X is continuous

$$P(a < X < b) = P(a \le X \le b) = P(a \le X < b) = P(a < X \le b)$$
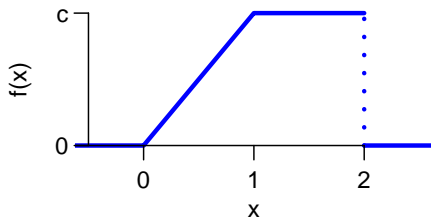
A PDF $f(x)$ May Not be Continuous

The PDF $f(x)$ of a continuous random variable might not be continuous.

See the example on the next page.

Example 1

Consider a continuous random variable X with the PDF

$$f(x) = \begin{cases} cx & \text{if } 0 \leq x \leq 1 \\ c & \text{if } 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$
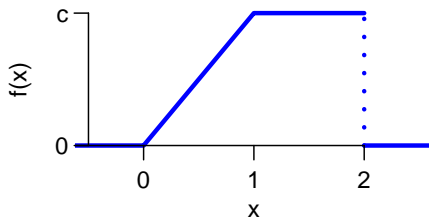

▶ Note $f(x)$ is not continuous at $x = 2$

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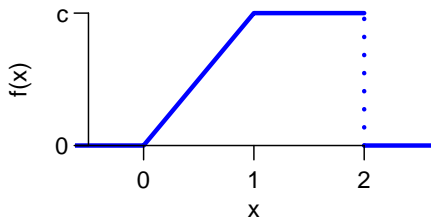

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▶ What is the value of c?

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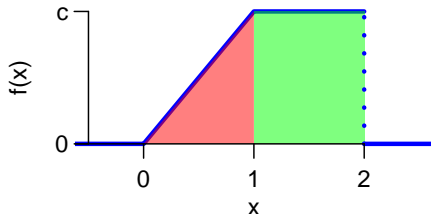
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▶ What is the value of c?

Total Area = Red + Green
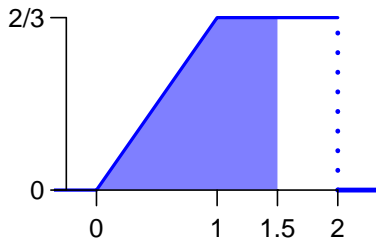$$= \frac{1 \cdot c}{2} + 1 \cdot c = \frac{3}{2}c = 1$$
$$\Rightarrow \quad c = \frac{2}{3}$$
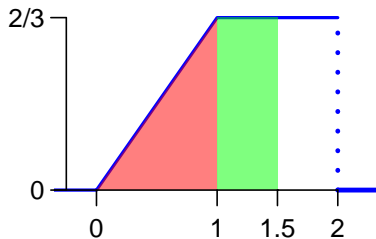
Example 1 (Cont'd)

What is $P(X \leq 1.5)$?

$$P(X \leq 1.5) =$$

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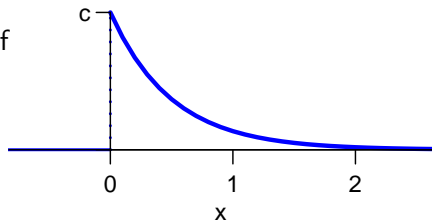

$$P(X \leq 1.5) =$$

$$= \text{Red} + \text{Green}$$

$$= \frac{1 \cdot (2/3)}{2} + (0.5)\frac{2}{3} = \frac{2}{3}$$

Example 2

Suppose the lifetime T (in days) of a certain type of batteries has the PDF below (plotted on the right).

$$f(t) = ce^{-2t}, \quad t > 0.$$

▶ Find the value of c so that $f(t)$ is a legitimate PDF.

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$$\int_{-\infty}^{\infty} f(t)dt = \int_{0}^{\infty} ce^{-2t}dt = -\frac{c}{2}e^{-2t}\Big|_{t=0}^{t=\infty} = \frac{c}{2} - 0 = 1$$

So $c = 2$!

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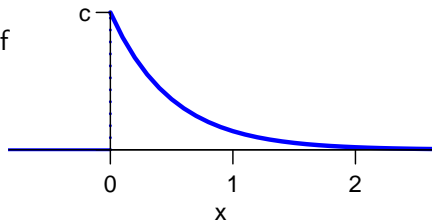
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▶ Observe that $f(0) = 2e^0 = 2 > 1$!?!
Can a PDF $f(x)$ exceed 1?

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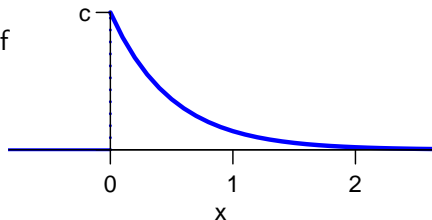▶ Find the value of c so that $f(t)$ is a legitimate PDF.

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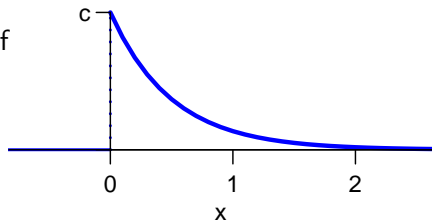
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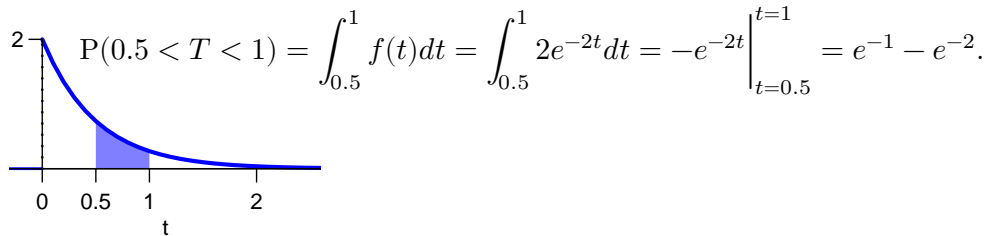Yes, the PDF $f(x)$ itself is not a probability.

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Example 2 (Cont'd)

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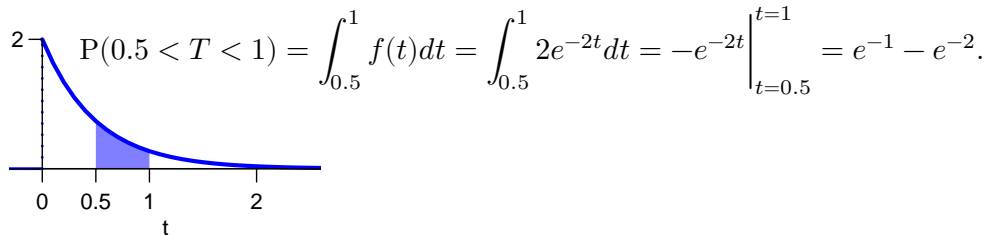
What is the chance that the battery last over one day, $P(T > 1)$?

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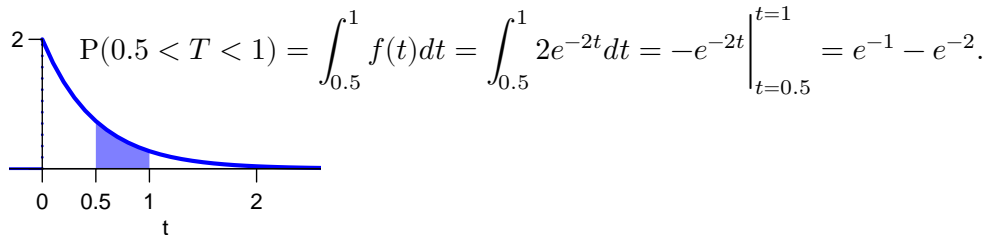
$$P(0.5 < T < 1) = \int_{0.5}^{1} f(t)dt = \int_{0.5}^{1} 2e^{-2t}dt = -e^{-2t}\Big|_{t=0.5}^{t=1} = e^{-1} - e^{-2}.$$

What is the chance that the battery last over one day, $P(T > 1)$?

$$P(T > 1) = \int_{1}^{\infty} f(x)dx = \int_{1}^{\infty} 2e^{-2x}dx = -e^{-2x}\Big|_{t=1}^{t=\infty} = e^{-2}.$$
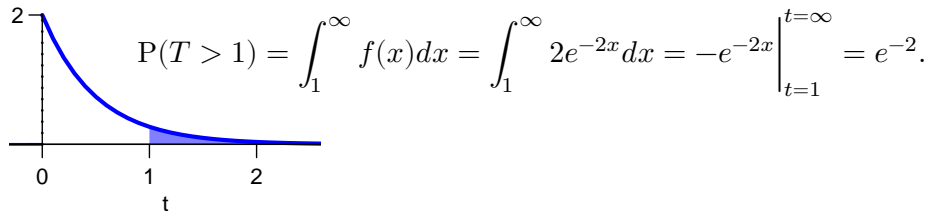
Cumulative Distribution Function (CDF)

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For any random variable X, its *cumulative distribution function* (*CDF*) is the function defined by

$$F(x) = F_X(x) = \mathrm{P}(X \leq x).$$

One can get the CDF of a random variable by **integrating** its PDF:

$$F(x) = \int_{-\infty}^{x} f(u)\, du$$

PDF

$f(x)$

$F(a)$ = shaded area

0.2

0.1

0.0

a

x

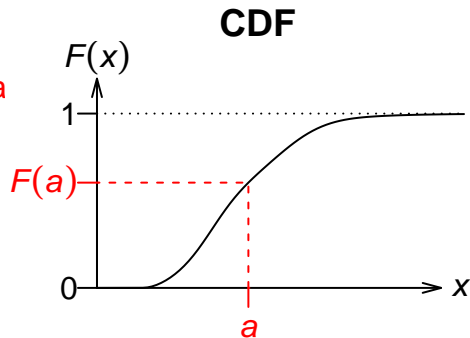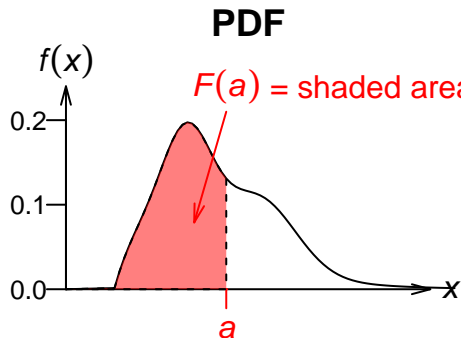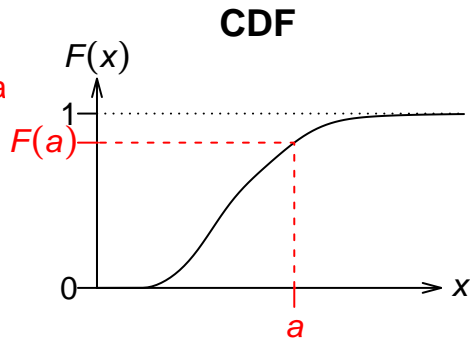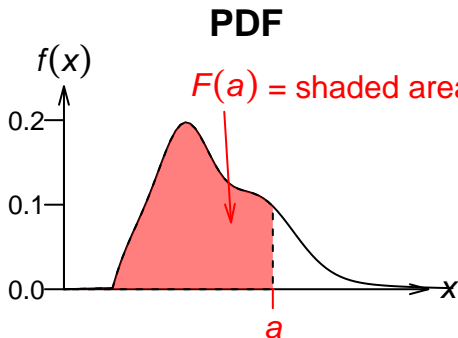**CDF**

$F(x)$

1

$F(a)$

0

a

x

Cumulative Distribution Function (CDF)

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Example 1 (CDF)

$$f(x) = \begin{cases} 2x/3 & \text{if } 0 \le x \le 1 \\ 2/3 & \text{if } 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$
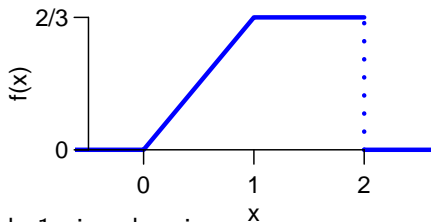
Let's find the CDF $F(x)$ for the density in Example 1 piece by piece.

▶ For $x < 0$, $F(x) = \int_{-\infty}^{x} f(u)du = 0$ since $f(u) = 0$ for $u < 0$.

▶ For $0 \le x < 1$,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)du$$

$$= \text{shaded area on the right}$$

$$= \frac{x \cdot (2x/3)}{2} = \frac{x^2}{3}$$
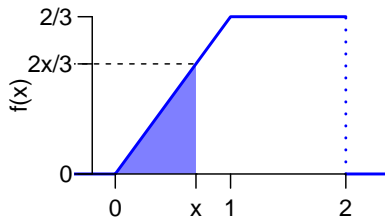
For $1 \le x \le 2$,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)du$$
$$= \text{Red} + \text{Green}$$
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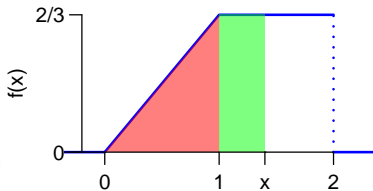
For $x > 2$, $F(x) = \int_{-\infty}^{x} f(u)du = 1$ since the entire area is included.

To sum up, the CDF is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{3}x^2 & \text{if } 0 \le x \le 1 \\ \frac{1}{3} + \frac{2}{3}(x-1) & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$
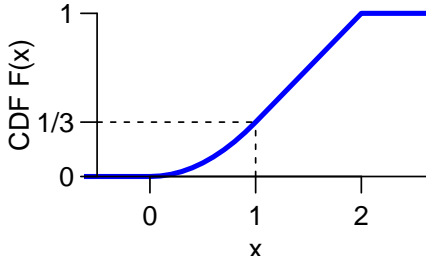
Example 2 (CDF)

Recall the PDF for the lifetime T (in days) of a certain type of batteries is

$$f(t) = 2e^{-2t}, \quad t > 0$$

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The CDF $F(t)$ is

$$F(t) = \begin{cases} 0 & \text{if } t < 0 \\ \int_{-\infty}^{t} f(x)dx = \int_{0}^{t} 2e^{-2u}du = -e^{-2u}\Big|_{0}^{t} = 1 - e^{-2t} & \text{for } t \geq 0 \end{cases}$$
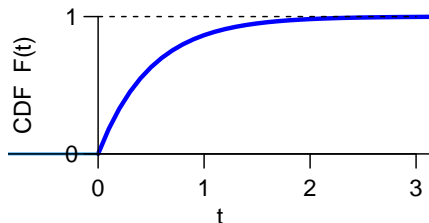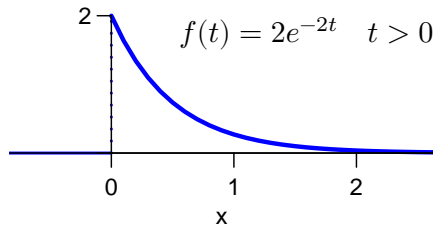
Obtaining the PDF from the CDF

The PDF can be obtained from the CDF by differentiation.

$$f(x) = \frac{d}{dx}F(x).$$

Example 1

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{3}x^2 & \text{if } 0 \leq x \leq 1 \\ \frac{1}{3} + \frac{2}{3}(x-1) & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } x > 2 \end{cases} \Rightarrow \frac{d}{dx}F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{2}{3}x & \text{if } 0 \leq x \leq 1 \\ \frac{2}{3} & \text{if } 1 \leq x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$

Observe $\frac{d}{dx}F(x)$ is exactly the PDF $f(x)$.

Example 2. For the CDF of the battery life distribution

$$F(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-2t} & \text{if } t \geq 0 \end{cases} \Rightarrow \frac{d}{dx}F(x) = \begin{cases} 0 & \text{if } t < 0 \\ 2e^{-2t} & \text{if } t \geq 0 \end{cases}$$

Computing Probabilities Using the CDF

Let X be a continuous rv with PDF $f(x)$ and CDF $F(x)$. Then for any number a,

$$P(X > a) = 1 - F(a)$$

and for any two numbers a and b with $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a)$$
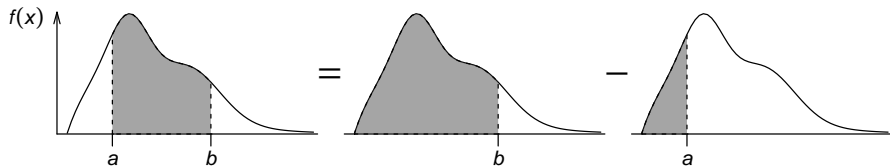
Recall in **Example 2**, we computed $P(0.5 < T < 1)$ by integrating the PDF. We can also compute it using the CDF, $F(t) = 1 - e^{-2t}$, $t > 0$.

$$P(0.5 < T < 1) = F(1) - F(0.5) = (1 - e^{-2}) - (1 - e^{-1}) = e^{-1} - e^{-2}$$

which agrees with our prior calculation.

CDFs for Discrete Random Variables

CDFs $F(x) = \mathrm{P}(X \leq x)$ are also defined for discrete random variables.
Below are the PMF and CDF for Binomial($n = 10,\ p = 0.3$).
PMF:

$$\mathrm{P}(x) = \mathrm{P}(X = x)$$
$$= \binom{n}{x} p^x (1-p)^{n-x}$$
for $x = 1, 2, \dots, n.$

CDF:
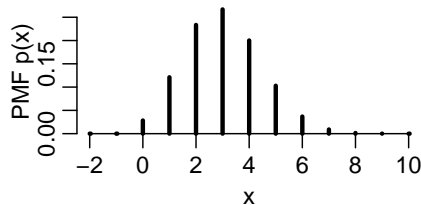
$$F(x) = \mathrm{P}(X \leq x)$$
$$= \sum_{i:0 \leq i \leq x} \binom{n}{i} p^i (1-p)^{n-i}$$
for $-\infty < x < \infty$
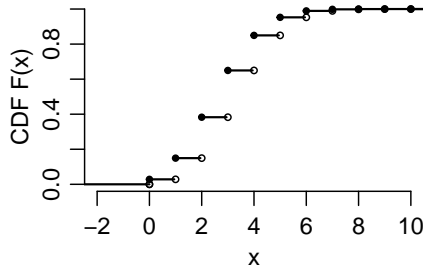
CDFs for Geometric Random Variables

Below are the PMF and CDF for Geometric($p = 0.3$):

PMF:

$$p(x) = \mathrm{P}(X = x)$$
$$= (1-p)^{x-1}p$$
$$\text{for } x = 1, 2, ...$$
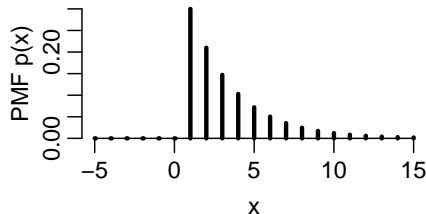
CDF:

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - (1-p)^{\lfloor x \rfloor} & \text{if } x > 1 \end{cases}$$

where $\lfloor x \rfloor$ is the integer part of x.
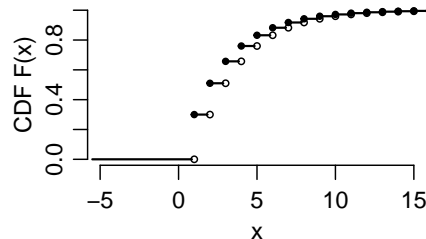
Note the CDF of a discrete r.v. is a discontinuous but right-continuous step function.

Summary: Properties of CDFs

▶ The CDF $F(x) = \mathrm{P}(X \le x)$ is a probability $\Rightarrow$ It must be between 0 and 1.

$$0 \le F(x) \le 1$$

▶ CDFs are always non-decreasing. For $a < b$,

$$F(b) - F(a) = \mathrm{P}(X \le b) - \mathrm{P}(X \le a) = \mathrm{P}(a < X \le b) \ge 0$$

▶ The CDF of a continuous r.v. must be continuous. As $\delta \to 0$

$$F(x + \delta) - F(x) = \int_x^{x+\delta} f(u)du \to 0$$

▶ The CDF of a discrete r.v. is discontinuous but right-continuous step function.

Common Continuous Distributions

Uniform Distribution

A random variable is said to be *uniform* over the interval $[a, b]$ if its density is constant over the interval $[a, b]$,

PDF:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$
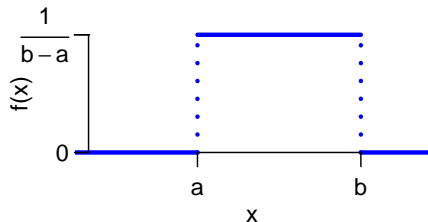
CDF:

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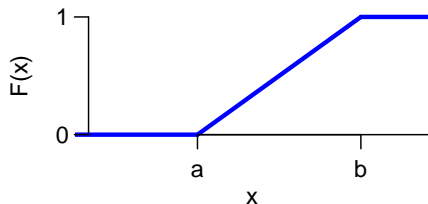
Exponential Distribution

A random variable X is said have an *exponential distribution* with *rate* λ, denoted as $X \sim \text{Exp}(\lambda)$, if its PDF is

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for } x \geq 0.$$

CDF:

$$F(x) = \int_{t=\infty}^{x} f(t) \, dt = \int_{t=0}^{x} \lambda e^{-\lambda t} \, dt$$
$$= \left[-e^{-\lambda t}\right]_{t=0}^{x} = 1 - e^{-\lambda x}.$$
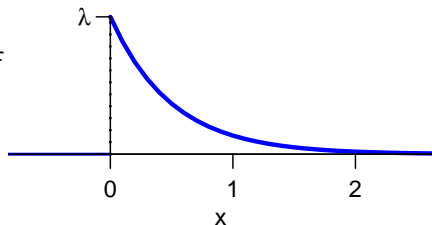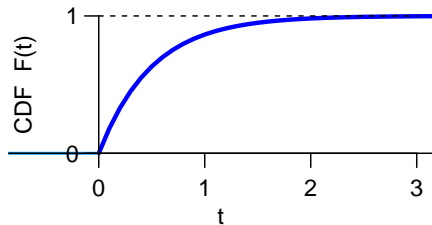
The Exponential Distribution is Memoryless

$$P(X > t + s \mid X > t) = P(X > s)$$

Proof.

$$
\begin{aligned}
P(X > t + s \mid X > t) &= \frac{P(X > t + s \cap X > t)}{P(X > t)} \\
&= \frac{P(X > t + s)}{P(X > t)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)
\end{aligned}
$$

Implication. If the lifetime of batteries has an Exponential distribution, then a used battery is as good as a new one, as long as it's not dead!

Gamma Distribution

Gamma(α, λ) distribution with the "shape" parameter $\alpha > 0$ and "rate" parameter $\lambda > 0$ has the PDF:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \text{ for } x \geq 0$$

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where

$$\Gamma(\alpha) = \int_{z=0}^{\infty} z^{\alpha-1} e^{-z} dz$$

is a normalizing constant (so that density integrates to 1).
Note:$\Gamma(k) = (k-1)!$ for integers $k \geq 1$.

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Note:$\Gamma(k) = (k-1)!$ for integers $k \geq 1$.

▶ Gamma$(\alpha = 1, \lambda) = \text{Exp}(\lambda)$
▶ Note: the textbook calls λ the "scale" but this does not agree with standard terminology.

Normal Distributions

A random variable X is said to have a normal distribution (aka. Gaussian distributions) with a mean μ, and an standard deviation (SD) σ denoted as

$$X \sim N(\mu, \sigma^2)$$

if its PDF is

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$
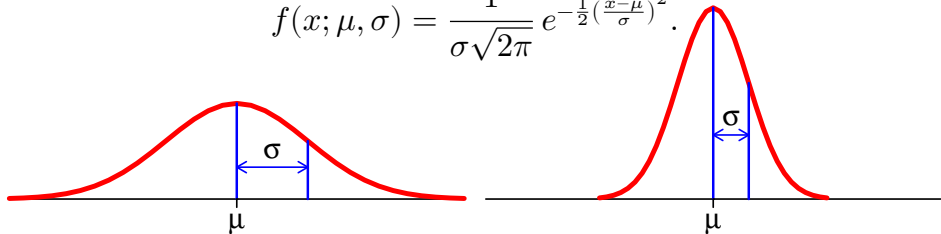
The density curve is bell-shaped and symmetric about its mean μ.

A normal distribution with $\mu = 0$, and $\sigma = 1$ is called the standard normal distribution, denoted as $N(0, 1)$

PDF & CDF of the Standard Normal Distribution

The PDF & CDF of the **standard normal** $N(0,1)$ are respectively

PDF: $\phi(z) = \dfrac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$,

$$-\infty < z < \infty$$

CDF: $\Phi(z) = \displaystyle\int_{-\infty}^{z} \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}\, du$,

$$-\infty < z < \infty.$$

Shaded Area = Φ(z)
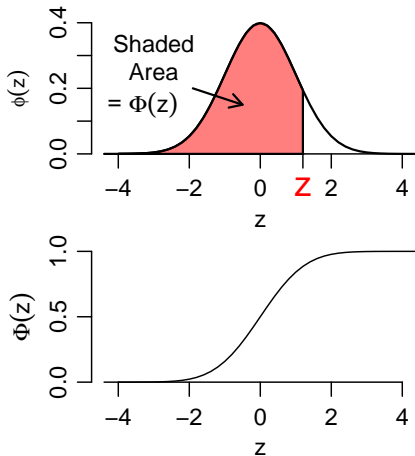
▶ No close-form formula for the CDF $\Phi(z)$
▶ The normal probability table (on p.A7 in Textbook) gives the values of the CDF $\Phi(z)$ for different z's

Beta Distributions

The random variable U is said to have a *beta distribution* with parameters α, β if its density is given by

$$f(u) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1}, \quad \text{for } 0 \leq u \leq 1,$$

denoted as $U \sim \text{BETA}(\alpha, \beta)$.

▶ $\text{BETA}(\alpha = 1, \beta = 1)$ is $\text{Uniform}(0, 1)$

Functions/Transformation of a Random Variable

Functions/Transformation of a Random Variable

If X is a continuous random variable with density $f_X(x)$, and $Y = g(X)$, what is the distribution of Y?

The general method is to find the CDF for $Y = g(X)$ first.

Ex1. Suppose $X \sim \text{Exp}(\lambda)$. Find the PDF for $Y = e^X$.

Functions/Transformation of a Random Variable

If X is a continuous random variable with density $f_X(x)$, and $Y = g(X)$, what is the distribution of Y?

The general method is to find the CDF for $Y = g(X)$ first.

Ex1. Suppose $X \sim \text{Exp}(\lambda)$. Find the PDF for $Y = e^X$.

Sol. First, recall the CDF for $\text{Exp}(\lambda)$ is

$$F_X(x) = 1 - e^{-\lambda x} \quad \text{for } x > 0, \text{ and } 0 \text{ otherwise.}$$

We can find the CDF for $Y = e^X$ as follows.

$$\begin{aligned} F_Y(y) = \text{P}(Y \leq y) = \text{P}(e^X \leq y) &= \text{P}(X \leq \log(y)) \\ &= F_X(\log(y)) = 1 - e^{-\lambda \log(y)} = 1 - y^{-\lambda} \quad \text{for } y \geq 1. \end{aligned}$$

We then differentiate the CDF to obtain the PDF.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \lambda y^{-\lambda - 1} & \text{for } y \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Ex2. Suppose $X \sim N(0, 1)$. Find the PDF for $Y = X^2$.

Ex2. Suppose $X \sim N(0,1)$. Find the PDF for $Y = X^2$.

Sol. We can find the CDF for $Y = X^2$ as follows.

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$
$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

We then differentiate the CDF to obtain the PDF using the chain rule. Recall the CDF for N(0,1) is $\Phi(x)$, and $\Phi'(x) = \phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \phi(\sqrt{y})\frac{d}{dy}\sqrt{y} - \phi(-\sqrt{y})\frac{d}{dy}(-\sqrt{y})$$
$$= \frac{1}{2\sqrt{y}}(\phi(\sqrt{y}) + \phi(-\sqrt{y}))$$
$$= \frac{1}{\sqrt{y}}\phi(\sqrt{y}) \quad \text{since } \phi(x) = \phi(-x)$$
$$= \frac{1}{\sqrt{2\pi y}}e^{-y/2} \quad \text{for } y \geq 0.$$

Linear Transformation of Random Variables

Suppose X is a continuous r.v. with the PDF $f_X(x)$. The PDF for $Y = aX + b$ is

$$f_Y = \frac{1}{|a|} f_X \left(\frac{y-b}{a} \right), \quad \text{if } a \neq 0.$$

Proof. Denote the CDF for X as $F_X(x)$. The CDF for $Y = aX + b$ would be

$$F_Y(y) = \mathrm{P}(\underbrace{aX + b}_{=Y} \leq y) = \begin{cases} \mathrm{P}\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right) & \text{if } a > 0 \\ \mathrm{P}\left(X \geq \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right) & \text{if } a < 0. \end{cases}$$

We then differentiate the CDF to obtain the PDF using the chain rule.
Recall $F_X'(x) = f_X(x)$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} f_X\left(\frac{y-b}{a}\right) \frac{d}{dy}\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & \text{if } a > 0 \\ -f_X\left(\frac{y-b}{a}\right) \frac{d}{dy}\left(\frac{y-b}{a}\right) = \frac{1}{-a} f_X\left(\frac{y-b}{a}\right) & \text{if } a < 0 \end{cases}$$

Linear Transformation of Random Variables — Examples

▶ If $Z \sim N(0,1)$, the PDF for $X = \sigma Z + \mu$ with $\sigma > 0$ is

$$f_X(x) = \frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \Rightarrow \boxed{X \sim N(\mu, \sigma^2)}$$

▶ If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ with $a \neq 0$,

$$\frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-b-a\mu}{a\sigma}\right)^2\right] \Rightarrow \boxed{Y \sim N(a\mu + b, a^2\sigma^2)}$$

▶ If $X \sim \mathsf{Exp}(\lambda)$ and $Y = aX$ for $a > 0$, then their PDFs are

$$f_X(x) = \lambda e^{-\lambda x}, \quad \text{for } x \geq 0,$$
$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y}{a}\right) = \frac{\lambda}{a} e^{-\lambda x/a}, \quad \text{for } x \geq 0 \quad \Rightarrow \boxed{Y \sim \mathsf{Exp}(\lambda/a)}.$$

▶ If $X \sim \mathsf{Gamma}(\alpha, \lambda)$ and $Y = aX$ for $a > 0$, then $Y \sim \mathsf{Gamma}(\alpha, \lambda/a)$ (HW)

Differentiable & Strictly Monotone Transformations

Suppose f_X is the PDF of X and $g()$ is differentiable & strictly monotone.

$\Rightarrow Y = g(X)$ is a continuous r.v. with PDF

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|.$$

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Suppose f_X is the PDF of X and $g()$ is differentiable & strictly monotone.

$\Rightarrow Y = g(X)$ is a continuous r.v. with PDF

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|.$$

Proof. The CDF of Y is

$$
\begin{aligned}
F_Y(y) &= \mathrm{P}(Y \le y) = \mathrm{P}(g(X) \le y) \\
&= \begin{cases}
\mathrm{P}(X \le g^{-1}(y)) = F_X(g^{-1}(y)) & \text{if } g \text{ is increasing} \\
\mathrm{P}(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y)) & \text{if } g \text{ is decreasing}
\end{cases}
\end{aligned}
$$

We then differentiate the CDF to obtain the PDF using the chain rule.

$$
f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases}
f_X(g^{-1}(y)) \dfrac{d}{dy} g^{-1}(y) & \text{if } g \text{ is increasing} \\[2mm]
-f_X(g^{-1}(y)) \dfrac{d}{dy} g^{-1}(y) & \text{if } g \text{ is decreasing}
\end{cases}
$$

Example (Random Laser Beams)

A laser device shoots laser beams at a random angle Θ uniform on $(-\pi/2, \pi/2)$ to a wall that is 1 unit away from the device. Find the PDF for the location $Y = \tan(\Theta)$ where the laser beam points to on the wall.
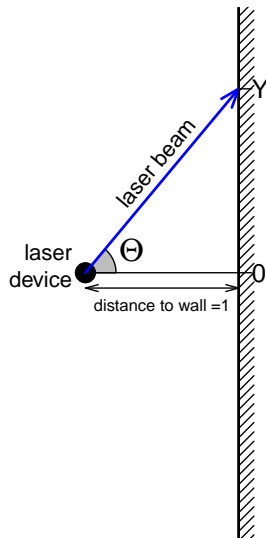
Example (Random Laser Beams)

The PDF for $\Theta \sim \mathsf{Uniform}(-\pi/2, \pi/2)$ is

$$f_\Theta(\theta) = \frac{1}{\pi}, \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

For $Y = g(\Theta) = \tan(\Theta)$, $g^{-1}(y) = \arctan(y)$, its derivative is

$$\frac{d}{dy} \arctan(y) = \frac{1}{1 + y^2}.$$

The PDF for $Y = g(\Theta) = \tan(\Theta)$ is

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\pi} \frac{1}{1 + y^2}, \; -\infty < y < \infty.$$

The distribution with the PDF above is called the *Cauchy distribution*.

laser beam

laser device

Θ

Y

0

distance to wall =1
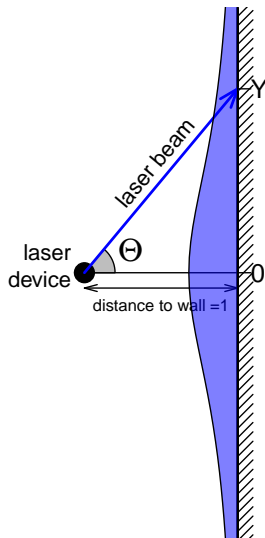
Caution

Watch out that the formula $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$ only works for strictly monotone transformation.

For other cases, like $g(x) = |x|$ or $g(x) = x^2$, use the CDF method.

Ex2 Revisit. For $Y = X^2$ where $X \sim N(0, 1)$,

$$g(x) = x^2, \quad g^{-1}(y) = \sqrt{y}, \quad \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Applying the formula $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$, we'll obtain the incorrect PDF

$$f_Y(y) = \phi(\sqrt{y}) \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{2\pi y}} e^{-y/2} \quad \text{for } y \geq 0,$$

rather than the correct PDF

$$f_Y(y) = \phi(\sqrt{y}) \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad \text{for } y \geq 0.$$

Transforming to Uniform

Suppose X is a continuous r.v. with CDF F, where

▶ F is strictly increasing on some interval I,
▶ $F = 0$ to the left of I, and $F = 1$ to the right of I.
▶ I may be a bounded interval or an unbounded interval such as the whole real line.

then $F^{-1}(x)$ is then well defined for $x \in I$.

Let $Y = F(X)$. What is the distribution of Y?

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then $F^{-1}(x)$ is then well defined for $x \in I$.

Let $Y = F(X)$. What is the distribution of Y?

$$
\begin{aligned}
F_Y(y) = P(Y \le y) &= P(F(X) \le y) \\
&= P(X \le F^{-1}(y)) \\
&= F(F^{-1}(y)) \\
&= y, \quad \text{for } 0 < y < 1.
\end{aligned}
$$

This means that $Y \sim \text{Uniform}(0, 1)$ (since its CDF is the Uniform CDF)

Example

Recall the CDF For $X \sim \text{Exp}(\lambda)$ is

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0.$$

Then $1 - \exp(-\lambda X) \sim \text{Uniform(0,1)}$, which also implies

$$\exp(-\lambda X) \sim \text{Uniform}(0, 1)$$

since U and $1 - U$ have identical distribution if $U \sim \text{Uniform}(0, 1)$.

How to Generate a Random Variable with a Given CDF from Uniform?

Let F be the CDF for some continuous distribution satisfying the conditions below

▶ F is strictly increasing on some interval I,
▶ $F = 0$ to the left of I, and $F = 1$ to the right of I.
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and let $U \sim \mathsf{Uniform}(0, 1)$.
What is the distribution of $X = F^{-1}(U)$?

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and let $U \sim \mathsf{Uniform}(0, 1)$.
What is the distribution of $X = F^{-1}(U)$?

$$\mathrm{P}(X \leq x) = \mathrm{P}(F^{-1}(U) \leq x) = \mathrm{P}(U \leq F(x)) = F(x)$$

where the last equality holds since U is $\mathsf{Uniform}[0, 1]$.

This means that X has CDF equal to F.