STAT 24400 Lecture 2
Conditional Probabilities, Independence, Bayes Rule

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Outline

Coverage: Section 1.5-1.6 of Rice's Book

- 1.5 Conditional Probability
  - Definition of Conditional Probability
  - Multiplication Law
  - Law Of Total Probability
  - Bayes' Rule
- 1.6 Independence

Conditional Probability

Example – Conditional Probability

A pair of dice is rolled. The sample space Ω is

$$\left\{\begin{array}{cccccc} \boxed{(1,1)} & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & \boxed{(2,2)} & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & \boxed{(3,3)} & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & \boxed{(4,4)} & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & \boxed{(5,5)} & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & \boxed{(6,6)} \end{array}\right\}$$

- What is the probability of **doubles** $=$ same number on both dice?

- If total is known to be $10+$, what is the probability of getting a double? $\frac{2}{6} = \frac{1}{3}$

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- What is the probability of **doubles** = same number on both dice?
  double =
  $\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

  $$\mathrm{P}(\text{double}) = \frac{\#(\text{double})}{\#(\Omega)} = \frac{6}{36}.$$

- If total is known to be $10+$, what is the probability of getting a double? $\frac{2}{6} = \frac{1}{3}$

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Conditional Probabilities

The **conditional probability of A happens <u>given</u> that B has occurred** is denoted

$$P(A \mid B),$$

and read as the probability of "*A given B*."

For the example on the previous slide, let

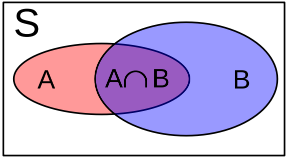
$$\begin{cases} A = \text{getting a double}, \\ B = \text{total is } 10+, \end{cases} \qquad \text{we have } P(A \mid B) = \frac{2}{6} \neq P(A) = \frac{6}{36}.$$

The given info (total is 10+) changed (restricted) the sample space.

$$\Omega = \left\{ \begin{matrix} \boxed{(1,1)} & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & \boxed{(2,2)} & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & \boxed{(3,3)} & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & \boxed{(4,4)} & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & \boxed{(5,5)} & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & \boxed{(6,6)} \end{matrix} \right\} \to \underbrace{\left\{ \begin{matrix} & & (4,6) \\ & \boxed{(5,5)} & (5,6) \\ (6,4) & (6,5) & \boxed{(6,6)} \end{matrix} \right\}}_{\text{restricted sample space}}$$

Definition of Conditional Probability

The **conditional probability** $\mathrm{P}(A \mid B)$ is defined as as



$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \quad \text{if } \mathrm{P}(B) > 0.$$

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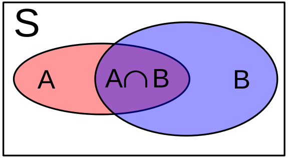


Example.

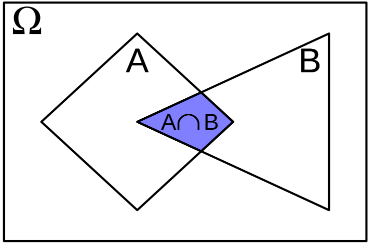
- $\mathrm{P}(\text{total is } 10+) = 6/36$
- $\mathrm{P}(\text{double } \cap \text{ total is } 10+) = \mathrm{P}(\{(5,5) \text{ or } (6,6)\}) = 2/36$

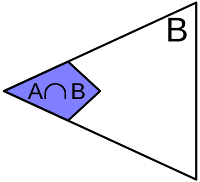
By definition of conditional probability,

$$\mathrm{P}(\text{double} \mid \text{total is } 10+) = \frac{\mathrm{P}(\text{double} \cap \text{total is } 10+)}{\mathrm{P}(\text{total is } 10+)} = \frac{2/36}{6/36} = \frac{2}{6}.$$

$\mathrm{P}(A \mid B)$ v.s. $\mathrm{P}(A \cap B)$

- $\mathrm{P}(A \cap B)$ is the probability that A and B both occur (we are unsure whether B will occur)
- $\mathrm{P}(A \mid B)$ is the probability that A occurs given that B has occurred
- $\mathrm{P}(A \cap B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(\Omega)} \to$ The sample space is Ω
- $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \to$ The sample space is B.





Example — Red or Black

You have 3 cards.

- Card 1 is Red on both sides, $\boxed{R_1}\boxed{R_2}$,
- Card 2 is Black on both sides, $\boxed{B_1}\boxed{B_2}$,
- Card 3 is Red on one side and Black on the other, $\boxed{R_3}\boxed{B_3}$,

After shuffling the cards behind your back, you select one of them at random and place it on your desk with your hand covering it. Upon lifting your hand, you observe that the face showing is red. Which of the following is the correct conditional probability

$$\mathrm{P}(\text{the other side is Red} \mid \text{the up side is Red})?$$

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1. Sample space $= \{\text{Card 1, Card 2, Card 3}\}$. Given the face is Red, it can only be Cards 1 or 3 and their flip sides are Red and Black, $\Rightarrow$ Answer $= 1/2$.

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1. Sample space $= \{$Card 1, Card 2, Card 3$\}$. Given the face is Red, it can only be Cards 1 or 3 and their flip sides are Red and Black, $\Rightarrow$ Answer $= 1/2$.
2. Sample space $= \{R_1, R_2, B_1, B_2, R_3, B_3\}$. Given the face is Red, it could be R_1, R_2, or R_3 and their flip sides are R_2, R_1 and B_3, $\Rightarrow$ Answer $= 2/3$.

Example — Red or Black (Cont'd)

By the definition of conditional probability

$$\mathrm{P}(\text{the other side is Red} \mid \text{the up side is Red}) = \frac{\mathrm{P}(\text{both sides are Red})}{\mathrm{P}(\text{the up side is Red})}.$$

Which sample space allows us to compute $\mathrm{P}(\text{the up side is Red})$?

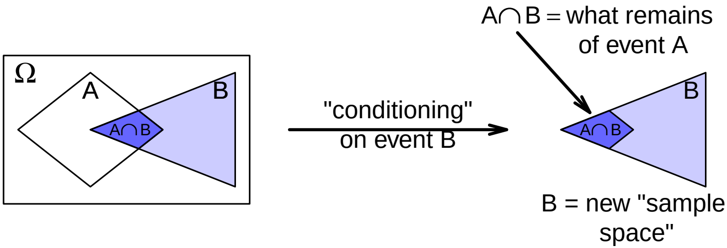
- $\{$Card 1, Card 2, Card 3$\}$
- $\{R_1, R_2, B_1, B_2, R_3, B_3\}$

Calculation of Conditional Probabilities

Do NOT always calculate conditional probabilities by the definition.

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Sometimes, it's more straightforward to find $\mathrm{P}(A \mid B)$ by thinking about how B has changed the sample space instead of finding $\mathrm{P}(A \cap B)$, $\mathrm{P}(B)$ and their ratio.



Ex. A deck of cards is well-shuffled and two cards are drawn w/o replacement. Find the probability that second card is a King given the first card is a King.

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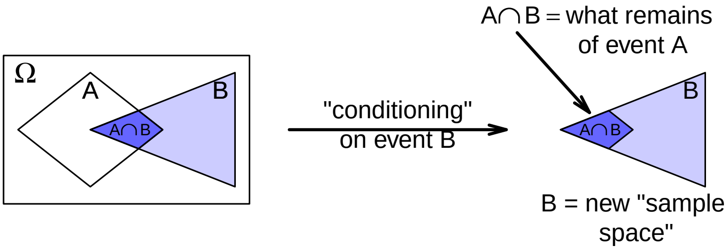

Ex. A deck of cards is well-shuffled and two cards are drawn w/o replacement. Find the probability that second card is a King given the first card is a King. $3/51$

Multiplication Law

Multiplication Law

The definition of conditional probability

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$$

can be used the other way around. Multiplying both sides by $\mathrm{P}(A)$, we get the *Multiplication Law*:

$$\boxed{\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \mid A)}$$

If we want $\mathrm{P}(A \cap B)$, and both $\mathrm{P}(A)$, $\mathrm{P}(B \mid A)$ are known or are easy to compute, we can use the Multiplication Law.

Example: Multiplication Law

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that both cards are Kings?

Solution. Let

$$\begin{aligned} A = & \text{ 1st card is a King,} \\ B = & \text{ 2nd card is a King.} \end{aligned}$$

- $\mathrm{P}(A) = \mathrm{P}(\text{the 1st card is a King}) = \qquad .$
- Given that the 1st card is a King, the conditional probability that the 2nd card is a King $= ?$.
- So the probability that both cards are Kings $= ?$

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- $\mathrm{P}(A) = \mathrm{P}(\text{the 1st card is a King}) = 4/52$.
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- $\mathrm{P}(A) = \mathrm{P}(\text{the 1st card is a King}) = 4/52$.
- Given that the 1st card is a King, the conditional probability that the 2nd card is a King $=$? $\mathrm{P}(B \mid A) = \dfrac{3}{51}$.
- So the probability that both cards are Kings $=$?

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \mid A) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221} \approx 0.0045.$$

Multiplication Law for Several Events

$$\begin{aligned}
\mathrm{P}(ABC) &= \mathrm{P}(A) \cdot \mathrm{P}(B \mid A) \cdot \mathrm{P}(C \mid AB) \\
\mathrm{P}(ABCD) &= \mathrm{P}(A) \cdot \mathrm{P}(B \mid A) \cdot \mathrm{P}(C \mid AB) \cdot \mathrm{P}(D \mid ABC) \\
\mathrm{P}(ABCDE) &= \mathrm{P}(A) \cdot \mathrm{P}(B \mid A) \cdot \mathrm{P}(C \mid AB) \cdot \mathrm{P}(D \mid ABC) \cdot \mathrm{P}(E \mid ABCD)
\end{aligned}$$

and so on

Example: General Multiplication Rule for Several Events

Five cards are dealt from a deck of well-shuffled card. What is the chance that none of them are hearts $\heartsuit$?

Sol. Let A_i be the event that the ith card dealt is not a $\heartsuit$.

- $\mathrm{P}(A_1) = \mathrm{P}(\text{1st card is not a } \heartsuit) = 39/52$

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- Given the 1st card is not a $\heartsuit$, the conditional probability that the 2nd is not a $\heartsuit = \mathrm{P}(A_2 \mid A_1) = \dfrac{38}{51}$.

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- Given the 1st card is not a ♡, the conditional probability that the 2nd is not a ♡ $= \mathrm{P}(A_2 \mid A_1) = \dfrac{38}{51}$.
- Given neither of the first two cards is a ♡, the condition probability that the 3rd is not a ♡ $= \mathrm{P}(A_3 \mid A_1 A_2) = \dfrac{37}{50}$.

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- Given neither of the first two cards is a $\heartsuit$, the condition probability that the 3rd is not a $\heartsuit = \mathrm{P}(A_3 \mid A_1 A_2) = \dfrac{37}{50}$.
- Likewise, $\mathrm{P}(A_4 \mid A_1 A_2 A_3) = \dfrac{36}{49}$, $\mathrm{P}(A_5 \mid A_1 A_2 A_3 A_4) = \dfrac{35}{48}$

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- Likewise, $\mathrm{P}(A_4 \mid A_1 A_2 A_3) = \dfrac{36}{49}$, $\mathrm{P}(A_5 \mid A_1 A_2 A_3 A_4) = \dfrac{35}{48}$
- By the General Multiplication Rule,

$$\mathrm{P}(A_1 A_2 A_3 A_4 A_5) = \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50} \times \frac{36}{49} \times \frac{35}{48} \approx 0.222$$

Law of Total Probability and Bayes' Rule

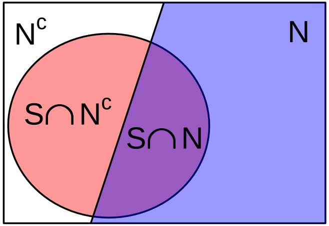
Example – A Nervous Job Applicant

Suppose an job applicant has been invited for an interview.
The probability that

- he is nervous is $\mathrm{P}(N) = 0.7$,
- he succeeds in interview given he is nervous is $\mathrm{P}(S \mid N) = 0.2$,
- he succeeds in interview given he is not nervous is $\mathrm{P}(S \mid N^c) = 0.9$.

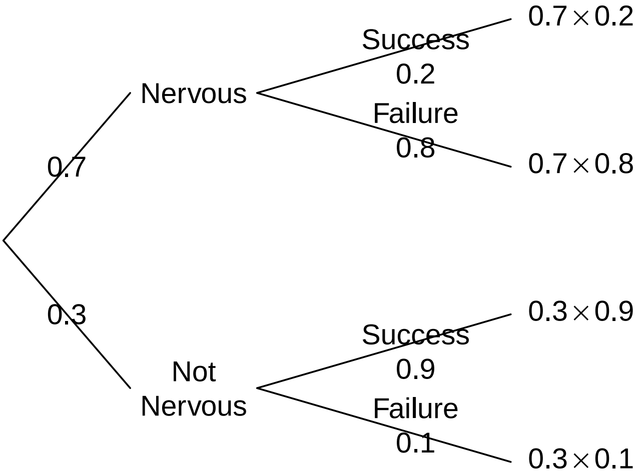
What is the probability that he succeeds in the interview?

$$\begin{aligned}\mathrm{P}(S) &= \mathrm{P}(S \cap N) + \mathrm{P}(S \cap N^c) \\ &= \mathrm{P}(N)\mathrm{P}(S \mid N) + \mathrm{P}(N^c)\mathrm{P}(S \mid N^c) \\ &= 0.7 \times 0.2 + 0.3 \times 0.9 = 0.41.\end{aligned}$$



Tree Diagram for the Nervous Job Applicant Example

Another look at the nervous job applicant example:



Nervous Job Applicant Example Continued

Conversely, given the interview is successful, what is the probability that the job applicant is nervous during the interview?

$$
\begin{aligned}
\mathrm{P}(N \mid S) &= \frac{\mathrm{P}(N \cap S)}{\mathrm{P}(S)} \\
&= \frac{\mathrm{P}(N \cap S)}{0.41} \qquad \left(\begin{array}{c} \text{where } \mathrm{P}(S) = 0.41 \text{ was} \\ \text{found in the previous page} \end{array} \right) \\
&= \frac{\mathrm{P}(N)\mathrm{P}(S \mid N)}{0.41} \qquad \text{since } \mathrm{P}(N \cap S) = \mathrm{P}(N)\mathrm{P}(S \mid N) \\
&= \frac{0.7 \times 0.2}{0.41} = \frac{14}{41} \approx 0.34.
\end{aligned}
$$

Bayes' Rule (or Bayes' Theorem)

The problem in the previous slide is an example of **Bayes' Rule**.

Knowing $\mathrm{P}(B \mid A), \mathrm{P}(B \mid A^c)$, and $\mathrm{P}(A)$, is there a way to know $\mathrm{P}(A \mid B)$?

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&= \frac{\mathrm{P}(A)\mathrm{P}(B \mid A)}{\mathrm{P}(B \cap A) + \mathrm{P}(B \cap A^c)} \quad \text{since } B = (B \cap A) \cup (B \cap A^c)
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&= \frac{\mathrm{P}(A)\mathrm{P}(B \mid A)}{\mathrm{P}(A)\mathrm{P}(B \mid A) + \mathrm{P}(A^c)\mathrm{P}(B \mid A^c)}
\end{aligned}
$$

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- $\mathrm{P}(T- \mid D^c)$ is called the *specificity* of the test

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  and $T-$ denote the event that the test result is *negative*
- $\mathrm{P}(T+ \mid D)$ is called the *sensitivity* of the test
- $\mathrm{P}(T- \mid D^c)$ is called the *specificity* of the test
- Ideally, we hope $\mathrm{P}(T+ \mid D)$ and $\mathrm{P}(T- \mid D^c)$ both equal 1.
  However, medical tests are not perfect.
  They may give false positives and false negatives.

Enzyme Immunoassay Test for HIV

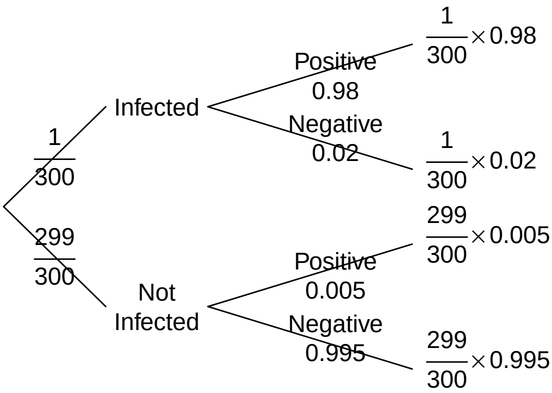
- $\mathrm{P}(T+ \mid D) = 0.98$ (sensitivity - positive for infected)
- $\mathrm{P}(T- \mid D^c) = 0.995$ (specificity - negative for not infected)
- $\mathrm{P}(D) = 1/300$ (prevalence of HIV in USA)

What is the probability that the tested person is infected if the test was positive?

$$\begin{aligned}
\mathrm{P}(D \mid T+) &= \frac{\mathrm{P}(D)\mathrm{P}(T+ \mid D)}{\mathrm{P}(D)\mathrm{P}(T+ \mid D) + \mathrm{P}(D^c)\mathrm{P}(T+ \mid D^c)} \\
&= \frac{1/300 \times 0.98}{(1/300) \times 0.98 + (299/300) \times 0.005} \\
&= 39.6\%
\end{aligned}$$

This test is not confirmatory. Need to confirm by a second test.

Tree Diagram for the HIV Test

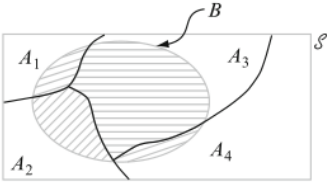


$$\mathrm{P}(D \mid T+) = \frac{(1/300) \times 0.98}{(1/300) \times 0.98 + (299/300) \times 0.005}$$

Bayes' Rule for 3 or More Cases

- The 2 examples above both split the sample space into 2 parts A or A^c (nervous or not nervous, infected or not infected)
- In many cases, we need to calculate $\mathrm{P}(B)$ by splitting it into several parts, using the Law of Total Probability:

Suppose $A_1, A_2, \ldots, A_k$ are disjoint and $A_1 \cup A_2 \cup \cdots \cup A_k = \Omega$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$, then



$$
\begin{aligned}
\mathrm{P}(B) &= \mathrm{P}(B \cap A_1) + \mathrm{P}(B \cap A_2) + \cdots + \mathrm{P}(B \cap A_k) \\
&= \mathrm{P}(A_1)\mathrm{P}(B \mid A_1) + \mathrm{P}(A_2)\mathrm{P}(B \mid A_2) + \cdots + \mathrm{P}(A_k)\mathrm{P}(B \mid A_k).
\end{aligned}
$$

Using the Law of Total Probability, Bayes Rule becomes

$$
\mathrm{P}(A_i \mid B) = \frac{\mathrm{P}(A_i)\mathrm{P}(B \mid A_i)}{\mathrm{P}(A_1)\mathrm{P}(B \mid A_1) + \mathrm{P}(A_2)\mathrm{P}(B \mid A_2) + \cdots + \mathrm{P}(A_K)\mathrm{P}(B \mid A_K)}
$$

Example (Bayes' Rule for 3 Cases)

At a gas station,

- 40% of the customers use regular gas (A_1),
- 35% use mid-grade gas (A_2), and
- 25% use premium gas (A_3).

Moreover,

- of those customers using regular gas, only 30% fill their tanks;
- of those using mid-grade, 60% fill their tanks;
- of those using premium, 50% fill their tanks.

Let B denote the event that the next customer fills the tank.

Example (Bayes' Rule for 3 Cases)

At a gas station,

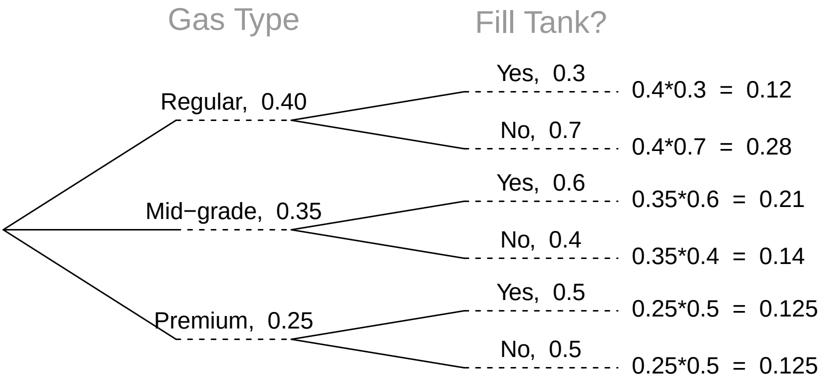
- 40% of the customers use regular gas (A_1),
- 35% use mid-grade gas (A_2), and
- 25% use premium gas (A_3).

Moreover,

- of those customers using regular gas, only 30% fill their tanks; $\mathrm{P}(B \mid A_1) = 0.3$
- of those using mid-grade, 60% fill their tanks; $\mathrm{P}(B \mid A_2) = 0.6$
- of those using premium, 50% fill their tanks. $\mathrm{P}(B \mid A_3) = 0.5$

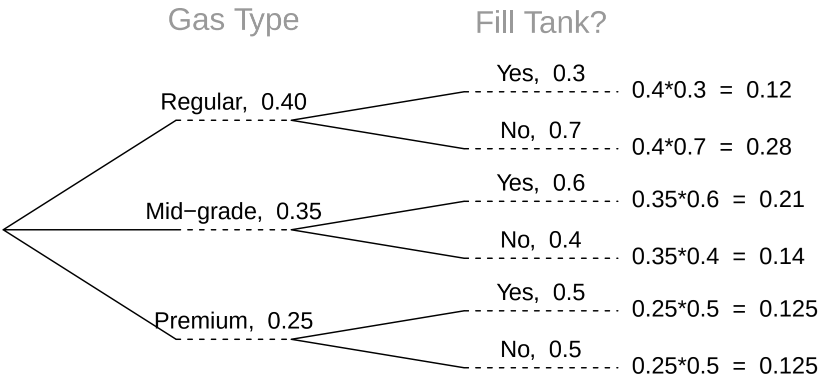
Let B denote the event that the next customer fills the tank.

Gas Station Example — Tree Diagram



Q1: What is the probability that the next customer request premium gas and fill the tank.

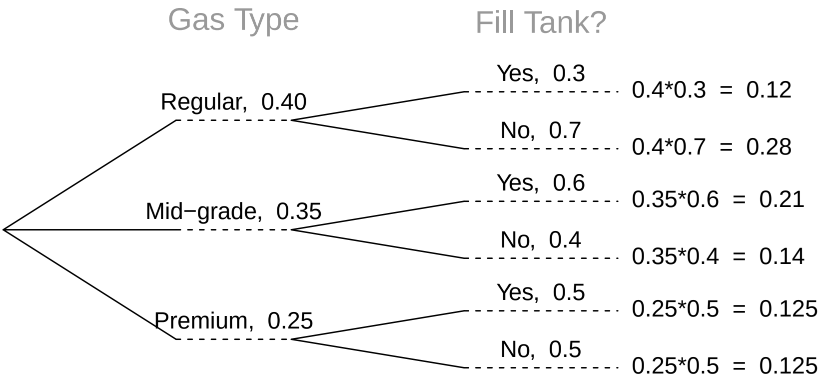
Gas Station Example — Tree Diagram



Q1: What is the probability that the next customer request premium gas and fill the tank.

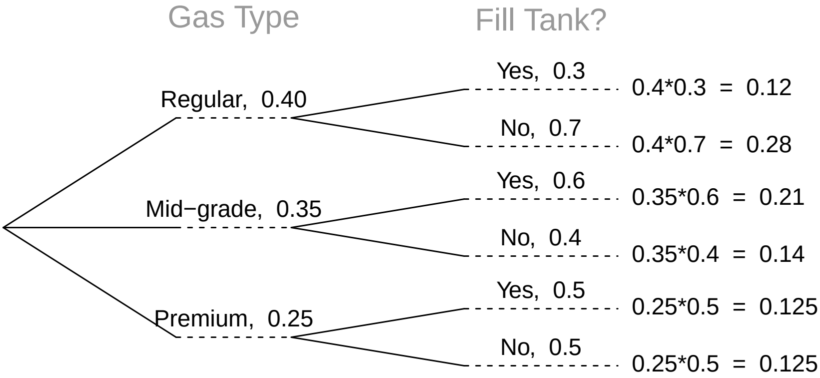
$$\mathrm{P}(A_3 \cap B) = \mathrm{P}(A_3)\mathrm{P}(B \mid A_3) = 0.25 \times 0.5 = 0.125.$$

Gas Station Example — Tree Diagram



Q2: What is the probability that the next customer fills the tank.

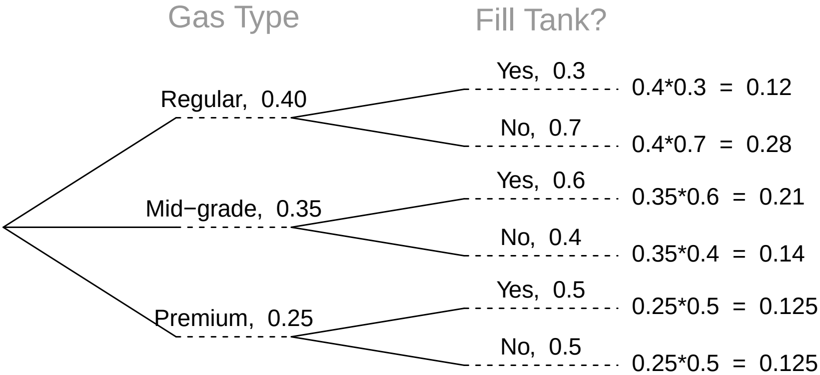
Gas Station Example — Tree Diagram



Q2: What is the probability that the next customer fills the tank.

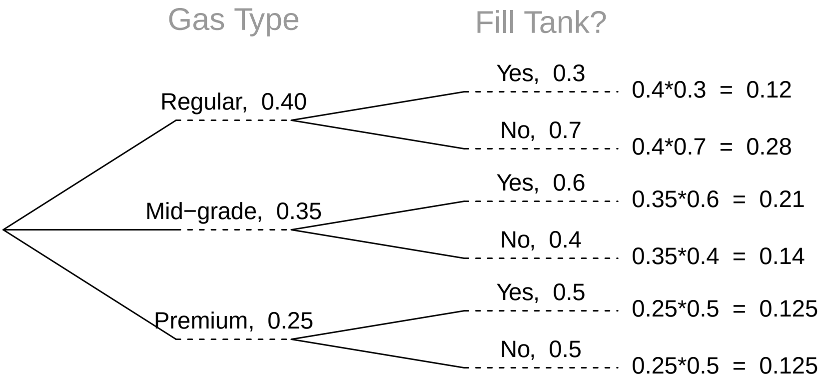
$$\mathrm{P}(B) = \mathrm{P}(A_1)\mathrm{P}(B \mid A_1) + \mathrm{P}(A_2)\mathrm{P}(B \mid A_2) + \mathrm{P}(A_3)\mathrm{P}(B \mid A_3)$$
$$= 0.4 \times 0.3 + 0.35 \times 0.6 + 0.25 \times 0.5 = 0.455$$

Gas Station Example — Tree Diagram



Q3: If the next customer fills the tank, what is the probability that premium gas is requested?

Gas Station Example — Tree Diagram



Q3: If the next customer fills the tank, what is the probability that premium gas is requested?

$$P(A_3 \mid B) = \frac{P(A_3 \cap B)}{P(B)} = \frac{0.125}{0.455} \approx 0.275.$$

Independence

Independence

Two events A and B are said to be independent if any of the following is true

- $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ . . . . B happens doesn't affect how likely A happens
- $\mathrm{P}(A \mid B) = \mathrm{P}(A \mid B^c)$ . . . . How likely A happens is not affected by B happens or not
- $\mathrm{P}(B \mid A) = \mathrm{P}(B)$ . . . . A happens doesn't affect how likely B happens
- $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$

If any of the identities above is true, then all remaining identities will also be true.

Proof of $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ implies $\mathrm{P}(B \mid A) = \mathrm{P}(B)$

$$
\begin{aligned}
\mathrm{P}(B \mid A) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} && \text{definition of conditional prob.} \\
&= \frac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(A)} && \text{Multiplication Law} \\
&= \frac{\mathrm{P}(B)\mathrm{P}(A)}{\mathrm{P}(A)} && \text{since } \mathrm{P}(A \mid B) = \mathrm{P}(A) \\
&= \mathrm{P}(B)
\end{aligned}
$$

Thus, $\mathrm{P}(A \mid B) = \mathrm{P}(A)$ implies $\mathrm{P}(B \mid A) = \mathrm{P}(B)$.

Proof of $\mathrm{P}(B \mid A) = \mathrm{P}(B)$ implies $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

$$
\begin{aligned}
\mathrm{P}(A \cap B) &= \mathrm{P}(A)\mathrm{P}(B \mid A) && \text{(by Multiplication Law)} \\
&= \mathrm{P}(A)\mathrm{P}(B) && \text{(since } \mathrm{P}(B \mid A) = \mathrm{P}(B)\text{)}
\end{aligned}
$$

Independent Events vs Disjoint Events

- If A and B are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$.
- If A and B are disjoint: $A \cap B = \emptyset \Rightarrow \mathrm{P}(A \cap B) = 0$.
- If $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$,
  - Independent events **cannot** be disjoint.
  - Disjoint events **cannot** be independent.
- Conceptually, A and B are disjoint means that one happens prevents the other from happening, so one's occurrence definitely affects the other's.

Multiplication Law for Independent Events

When A and B are independent

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$$

- This is simply the Multiplication Law: $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \mid A)$ in which $\mathrm{P}(B \mid A)$ reduce to $\mathrm{P}(B)$ when A and B are independent
- More generally,

$$\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = \mathrm{P}(A_1) \times \mathrm{P}(A_2) \times \cdots \times \mathrm{P}(A_k)$$

if $A_1, \ldots, A_k$ are independent.

Example: Tossing a Coin Until Heads Come Up

Recall the example of tossing a fair coin repeatedly until heads come up. The sample space is

$$\Omega = \{\mathtt{H}, \mathtt{TH}, \mathtt{TTH}, \mathtt{TTTH}, \dots\} = \{\underbrace{1, 2, 3, 4, \dots}_{\text{all positive integers}}\}.$$

As the tosses are independent,

$$\begin{aligned}
\mathrm{P}(1) &= \mathrm{P}(\mathtt{H}) = 1/2 \\
\mathrm{P}(2) &= \mathrm{P}(\mathtt{TH}) = \mathrm{P}(\mathtt{T})\mathrm{P}(\mathtt{H}) = (1/2)(1/2) = 1/2^2 \\
&\vdots \\
\mathrm{P}(k) &= \mathrm{P}(k-1 \ \mathtt{T}\text{'s followed by an } \mathtt{H}) \\
&= \underbrace{\mathrm{P}(\mathtt{T}) \cdots \mathrm{P}(\mathtt{T})}_{k-1 \text{ times}} \mathrm{P}(\mathtt{H}) \\
&= \underbrace{(1/2) \dots (1/2)}_{k-1 \text{ times}}(1/2) = 1/2^k, \quad k = 1, 2, 3 \dots
\end{aligned}$$

Back to the Warmup Puzzle

Puzzle: A fair coin is flipped repeatedly
until the first time we see the sequence HH or TH.

- Player A wins if HH comes up first.
- Player B wins if TH comes up first.

What is the chance of winning for Player A?

Back to the Warmup Puzzle

Puzzle: A fair coin is flipped repeatedly
until the first time we see the sequence HH or TH.

- Player A wins if HH comes up first.
- Player B wins if TH comes up first.

What is the chance of winning for Player A?

What is Ω?

HH	TH
HTH	TTH
HTTH	TTTH
HTTTH	TTTTH
HTTTTH	TTTTTH
...	...