STAT 234 Lecture 19
Hypothesis Tests About a Population Mean
Section 9.2, 9.4

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University of Chicago

Hypothesis Tests about Means

Example: Number of College Applications

To know how many colleges students applied to, the dean of a certain university took a random sample of size 106 from their newly admitted students. This sample yielded an average of 9.7 college applications with a standard deviation of 7. College Board website states that counselors recommend students apply to roughly 8 colleges. Do these data provide convincing evidence that the average number of colleges all freshmen in this university apply to is higher than recommended?

http://www.collegeboard.com/student/apply/the-application/151680.html

Example: Number of College Applications – Hypotheses

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- *Null Hypothesis* H_0: $\mu = 8$ (Freshmen in this university have applied 8 schools on average, as recommended)
- *Alternative Hypothesis* H_A: $\mu > 8$ (Freshmen in this university have applied *over 8* schools on average)

Incorrect Statements of H_0 and H_A

H_0 and H_A are **ALWAYS** about population parameters, not sample statistics

Neither

$$H_0: \ \bar{x} = 8, \quad H_A: \ \bar{x} > 8$$

nor

- H_0: the 106 new students applied 8 schools on average
- H_A: the 106 new students applied 9.7 schools on average

is correct. The correct statements should be

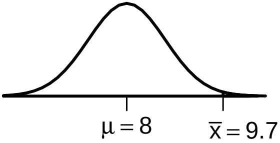
$$H_0: \ \mu = 8, \quad H_A: \ \mu > 8$$

Also please **clearly specify what is** μ.
e.g., μ is the average number of colleges freshmen in this university applied to.

Number of College Applications — Test Statistic

By CLT, under H_0: $\mu = 8$, the sampling distribution of the sample mean is

$$\bar{x} \sim N\left(\mu = 8, \text{SE} = \frac{7}{\sqrt{106}} = 0.68\right)$$



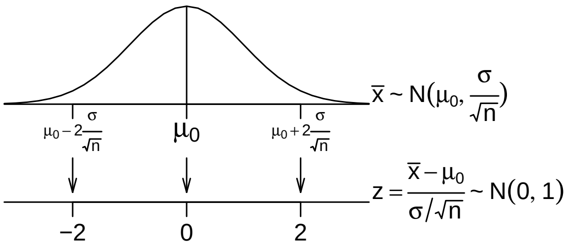
To gauge how unusual the observed sample mean $\bar{x} = 9.7$ is relative to its the hypothesized sampling distribution above, the *test statistic* we used is the z-statistic, which is the z-score of the sample mean relative to the distribution above

$$z\text{-statistic} = \frac{\bar{x} - \mu_0}{\text{SE}} = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} = \frac{9.7 - 8}{7/\sqrt{106}} \approx 2.5$$

$$\sim N(0, 1) \quad \text{under } H_0\text{: } \mu = \mu_0 = 8$$

Rejection Region, Critical Value, and P(Type 1 error)

To test H_0: $\mu = \mu_0$ against H_A: $\mu > \mu_0$, only a sample mean $\bar{x}$ far above μ_0 is evidence for H_A and only in such cases should H_0 be rejected.



To control the P(Type 1 error) $= P(\text{rejecting } H_0 | H_0 \text{ is true})$ at the significance level α, we should reject H_0 only when

$$z\text{-statistic} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha = \texttt{qnorm}(1 - \alpha).$$

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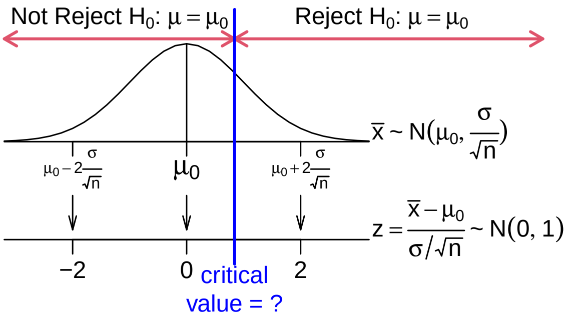

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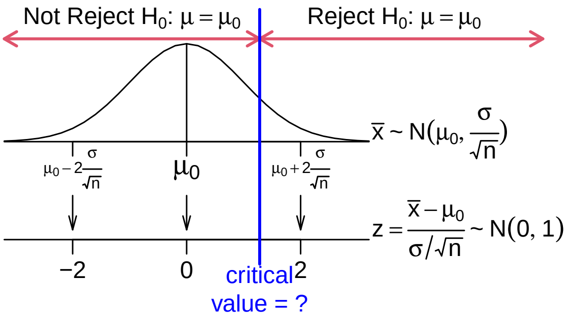

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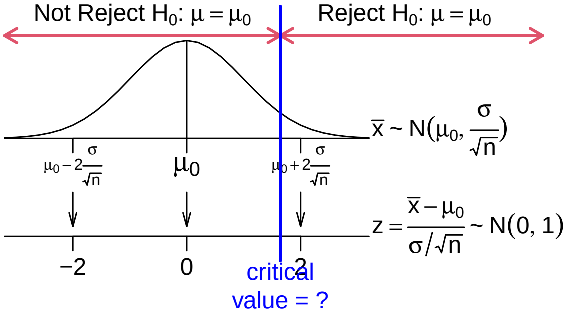

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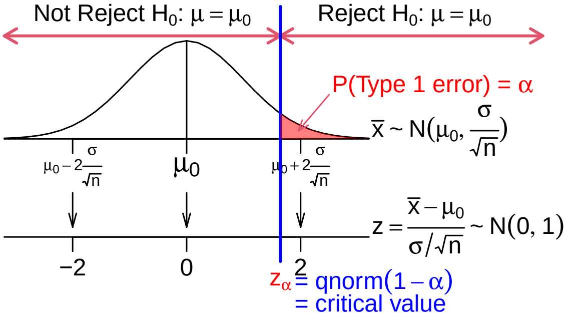

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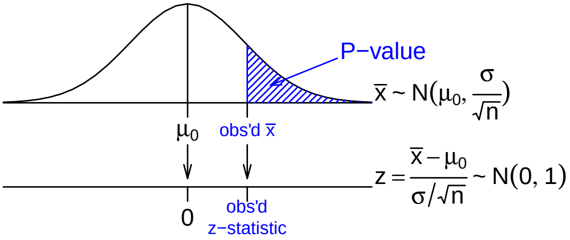
P-value for Upper One-Sided Test

To test H_0: $\mu = \mu_0$ against H_A: $\mu > \mu_0$, the P-value is $P(\bar{x} >$ observed value of $\bar{x} \mid \mu = \mu_0)$ or the blue shaded region below.



Example (College Applications) For testing H_0: $\mu = 8$ v.s. H_A: $\mu > 8$,

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\begin{aligned}
P\text{-value} &= P(\bar{x} > 9.7 \mid \mu = 8) \\
&= P\left(Z > \frac{9.7 - 8}{7/\sqrt{106}} \approx 2.500\right) \\
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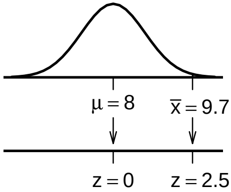

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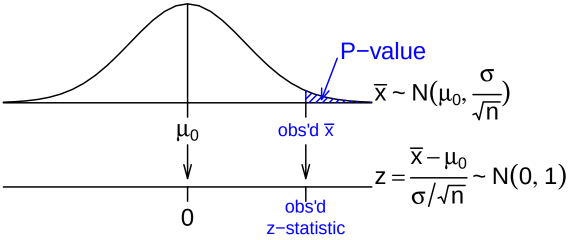

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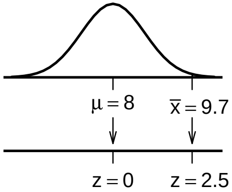
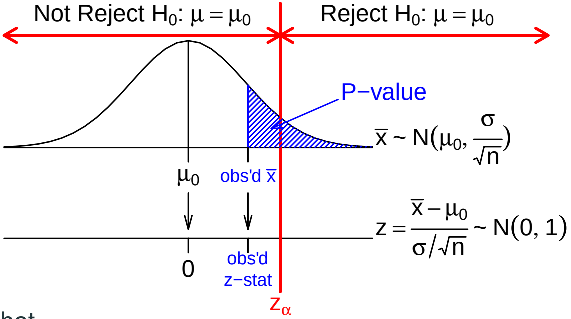


P-value and Critical Value Approaches for Hypotheses Tests



Observed that

if z-statistic $< z_\alpha$ then P-value $> \alpha$

Two equivalent approaches to test H_0: $\mu = \mu_0$ v.s. H_A: $\mu > \mu_0$ and control the P(Type 1 error) at a significance level α:

- *Critical value approach*: one can compute the z-stat $= \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ and the critical value $z_\alpha = \texttt{qnorm}(1-\alpha)$, and reject H_0 if the z-stat $> z_\alpha$.
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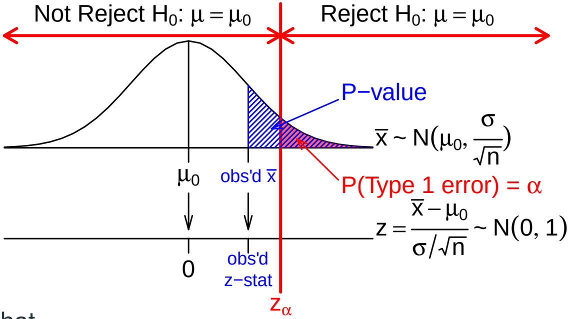

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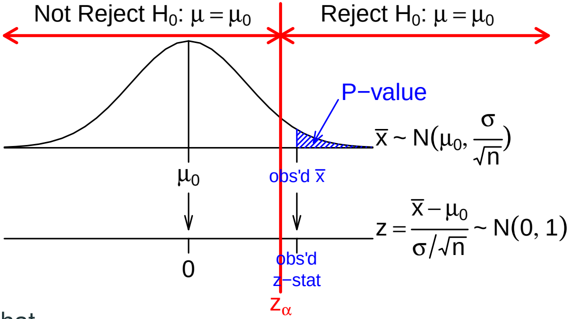

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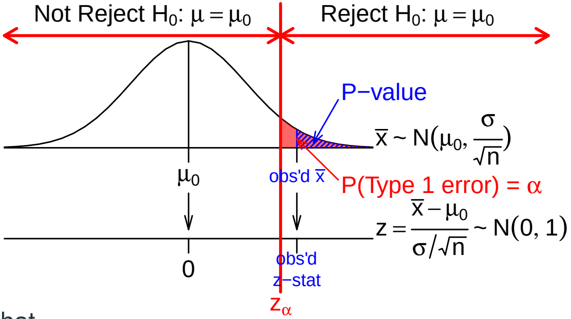

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Two-Sided Hypothesis Tests

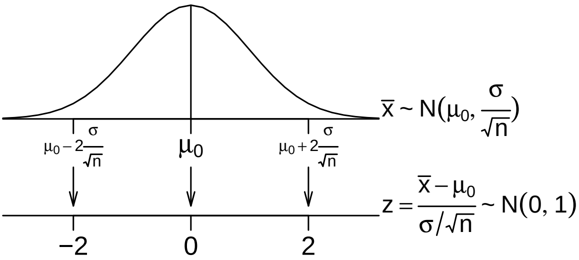
If the Dean wanted to know whether the data provide convincing evidence that the average number of colleges applied is *different* than the recommended 8 schools, the alternative hypothesis would be different.

$$H_A : \mu \neq 8$$

In this case, a sample mean $\bar{x}$ far below 8 would also be evidence in favor of H_A.

Two-Sided Hypothesis Tests

To test H_0: $\mu = \mu_0$ against the **two-sided alternative** $\boxed{H_A\colon \mu \neq \mu_0}$, both $\bar{x}$ far above or below μ_0 are evidence for H_A and hence H_0 should be rejected when $|\bar{x} - \mu_0|$ is large.



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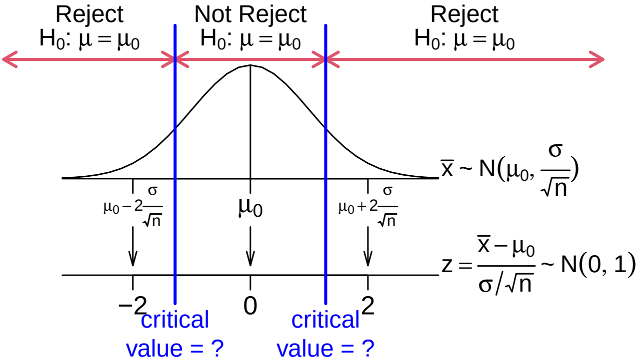

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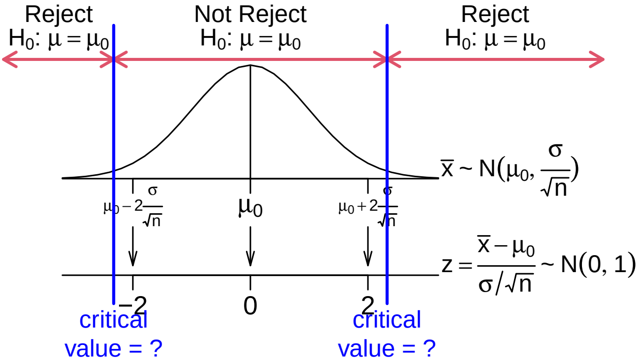

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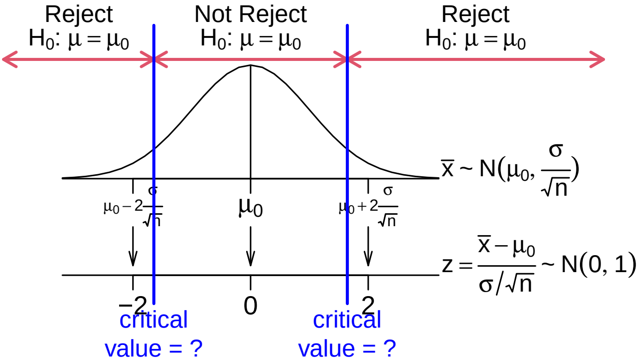

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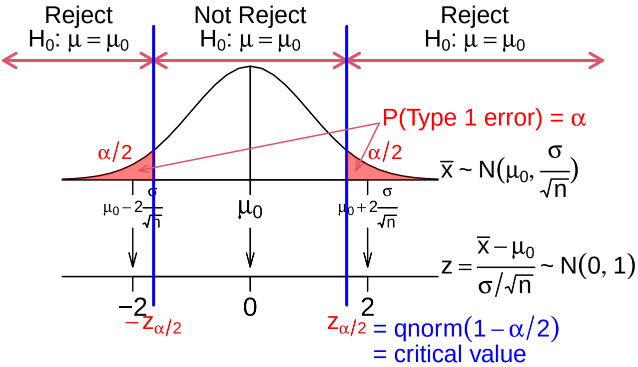

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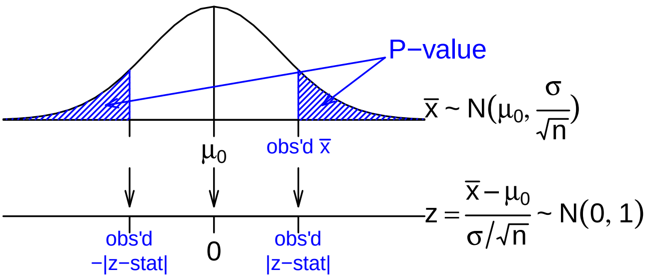

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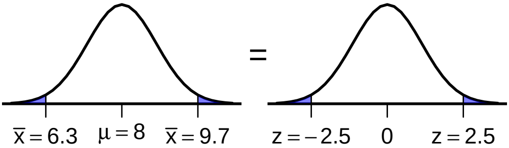

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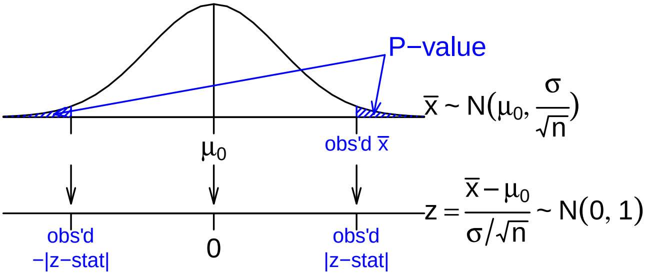

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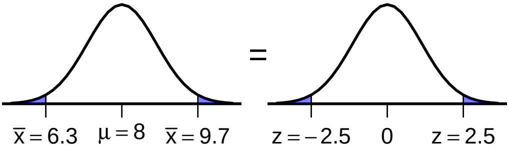

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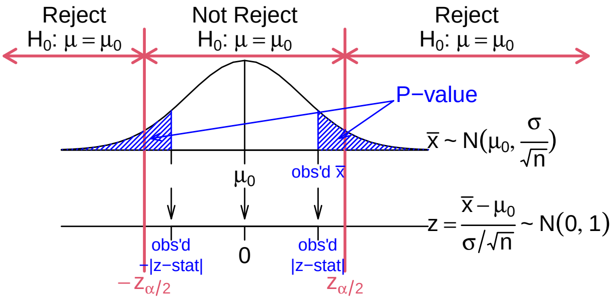

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Two equivalent approaches to test H_0: $\mu = \mu_0$ v.s. H_A: $\mu \neq \mu_0$ and control the P(Type 1 error) at the significance level α.

- *Critical value approach*: reject H_0 if the absolute value of the z-statistic $= \left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2} = \texttt{qnorm}(1-\alpha/2)$
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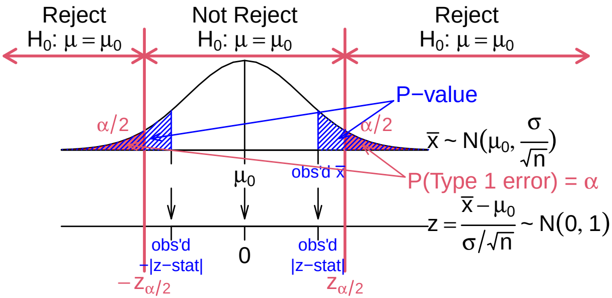

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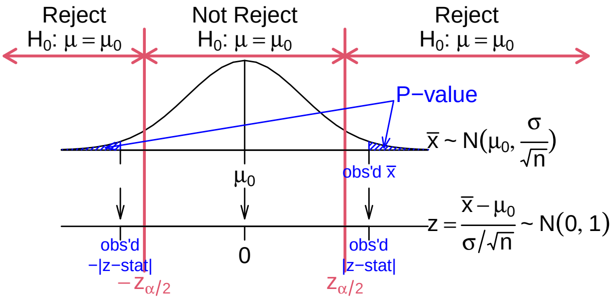

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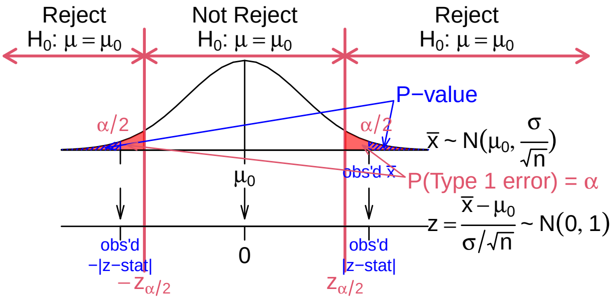

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Lower One-Sided Hypothesis Test

If the Dean wanted to know whether the data provide convincing evidence that the average number of colleges applied is lower than the recommended 8 schools, the alternative hypothesis would be

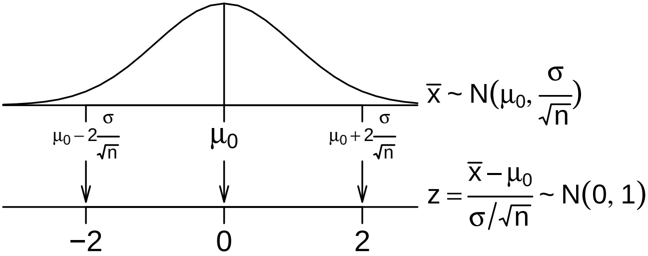
$$H_A : \mu < 8$$

Three types of alternative hypotheses:

- Upper one-sided: H_A: $\mu > 8$
- Lower one-sided: H_A: $\mu < 8$
- Two-sided: H_A: $\mu \neq 8$

Lower One-Sided Tests

To test H_0: $\mu = \mu_0$ against the **lower one-sided** alternative H_A: $\mu < \mu_0$, only a sample mean $\bar{x}$ far below μ_0 is evidence for H_A and H_0 should be rejected only in such cases.



To control P(Type 1 error) at the significance level α, we should reject H_0 only when the z-statistic $= \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} < -z_\alpha = \texttt{qnorm}(\alpha)$.

Lower One-Sided Tests

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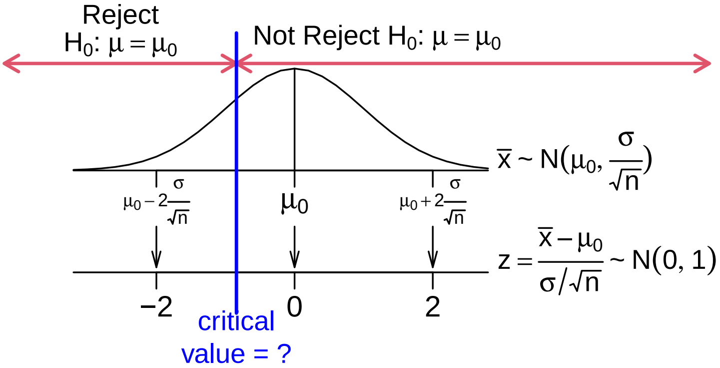

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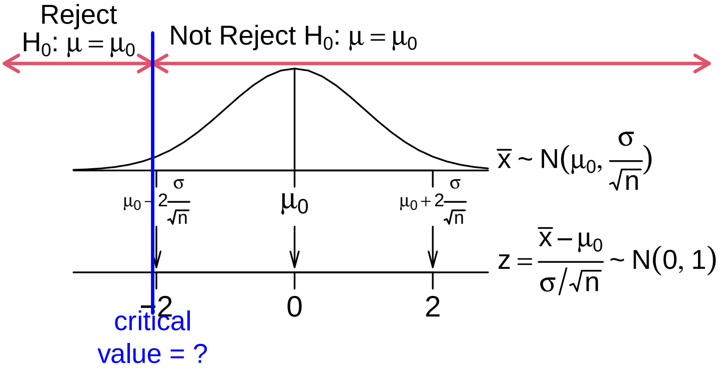

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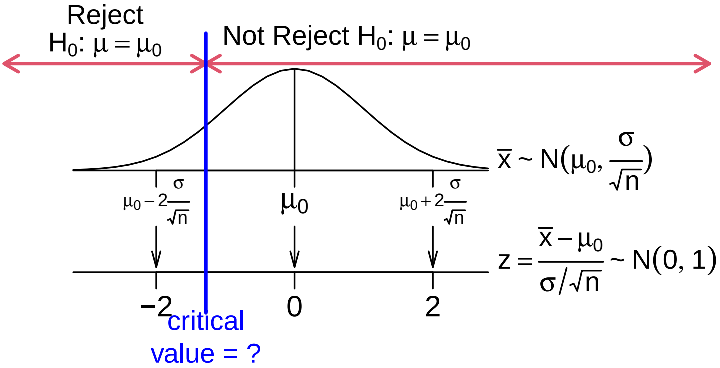

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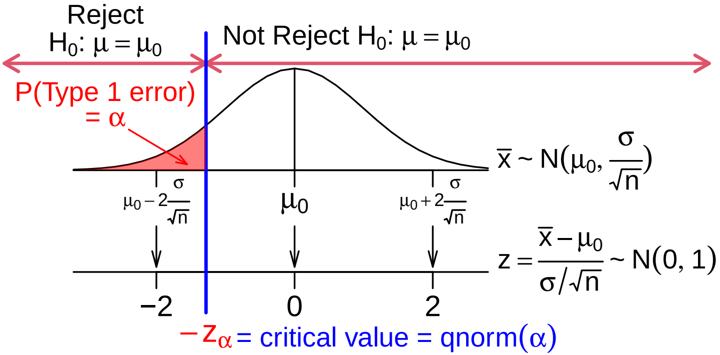

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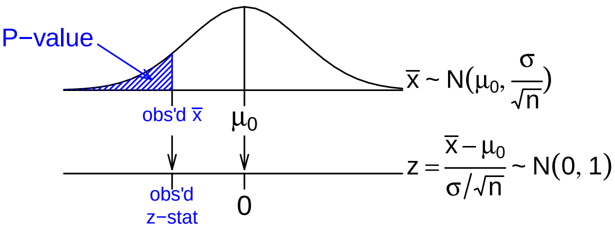
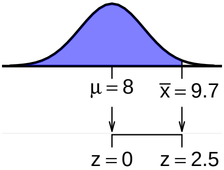
P-values for Lower One-Sided Hypothesis Tests

To test H_0: $\mu = \mu_0$ v.s. **lower one-sided alternative** H_A: $\mu < \mu_0$,
the P-value is the lower tail probability (blue shaded region) below.



Example (College Applications) For H_0: $\mu = 8$ v.s. H_A: $\mu < 8$, the P-value is lower tail area $1 - 0.0062 =$ 0.9938, which makes sense since H_A: $\mu < 8$ is less plausible than H_0: $\mu = 8$ given $\bar{x} = 9.7 > 8$. No reason to reject H_0.



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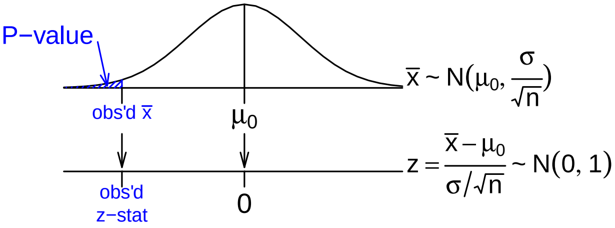

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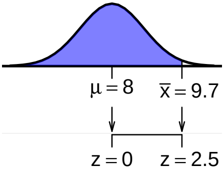
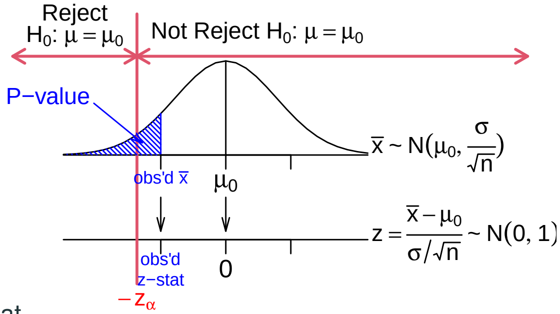


P-value & Critical Value Approaches for Lower One-Sided Tests



Observed that

if z-statistic $> -z_\alpha$ then P-value $> \alpha$

Two equivalent approaches to test H_0: $\mu = \mu_0$ v.s. H_A: $\mu < \mu_0$ and control the P(Type 1 error) at the significance level α:

- *Critical value approach*:
  reject H_0 if z-stat $= \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} < -z_\alpha = \texttt{qnorm}(\alpha)$
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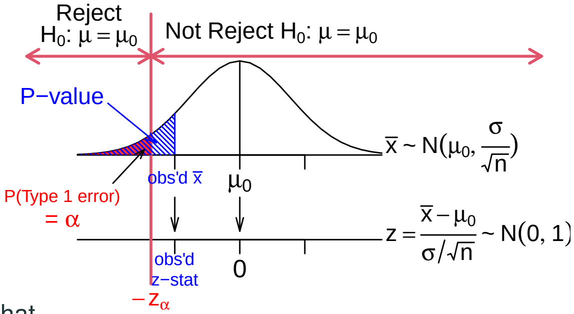

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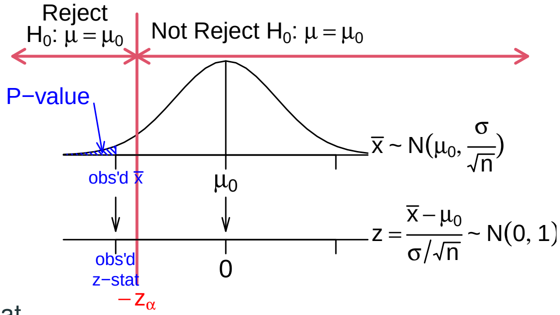

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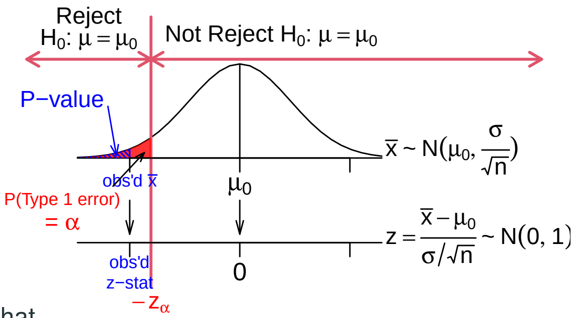

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P-value Approach or Critical Value Approach?

We introduced both the critical value approach and the P-value approach for hypothesis testing. They are equivalent but we generally *recommend the P-value approach*, for two reasons.

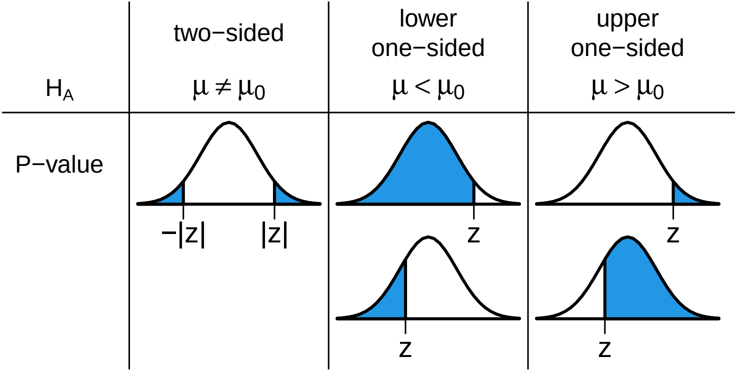
- The rejection rule is simpler, just compare the P-value with the significance level α
- More importantly, we can simply report the P-value and let people choose their own significance level α (= the P(Type 1 error)) and decide whether to reject or not to reject the H_0

From now on, we will just stick with the P-value approach.

Recap: How to Compute One-Sided & Two Sided P-values

The z-statistic for testing $\mathrm{H}_0 : \mu = \mu_0$ is $z = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \sim N(0, 1)$.

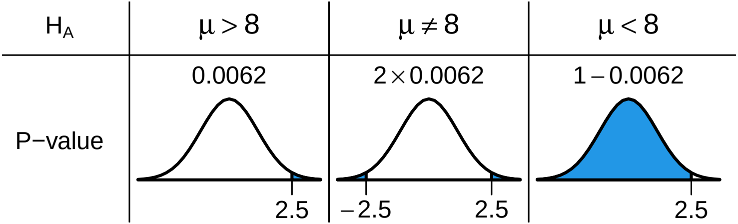
The P-value depends on H_A.



The bell-shape curve above is the standard normal curve.

Then we reject H_0 when P-value $< \alpha$.

Back to the College Applications Example



For H_A: $\mu > 8$ and H_A: $\mu \neq 8$, we *reject H_0* since P-value is *low* ($< 5\%$)

- The data provide convincing evidence that freshmen in this university have applied to more than (different from) 8 schools on average.
- The diff. betw. the null value of 8 schools and observed sample mean of 9.7 schools is beyond sampling variability.

For H_A: $\mu < 8$, there is no reason to reject H_0: $\mu = 8$ since the alternative H_A: $\mu < 8$ is even less plausible than H_0: $\mu = 8$ given the observed sample mean $9.7 > \mu = 8$.

Conclusion when the P-value is Low

When the P-value is below the significance level, we say

- H_0 is rejected
- There is strong evidence that freshmen in this university had applied to over 8 schools on average (H_A is true)
- The mean number of schools freshmen in this university had applied is *significantly* over 8

We don't say

- The H_A is accepted
- We fail to reject H_A

Conclusion when the P-value is Not Low

When the P-value exceeds the significance level, we say

- We fail to reject H_0
- No strong evidence that freshmen in this university had applied to over 8 schools on average (H_A is true)
- The mean number of schools freshmen in this university had applied is *not significantly* over 8

We don't say

- the H_0 is accepted
- we fail to accept H_A
- there is strong evidence that H_0 is true — because we might have made a Type 2 error, and the chance of making a Type 2 error is not controlled, which can be quite big

More Incorrect Statements of Hypotheses

Please note that the terms: *significant(ly)* and *reject*, are only used to state the conclusions of the hypotheses tests. Do NOT use them in the hypotheses. It's incorrect to state the hypotheses as

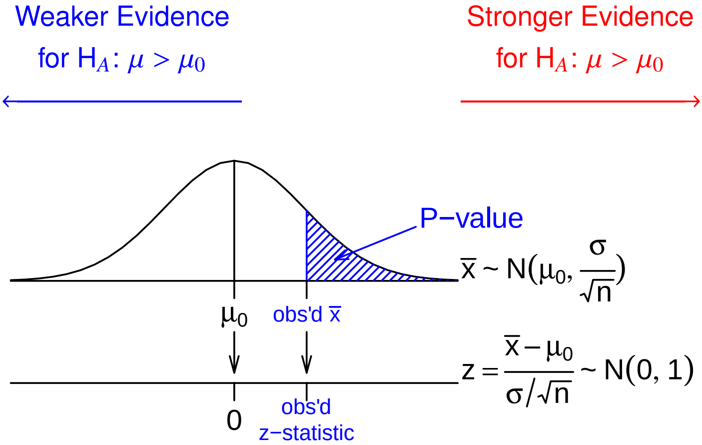
- H_0: The mean number of schools students have applied is *not significantly* over 8
- H_A: The mean number of schools students have applied is *significantly* over 8

or

- H_0: We don't reject that the mean number of schools students have applied is 8
- H_A: We reject that the mean number of schools students applied is 8

Interpretation of P-Values — Upper One-Sided Tests

The P**-value** is *the probability of getting data such that the evidence for the* H_A *is* ***at least as strong as*** *our observed data, if in fact* $H_0: \mu = \mu_0$ *were true*.

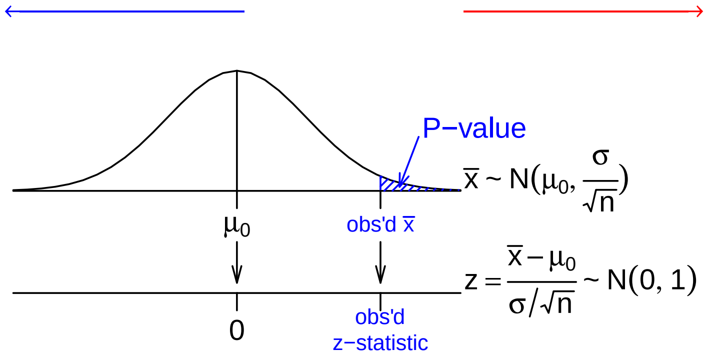


Interpretation of P-Values — Upper One-Sided Tests

The P**-value** is *the probability of getting data such that the evidence for the H_A is **at least as strong as** our observed data, if in fact H_0: $\mu = \mu_0$ were true*.

Weaker Evidence for H_A: $\mu > \mu_0$

Stronger Evidence for H_A: $\mu > \mu_0$



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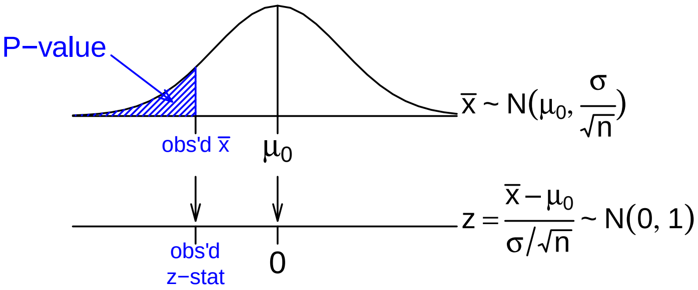
The P-**value** is *the probability of getting data such that the evidence for the H_A is **at least as strong as** our observed data, if in fact H_0: $\mu = \mu_0$ were true*.

Stronger Evidence
for H_A: $\mu < \mu_0$
←

Weaker Evidence
for H_A: $\mu < \mu_0$
→



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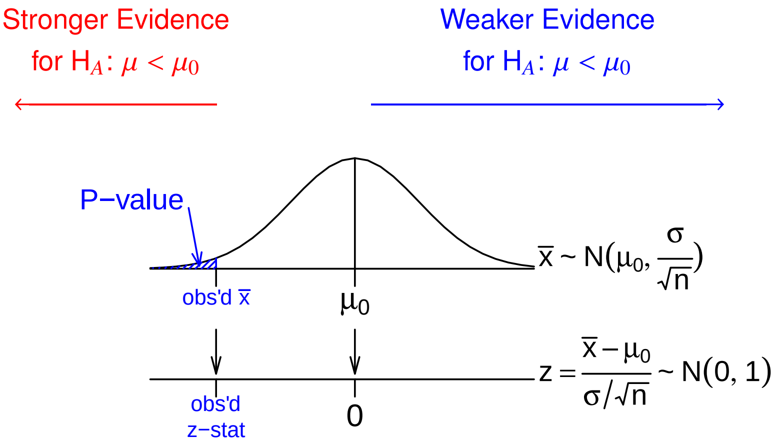
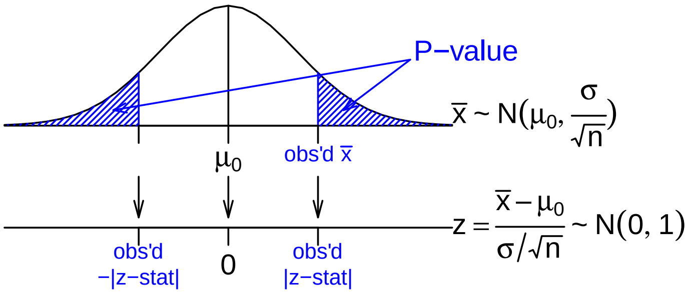


Interpretation of P-Values — Two-Sided Tests

The P-**value** is *the probability of getting data such that the evidence for the H_A is **at least as strong as** our observed data, if in fact H_0: $\mu = \mu_0$ were true*.

Stronger evidence for H_A: $\mu \neq \mu_0$	Weak evidence for H_A: $\mu \neq \mu_0$	Stronger evidence for H_A: $\mu \neq \mu_0$
⟵	⟶ ⟵	⟶



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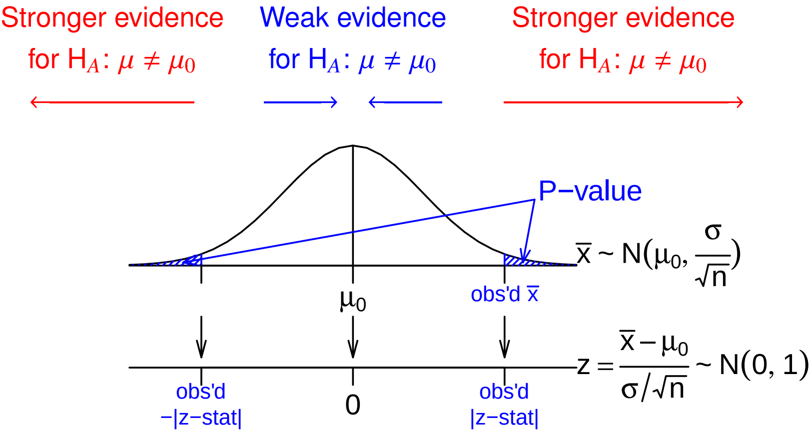

Example: Number of College Applications – Conditions

As CLT is used in the hypothesis test above, we need to check the same conditions as we construct confidence intervals for the population mean.

- Observations must be *independent*
  - Use your knowledge to judge if the data might be dependent
- The population distribution of the number of colleges students apply to should not be extremely skewed.
- In the z-statistic $= \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$, if the unknown population SD σ is replaced with the sample SD s, we need to further check that
  - sample size cannot be too small (MMSA said at least 40)
  - no outliers & not too skewed $\Rightarrow$ Check the histogram of data!

Recap: Hypothesis Testing for a Population Mean

1. Set the hypotheses
   - $H_0 : \mu = \mu_0$
   - $H_A : \mu <$ or $>$ or $\neq \mu_0$
2. Check assumptions and conditions
   - Independence
   - Normality: nearly normal population or $n \geq 40$, no extreme skew – or use the t distribution (next lecture)
3. Calculate a *test statistic*

$$Z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

Recap: Hypothesis Testing for a Population Mean (Cont'd)

"Rejection region" approach:

4. Choose a significance level α and reject H_0 when
   - z-stat $> z_\alpha$ for H_A: $\mu > \mu_0$
   - z-stat $< -z_\alpha$ for H_A: $\mu < \mu_0$
   - $|z\text{-stat}| > z_{\alpha/2}$ for H_A: $\mu \neq \mu_0$

"P-value" approach

4. Compute the P-value as on p.18
5. (optional) Choose a significance level α and make a decision
   - If P-value $< \alpha$, reject H_0
   - If P-value $> \alpha$, do not reject H_0