Stat22200 Additional Exercises for Chapter 3 & 4

1. The table below shows the start of an ANOVA table. Fill in the whole table from what is given here. How many groups were there? Is there evidence that the group means are different?

Source	df	Sum of Squares	Mean Squares	F-statistic	P-value
Treatment	?	?	?	?	?
Error	24	35088	?		
Total	31	70907			

Answer: The df for treatment is $31 - 24 = 7 = g - 1$. So there are $g = 8$ groups in total.

Source	df	Sum of Squares	Mean Squares	F-statistic	P-value
Treatment	$31 - 24 = 7$	$70907 - 35088 = 35819$	$35819/7 = 5117$	$5117/1462 = 3.5$	0.00994
Error	24	35088	$35088/24 = 1462$		
Total	31	70907			

The P-value is found using the R command below:

```
> pf(3.5, 7, 24, lower.tail=F)
[1] 0.009941808
```

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2. (**Exercise 3.2**, p.60, Oehlert textbook) An experimenter randomly allocated 125 male turkeys to five treatment groups: control and treatments A, B, C, and D. There were 25 birds in each group, and the mean results were 2.16, 2.45, 2.91, 3.00, and 2.71, respectively. The sum of squares for experimental error was 153.4. Test the null hypothesis that the five group means are the same against the alternative that one or more of the treatments differs from the control.

   Create an ANOVA table based on the information provided and answer the question. Show your work.

   To find the p-value, you can use the R command `pf`

   `1 - pf(F, df1, df2),`

   where `F` is the value of the F-statistic and `df1` , `df2` are the two degrees of freedom.

Answer:

The treatment means are 2.16, 2.45, 2.91, 3.00, and 2.71. Since each group has the same number of turkeys, the average of these five numbers will give the overall mean:

$$\overline{y}_{\bullet\bullet} = \frac{2.16 + 2.45 + 2.91 + 3.00 + 2.71}{5} = 2.646$$

We then use this to compute the treatment sum of square SS_{Trt}:

$$\begin{aligned} SS_{Trt} &= \sum_{i=1}^{g} n_i(\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet})^2 = 25\sum_{i=1}^{g}(\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet})^2 \\ &= 25[(2.16 - 2.646)^2 + (2.45 - 2.646)^2 + (2.91 - 2.646)^2 + (3.00 - 2.646)^2 + (2.71 - 2.646)^2] \\ &= 25[(-0.486)^2 + (-0.196)^2 + (0.264)^2 + (0.354)^2 + (0.064)^2] \\ &= 11.843 \end{aligned}$$

The ANOVA table is

Source	df	SS	MS	F-statistic
Treatment	$g-1=5-1=4$	11.843	$\text{MS}_{Trt}=\frac{SS_{Trt}}{g-1}=\frac{11.843}{4}=2.961$	$F=\frac{\text{MS}_{Trt}}{\text{MSE}}=\frac{2.971}{1.278}=2.316$
Error	$N-g=125-5=120$	153.396	$\text{MSE}=\frac{\text{SSE}}{N-g}=\frac{153.4}{120}=1.278$	

```
> pf(2.316, 4, 120, lower.tail=F)
[1] 0.06116433
```

Alternatively, use Table D.5 in the textbook. The 5% cutoff value for an $F_{4,100}$ is 2.46 and the cutoff for 4 and 200 df is $F_{0.05,4,200} = 2.42$.

Because the P-value 0.061 is greater than 0.05 or because the F-statistic 2.316 is less than the critical value $F_{0.05,4,200} = 2.42$, we fail reject the H_0 at the 0.05 level. However, the p-value is very close to 0.05, so perhaps we should be skeptical of H_0 even though we cannot reject it.

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3. (**Problem 4.1 on p. 75-76, Revised**) A consumer testing agency obtains **four** cars from each of six makes:

Make		Domestic	Manufacturer	Expensive	$\overline{y}_{i\bullet}$
1	(Ford)	domestic	Ford	N	4.6
2	(Chevrolet)	domestic	GM	N	4.3
3	(Nissan)	imported	Other	N	4.4
4	(Lincoln)	domestic	Ford	Y	4.7
5	(Cadillac)	domestic	GM	Y	4.8
6	(Mercedes)	imported	Other	Y	6.2

Make 1 and 4 are Ford products, while 2 and 5 are GM products. We wish to compare the six makes on their oil use per 100,000 miles driven. The mean responses by make of car were 4.6, 4.3, 4.4, 4.7, 4.8, 6.2, and the sum of squares for error (SSE) was 2.25.

(a) Compute the ANOVA table for this experiment. What would you conclude?

Answer: The group means are 4.6, 4.3, 4.4, 4.7, 4.8, and 6.2. Since each group has the same number of cars, the average of these six group means will give the overall mean:

$$\overline{y}_{\bullet\bullet} = \frac{4.6 + 4.3 + 4.4 + 4.7 + 4.8 + 6.2}{6} = 4.83$$

$$\begin{aligned} SS_{Trt} &= \sum_{i=1}^{6} n_i(\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet})^2 = 4\sum_{i=1}^{6}(\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet})^2 \\ &= 4[(4.6-4.83)^2 + (4.3-4.83)^2 + (4.4-4.83)^2 + (4.7-4.83)^2 + (4.8-4.83)^2 + (6.2-4.83)^2] \\ &= 9.654. \end{aligned}$$

The df for treatments is $g - 1 = 6 - 1 = 5$, and for error is $N - g = 24 - 6 = 18$.
Then, $\text{MS}_{Trt} = \text{SS}_{Trt}/(g-1) = 9.654/5 = 1.93$ and $\text{MSE} = \text{SSE}/(N-g) = 2.25/18 = 0.125$. Finally, $F = \text{MS}_{Trt}/\text{MSE} = 1.93/0.125 = 15.44$, and we can find the p-value in R:

```
> 1 - pf(15.44, 5, 18)
[1] 5.759711e-06
```

Thus, the ANOVA table is

Source	df	SS	MS	F-stat	P-value
Treatment	5	9.65	1.93	15.44	5.76×10^{-6}
Error	18	2.25	0.125		

From the ANOVA table and small p-value, we can conclude that the 6 makes have significantly different mean oil uses per 100000 miles driven.

(b) Make pairwise comparisons of all six makes. (i) Explain why the standard errors of the mean difference between all pairs of makes are all the same. (ii) How large the sample mean difference $\overline{y}_{j\bullet} - \overline{y}_{i\bullet}$ needs to be so that is can be significant at 5% level using a t-test (This is called the "least significant difference (LSD)")? (iii) Identify all the pairs of makes that are significant at 5% level. (v) Finally, create an underline diagram to summarize the result.

Answer: (i) For cars from Makes i and j, the estimated mean difference is $\overline{y}_{j\bullet} - \overline{y}_{i\bullet}$ with standard error

$$\text{SE} = \sqrt{\text{MSE}\left(\frac{1}{n_j} + \frac{1}{n_i}\right)} = \sqrt{0.125\left(\frac{1}{4} + \frac{1}{4}\right)} = 0.25.$$

which is identical for all pairs since the six groups have equal number of cars.
(ii) The t-statistic for pairwise comparison

$$\frac{\overline{y}_{j\bullet} - \overline{y}_{i\bullet}}{\text{SE}} = \frac{\overline{y}_{j\bullet} - \overline{y}_{i\bullet}}{0.25}$$

as a t-distribution with df = 18 (same as the df for MSE). It is significant at 5% level if and only if the absolute value of the t-statistic is greater than the critical value $t_{0.025,df=18} = 2.101$, or equivalently,

$$|\overline{y}_{j\bullet} - \overline{y}_{i\bullet}| > \text{SE} \times t_{0.025,df=18} = 0.25 \times 2.101 = 0.52525.$$

That is, to be significantly different at 5% level, their mean difference must be least as large as 0.52525.
(iii) The 6 makes ordered by oil use from low to high are

Ch	Ni	Fo	Li	Ca	Me
4.3	4.4	4.6	4.7	4.8	6.2

where the car makes are denoted by the first two letters of the names.
We can see that out of the 6 makers, 5 of them are not significantly different at the 5% level ($\overline{y}_{j\bullet} - \overline{y}_{i\bullet}$ not greater than 0.52525). Only Mercedes is significantly higher than all other 5 makes.
(iv) We can then display the result of all pairwise comparison as an underline diagram below:

Ch	Ni	Fo	Li	Ca	Me
<u>4.3</u>	<u>4.4</u>	<u>4.6</u>	<u>4.7</u>	<u>4.8</u>	6.2

(c) Consider the contrast for comparing the mean oil use of domestic cars and imported cars as follows:

$$C = \frac{\mu_{Ni} + \mu_{Me}}{2} - \frac{\mu_{Fo} + \mu_{Ch} + \mu_{Li} + \mu_{Ca}}{4}.$$

Give a 95% confidence interval for this contrast.

Answer: The w_i for the contrast

$$C = \frac{\mu_{Ni} + \mu_{Me}}{2} - \frac{\mu_{Fo} + \mu_{Ch} + \mu_{Li} + \mu_{Ca}}{4}$$

is $(w_{Ni}, w_{Me}, w_{Fo}, w_{Ch}, w_{Li}, w_{Ca}) = (1/2, 1/2, -1/4, -1/4, -1/4, -1/4)$.
The estimator for the contrast C is

$$\begin{aligned}\widehat{C} = \sum_i w_i \widehat{\mu}_i = \sum_i w_i \overline{y}_{i\bullet} &= \frac{\overline{y}_{Ni\bullet} + \overline{y}_{Me\bullet}}{2} - \frac{\overline{y}_{Fo\bullet} + \overline{y}_{Ch\bullet} + \overline{y}_{Li\bullet} + \overline{y}_{Ca\bullet}}{4} \\ &= \frac{4.4 + 6.2}{2} - \frac{4.6 + 4.3 + 4.7 + 4.8}{4} = 5.3 - 4.6 = 0.7\end{aligned}$$

The estimator $\widehat{C}$ has the standard error

$$\begin{aligned}SE(\widehat{C}) &= \sqrt{\text{MSE}}\sqrt{\sum_i \frac{{w_i}^2}{n_i}} \\ &= \sqrt{0.125}\sqrt{\frac{(1/2)^2 + (1/2)^2 + (-1/4)^2 + (-1/4)^2 + (-1/4)^2 + (-1/4)^2}{4}} \\ &= 0.153\end{aligned}$$

Hence, the 95% confidence interval for the mean difference (imported minus domestic) is

$$\widehat{C} \pm t_{24-6,0.025} \times SE(\widehat{C}) = 0.7 \pm 2.1 \times 0.153 = (0.38, 1.02).$$

As 0 is not in the interval, it appears that the 2 imported makes use more oil than the 4 domestic makes.

(d) Define a contrast for comparing the mean oil use of Ford cars (Ford and Lincoln) and GM cars (Chevrolet and Cadillac). Test if this contrast is 0 at 5% significance level.

Answer: The mean difference (Ford minus GM) is a contrast,

$$C = \frac{\mu_{Fo} + \mu_{Li}}{2} - \frac{\mu_{Ch} + \mu_{Ca}}{2}$$

where the w_i's are $(w_{Fo}, w_{Li}, w_{Ch}, w_{Ca}, w_{Ni}, w_{Me}) = (1/2, 1/2, -1/2, -1/2, 0, 0)$.
The estimator for the contrast C is

$$\begin{aligned}\widehat{C} = \sum_i w_i \widehat{\mu}_i = \sum_i w_i \overline{y}_{i\bullet} &= \frac{\overline{y}_{Fo\bullet} + \overline{y}_{Li\bullet}}{2} - \frac{\overline{y}_{Ch\bullet} + \overline{y}_{Ca\bullet}}{2} \\ &= \frac{4.6 + 4.7}{2} - \frac{4.3 + 4.8}{2} = 4.64 - 4.55 = 0.1\end{aligned}$$

The estimator $\widehat{C}$ has the standard error

$$\begin{aligned}SE(\widehat{C}) &= \sqrt{\text{MSE} \sum_i \frac{{w_i}^2}{n_i}} \\ &= \sqrt{0.125 \times \frac{(1/2)^2 + (1/2)^2 + (-1/2)^2 + (-1/2)^2 + 0^2 + 0^2}{4}} \\ &= 0.178\end{aligned}$$

Hence, the t-statistic is

$$t = \frac{\widehat{C}}{\text{SE}(\widehat{C})} = \frac{0.1}{0.178} \approx 0.562$$

with $N - g = 24 - 6 = 18$ degrees of freedom. The two-sided P-value ≈ 0.58 is found in R as follows.

```
> 2*pt(0.562, df=18, lower.tail=F)
[1] 0.5810475
```

Or one could compare the t-statistic 0.562 with the critical value $t_{0.05/2,df=18} \approx 2.101$. As the t-statistic 0.562 with the critical value 2.101, we fail to reject H_0: $C = 0$.

Conclusion: Cars made by Ford or GM do not differ significantly in mean oil use.