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A CENTRAL LIMIT THEOREM FOR ITERATED RANDOM FUNCTIONS

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Abstract

A central limit theorem is established for additive functions of a Markov chain that can be constructed as an iterated random function. The result goes beyond earlier work by relaxing the continuity conditions imposed on the additive function, and by relaxing moment conditions related to the random function. It is illustrated by an application to a Markov chain related to fractals.

Keywords: Lipschitz continuity; fractals; indicator functions; Markov chains; Poisson's equation; transition operator

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1. Introduction

An iterated random function is a sequence of the form

$$X_n = F(X_{n-1}, \theta_n) \tag{1}$$

for $n = 1, 2, \dots,$ where $X_0, \theta_1, \theta_2, \dots$ are independent random elements for which $\theta_1, \theta_2, \dots \sim^{\text{ind}} H$ are i.i.d. with common marginal distribution H, say. Here X_n take values in a complete separable metric space $\mathcal{X}$, endowed with its Borel sets; $\theta_1, \theta_2, \dots$ take values in a second measurable space Θ; and $F : \mathcal{X} \times \Theta \to \mathcal{X}$ is a jointly measurable function. Then $X_0, X_1, X_2, \dots$ is a Markov chain with a stationary transition function

$$Q(x; B) = H\{\theta : F(x, \theta) \in B\}$$

for $x \in \mathcal{X}$ and Borel sets $B \subseteq \mathcal{X}$. Under quite general conditions, Markov chains of this form are ergodic and have unique proper stationary initial distributions. To state simple sufficient conditions, let $F_\theta(x) = F(x, \theta)$ and

$$L_\theta = \sup_{x' \neq x} \frac{\rho[F_\theta(x), F_\theta(x')]}{\rho(x, x')} \leq \infty$$

for $\theta \in \Theta$. Thus, F_θ is a θ-section of F, and L_θ is the Lipschitz constant of F_θ. Sufficient conditions for the existence and uniqueness of a stationary initial distribution are that

$$\int_\Theta \log(L_\theta) H\{\mathrm{d}\theta\} < 0, \tag{2}$$

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and that there are $x_0 \in \mathcal{X}$ and $\alpha > 0$ for which

$$\int_\Theta L_\theta^\alpha H\{\mathrm{d}\theta\} < \infty \tag{3}$$

and

$$\int_\Theta \rho[x_0, F_\theta(x_0)]^\alpha H\{\mathrm{d}\theta\} < \infty. \tag{4}$$

The integral in (2) may be $-\infty$. Clearly, if (3) and (4) hold for some $\alpha > 0$, then they hold for all smaller positive α.

Theorem 1. *If (2), (3), and (4) hold, then there is a unique stationary distribution π for the Markov chain (1), and the stationary processes obtained by letting $X_0 \sim \pi$ is ergodic.*

Results of this nature have a substantial history. See Theorem 5.1 in Diaconis and Freedman (1999) for a proof. The latter paper also contains examples and references.

Now let X_0 have the stationary distribution π; let $g \in L_0^2(\pi)$ be a square integrable function on $\mathcal{X}$ with mean 0; that is,

$$\int_\mathcal{X} g\,\mathrm{d}\pi = 0 \quad \text{and} \quad \|g\|_2^2 = \int_\mathcal{X} g^2\,\mathrm{d}\pi < \infty;$$

and let

$$S_n(g) = g(X_1) + \cdots + g(X_n).$$

Under what conditions on g and F_θ is $S_n(g)/\sqrt{n}$ asymptotically normal as $n \to \infty$? Benda (1998) showed that the desired conclusion holds if (4) holds with $\alpha = 2$,

$$\int_\Theta L_\theta^2 H\{\mathrm{d}\theta\} < 1,$$

and g is Lipschitz continuous; that is,

$$\sup_{x' \neq x} \frac{|g(x') - g(x)|}{\rho(x, x')} < \infty. \tag{5}$$

In the proof Benda (1998) constructed a solution $h \in L^2(\pi)$ to Poisson's equation

$$h = g + Qh, \tag{6}$$

where

$$Qh(x) = \int_\mathcal{X} h(y)Q(x; \mathrm{d}y) \tag{7}$$

for almost every x. It is then possible to write $S_n(g)$ in the form $S_n(g) = M_n + R_n$, where M_n is a martingale and R_n is stochastically bounded, as in Gordin and Lifsic (1978), and normality follows from the martingale central limit theorem. See Corollary 1 below for more detail.

Here it is shown that by strengthening the moment condition $g \in L^2(\pi)$ slightly (to the existence of pth moments for some $p > 2$), the continuity conditions on g and the moment conditions on $\rho[x_0, F_\theta(x_0)]$ and L_θ can be relaxed. Our conditions do not even require that g be continuous and are satisfied by the indicator functions of most balls. The main result is stated in Section 2 and proved in Section 4. In Section 3, the main result is illustrated by a Markov chain that is closely related to fractals. Since the nature of the extension is to allow discontinuous g, indicator functions are emphasized in the examples.

2. Main results

To state the conditions, let Ψ be the collection of non-decreasing, concave functions $\psi : [0, \infty) \to [0, \infty)$ for which $\psi(t) > 0$ for all $t > 0$ and

$$\int_0^1 \frac{\sqrt{\psi(t)}}{t}\,\mathrm{d}t < \infty. \tag{8}$$

For such ψ and $g \in L^2(\pi)$, let

$$K(g, \psi; x) = \sup_{x' : 0 < \rho(x, x') \le 1} \frac{|g(x') - g(x)|}{\sqrt{\psi[\rho(x, x')]}},$$

where the supremum of the empty set is understood to be zero, and

$$\kappa(g, \psi)^2 = \int_{\mathcal{X}} K(g, \psi; x)^2 \pi\{\mathrm{d}x\}.$$

Examples of $\psi \in \Psi$ that satisfy (8) include $\psi(t) = |t|^\alpha$ for any $0 < \alpha \le 1$, and any concave ψ for which $\psi(t) = 1/[\log^\alpha(1/t)]$ for $0 < t \le \mathrm{e}^{-(\alpha+1)}$, where $\alpha > 2$. (The latter function may be continued linearly beyond $\mathrm{e}^{-(\alpha+1)}$.) If g is uniformly Lipschitz continuous, as in (5), then $K(g, \psi; x)$ is bounded in x for any $\psi \in \Psi$. The definition of K does not require g to be continuous, however. For example, if $g = \mathbf{1}_B$ is the indicator function of a Borel set B, then

$$K(\mathbf{1}_B, \psi; x) \le \frac{1}{\sqrt{\psi[\rho(B, x) \vee \rho(B', x)]}}, \tag{9}$$

where $a \vee b$ is the larger of a and b, $\rho(C, x) = \inf\{\rho(x, y) : y \in C\}$, and the infimum of the empty set is understood to be ∞. If $\mathcal{X}$ is a convex subset of Euclidean space, then it is easily seen that $\rho(B, x) \vee \rho(B', x) = \rho(\partial B, x)$, where ∂B is the boundary of B.

Theorem 2. *Suppose that (2), (3), and (4) hold, and denote the stationary distribution by* π. *If* $g \in L_0^2(\pi) \cap L^p(\pi)$ *for some* $p > 2$, *and if there is a* $\psi \in \Psi$ *for which* $\kappa(g, \psi) < \infty$, *then there is a solution* $h \in L^2(\pi)$ *to Poisson's equation (6).*

The proof of Theorem 2 is presented in Section 4.

Corollary 1. *As* $n \to \infty$, $S_n(g)/\sqrt{n}$ *is asymptotically normal with mean* 0 *and variance* $\sigma^2(g) = \|h\|_2^2 - \|Qh\|_2^2$.

Proof. As in Benda (1998), the normality follows directly from the existence of a solution to Poisson's equation and the martingale central limit theorem. For

$$S_n(g) = M_n + R_n,$$

where

$$M_n = \sum_{k=1}^{n} [h(X_k) - Qh(X_{k-1})],$$
$$R_n = Qh(X_0) - Qh(X_n),$$

for $n \geq 1$. Here $h(X_k) - Qh(X_{k-1}), k \geq 1$, form a stationary sequence of square integrable martingale differences (with respect to $\sigma\{X_0, \ldots, X_n\}$), and $R_n = Qh(X_0) - Qh(X_n)$ is stochastically bounded in n. The corollary follows easily.

The same proof delivers even more. Let P^x denote conditional probability given $X_0 = x$; and let $A_n(0) = \mathbb{B}_n(0) = 0$,

$$A_n(t) = \frac{1}{\sqrt{n}} M_{\lceil nt \rceil} \quad \text{and} \quad \mathbb{B}_n(t) = \frac{1}{\sqrt{n}} S_{\lceil nt \rceil}$$

for $0 < t \leq 1$, where $\lceil x \rceil$ is the least integer that is greater than or equal to x. Further, let $\mathbb{B}$ denote a standard Brownian motion, and let Q_σ denote the induced distribution of $\sigma\mathbb{B}$ in $D[0, 1]$, the space of right continuous functions on $[0, 1]$ with left limits on $[0, 1)$.

Corollary 2. *For almost every $x(\pi)$, the conditional distribution of $\mathbb{B}_n$ in $D[0, 1]$ given $X_0 = x$ converges weakly to Q_σ as $n \to \infty$.*

Proof. The sequence $M_1, M_2, \ldots$ is a martingale with respect to P^x for almost every $x(\pi)$. To verify the conditions for the martingale central limit theorem, simply observe that

$$\lim_{n\to\infty} \frac{g(X_1)^2 + \cdots + g(X_n)^2}{n} = \|g\|_2^2$$

with probability 1 (P) and, therefore, with probability 1 (P^x) for almost every $x(\pi)$, by the ergodic theorem. That the conditional distribution of A_n given $X_0 = x$ converges to Q_σ for almost every x then follows from Theorem 2.3 of Durrett and Resnick (1978). For $\mathbb{B}_n$ observe that $Qh(X_0), Qh(X_1), Qh(X_2), \ldots$ is a square integrable stationary sequence. So, $Qh(X_n) = o(\sqrt{n})$ with probability 1 and therefore, $\max_{1\leq k\leq n} |R_k|/\sqrt{n} = o(1)$ with probability 1.

To illustrate how Theorem 2 relaxes the continuity condition in Benda (1998), consider the indicator functions of balls. For a given $x_0 \in \mathcal{X}$ (not necessarily the same x_0 as in (4)), let

$$B_r = B(x_0, r) = \{y \in \mathcal{X} : \rho(x_0, y) \leq r\}.$$

Further, let $\psi(t) = \sqrt{t},\ 0 \leq t < \infty$, and write $K_r(x)$ and κ_r for $K(\mathbf{1}_{B_r}, \psi; x)$ and $\kappa(\mathbf{1}_{B_r}, \psi)$.

Theorem 3. *For each $x_0 \in \mathcal{X}$, $\kappa_r < \infty$ for almost every r (Lebesgue).*

Proof. From (9), $K_r(x) \leq [\rho(B_r, x) \vee \rho(B'_r, x)]^{-1/4}$ for $x \in \mathcal{X}$. By considering the cases $\rho(x_0, x) < r$ and $\rho(x_0, x) > r$ separately, it is easily seen that $\rho(B_r, x) \vee \rho(B'_r, x) \geq |\rho(x_0, x) - r|$. For example, if $x \in B_r$ and $y \in B'_r$, then $r \leq \rho(x_0, y) \leq \rho(x_0, x) + \rho(x, y)$, so that $\rho(x, y) \geq r - \rho(x_0, x)$ and, therefore, $\rho(B'_r, x) \geq r - \rho(x_0, x)$; and a similar inequality may be obtained for $x \notin B_r$. So, if $0 < r < c < \infty$, then

$$\kappa_r^2 \leq \int_{\{x:\rho(x_0,x)\leq c+1\}} \frac{1}{\sqrt{|r - \rho(x_0, x)|}} \pi\{\mathrm{d}x\} + 1 \leq \infty$$

and

$$\begin{aligned}\int_0^c \kappa_r^2 \,\mathrm{d}r &\leq \int_{\{x:\rho(x_0,x)\leq c+1\}} \left[\int_0^c \frac{1}{\sqrt{|r - \rho(x_0, x)|}} \,\mathrm{d}r\right] \pi\{\mathrm{d}x\} + c \\ &\leq 2\int_{\{x:\rho(x_0,x)\leq c+1\}} [\sqrt{\rho(x_0, x)} + \sqrt{|c - \rho(x_0, x)|}] \pi\{\mathrm{d}x\} + c,\end{aligned}$$

which is finite. So, $\kappa_r < \infty$ for almost every r, and the theorem follows.

3. Fractals

Suppose that Θ is a finite set, say $\Theta = \{1, \dots, m\}$, $m \ge 2$, that $\mathcal{X}$ is a Euclidean space, say $\mathcal{X} = \mathbb{R}^d$, and that each F_θ is an affine function, say

$$F_\theta(x) = A_\theta x + b_\theta,$$

where A_θ are non-singular $d \times d$ matrices with operator norms $\|A_\theta\| < 1$ and $b_\theta \in \mathbb{R}^d$. Then π has a simple description. Let $\mathbb{N} = \{1, 2, \dots\}$ and let $\Theta^{\mathbb{N}}$ denote the product space consisting of lists $\boldsymbol{i} = (i_1, i_2, \dots)$, where $1 \le i_j \le m$ for all j. Then

$$\phi(\boldsymbol{i}) = \lim_{n\to\infty} F_{i_1} \circ \cdots \circ F_{i_n}(x)$$

exists for each $\boldsymbol{i} \in \Theta^{\mathbb{N}}$ and $x \in \mathbb{R}^d$ and is independent of x; and $\pi = H^{\mathbb{N}} \circ \phi^{-1}$, where $H^{\mathbb{N}}$ is the product measure. Let $p_i = H\{i\}$. If $p_i > 0$ for all $i \in \Theta$, then the support of π is $K = \phi(\Theta^{\mathbb{N}})$ and $K = \cup_{i=1}^m F_i(K)$. For the remainder of this subsection we suppose that $p_i > 0$ for all i and the *strong separation condition*: $F_i(K) \cap F_j(K) = \varnothing$ for all $i \ne j$. Then ϕ is a homeomorphism from $\Theta^{\mathbb{N}}$, endowed with the product topology, onto K. See Hutchinson (1981). If $x = \phi(\boldsymbol{i})$, then $\boldsymbol{i}$ is called the *code for* x.

Lemma 1. *Let λ_i^2 be the minimal eigenvalue of $A_i' A_i$ ($\lambda_i > 0$), where $'$ denotes transpose, and*

$$\gamma = \min_{i \le m} \frac{\log(p_i)}{\log(\lambda_i)}.$$

Then there is a constant C for which $\pi\{B(x, r)\} \le C r^\gamma$ for all $x \in K$ and $r > 0$.

Proof. Let d_0 be the minimum distance between $F_i(K)$ and $F_j(K)$ for $i \ne j$, and $\lambda_* = \min_{i \le m} \lambda_i$. Then $d_0 > 0$, by compactness, and $\lambda_* > 0$ because each A_i is assumed to be non-singular. Let $x \in K$ have code $\boldsymbol{i}$ and let $0 < r < \lambda_* d_0$. Then there is a unique integer ℓ depending on x and r for which

$$d_0 \lambda_{i_1} \times \cdots \times \lambda_{i_{\ell+1}} \le r < d_0 \lambda_{i_1} \times \cdots \times \lambda_{i_\ell}. \tag{10}$$

If $y \in B(x, r)$ has code $\boldsymbol{j}$, then it is easily seen that $j_k = i_k$ for $k = 1, \dots, \ell$ (cf. Falconer (1990), Proposition 9.7). So,

$$\pi[B(x, r)] \le p_{i_1} \times \cdots \times p_{i_\ell} = \frac{p_{i_1} \times \cdots \times p_{i_\ell}}{(\lambda_{i_1} \times \cdots \times \lambda_{i_\ell})^\gamma} (\lambda_{i_1} \times \cdots \times \lambda_{i_\ell})^\gamma.$$

Then the second factor on the right is at most $[r/(d_0 \lambda_*)]^\gamma$, by (10), and the first is at most one, because its logarithm is

$$\sum_{k=1}^{\ell} \left[\gamma \log\left(\frac{1}{\lambda_{i_k}}\right) - \log\left(\frac{1}{p_{i_k}}\right) \right] \le \sum_{k=1}^{\ell} \log\left(\frac{1}{\lambda_{i_k}}\right) \left[\gamma - \frac{\log(p_{i_k})}{\log(\lambda_{i_k})} \right] \le 0.$$

The lemma follows directly.

If $\varnothing \ne A \subseteq \mathbb{R}^d$, let $N(A, \epsilon)$ be the minimal number of balls of radius ϵ required to cover A. Then

$$c(A) = \limsup_{\epsilon \to 0} \frac{\log[N(A, \epsilon)]}{\log(\epsilon^{-1})}$$

is called *the capacity of* A. By convention, $c(\varnothing) = 0$.

Theorem 4. *If B is a Borel set for which $c(\partial B) < \gamma$, then there is a $\psi \in \Psi$ for which $\kappa(\mathbf{1}_B, \psi)^2 < \infty$.*

Proof. Let $\beta = c(\partial B)$, so that $\beta < \gamma$; let $\beta' = (\beta + \gamma)/2$ and $0 < \alpha < \gamma - \beta'$; and let $\psi(r) = r^\alpha$. Then $K(\mathbf{1}_B; \psi; x) \le 1/\rho(\partial B, x)^{\alpha/2}$, and

$$\begin{aligned}\kappa^2(\mathbf{1}_B, \psi) &\le \int_0^\infty \pi\{x : \rho(\partial B, x)^{-\alpha} > y\}\,\mathrm{d}y \\ &\le 1 + \int_1^\infty \pi\{x : \rho(\partial B, x) < y^{-1/\alpha}\}\,\mathrm{d}y \\ &\le 1 + \alpha \int_0^1 \pi\{x : \rho(\partial B, x) < r\}\frac{1}{r^{1+\alpha}}\,\mathrm{d}r.\end{aligned}$$

By Lemma 1 and the definition of capacity, $\pi\{x : \rho(\partial B, x) < r\} \le CN(\partial B, r)r^\gamma \le Cr^{\gamma-\beta'}$ for all sufficiently small r. That $\kappa^2(\mathbf{1}_B; \psi) < \infty$ follows.

Example 1. The condition of the theorem is satisfied if ∂B has capacity 0, in particular, if $d = 1$ and B is an interval.

Example 2. If each F_i is a similitude, so that λ_i^2 is also the maximum eigenvalue of $A_i'A_i$, then the Hausdorff dimension of K is the solution to the equation $\lambda_1^D + \cdots + \lambda_m^D = 1$. If also $p_i = \lambda_i^D$, then $\gamma = D$.

4. Proof of Theorem 2

To prove Theorem 2, it is necessary to recall a little of the proof of Theorem 1. If X_n are as in (1), then $X_n = F_{\theta_n} \circ \cdots \circ F_{\theta_1}(X_0)$. Let $Z_n(x) = F_{\theta_1} \circ \cdots \circ F_{\theta_n}(x)$ for $x \in \mathcal{X}$ and $n \ge 1$ and

$$Y_n = Z_n(X_0) = F_{\theta_1} \circ \cdots \circ F_{\theta_n}(X_0).$$

Then the conditional distributions of X_n and Y_n given $X_0 = x$ are the same, because the distributions of $\theta_1, \ldots, \theta_n$ are the same. The proof of Theorem 1 consists of showing that

$$Y_\infty = \lim_{n\to\infty} Z_n(x)$$

exists with probability 1 simultaneously for all x and is independent of x. The stationary distribution π is the distribution of Y_∞. The following two lemmas are implicit in the proof of Theorem 1. For completeness, their proofs are sketched.

Lemma 2. *If (2) and (3) hold, then*

$$r_\alpha := \int_\Theta L_\theta^\alpha H\{\mathrm{d}\theta\} < 1 \tag{11}$$

for all sufficiently small $\alpha > 0$.

Proof. The derivative of the left side of (11) at $\alpha = 0$ is the integral in (2), which is negative. So, the left side of (11) is less than 1 for sufficiently small $\alpha > 0$.

Lemma 3. *Suppose that (2), (3), and (4) hold and that $X_0 \sim \pi$ has the stationary distribution. Then, for sufficiently small $\alpha > 0$, there are $0 < C_\alpha < \infty$ for which*

$$E[\rho(Y_n, Y_\infty)^\alpha] \le C_\alpha r_\alpha^n. \tag{12}$$

Proof. Let $\alpha \in (0, 1)$ be so small that $r_\alpha < 1$, and let $C_\alpha = 2I_\alpha(x_0)/(1 - r_\alpha)$, where $I_\alpha(x_0)$ is the integral in (4). Then, using the independence of $X_0, \theta_1, \theta_2, \ldots$ and properties of the L^p spaces for $p < 1$ (Dunford and Schwartz (1964), p. 171),

$$\begin{aligned} E\{\rho[x_0, Y_\infty]^\alpha\} &\le \sum_{n=0}^{\infty} E\{\rho[Z_n(x_0), Z_{n+1}(x_0)]^\alpha\} \\ &\le \sum_{n=0}^{\infty} E\left\{\prod_{j=1}^{n} L_{\theta_j}^{\alpha} \times \rho[x_0, F_{\theta_{n+1}}(x_0)]^\alpha\right\} = \tfrac{1}{2}C_\alpha. \end{aligned}$$

Clearly,

$$\begin{aligned} \rho(Y_n, Y_\infty) &\le \rho[Y_n, Z_n(x_0)] + \rho[Z_n(x_0), Y_\infty] \\ &\le \prod_{j=1}^{n} L_{\theta_j} \times \rho(x_0, X_0) + \sum_{k=n+1}^{\infty} \prod_{j=1}^{k-1} L_{\theta_j} \times \rho[x_0, F_{\theta_k}(x_0)]. \end{aligned}$$

So, using (4), (6), the independence of $X_0, \theta_1, \theta_2, \ldots$, and $X_0 \sim \pi$, $Y_\infty \sim \pi$,

$$\begin{aligned} E[\rho(Y_n, Y_\infty)^\alpha] &\le r_\alpha^n E[\rho(x_0, X_0)^\alpha] + \sum_{k=n+1}^{\infty} r_\alpha^{k-1} E\{\rho[x_0, F_{\theta_k}(x_0)]^\alpha\} \\ &\le C_\alpha r_\alpha^n, \end{aligned}$$

as asserted.

Proof of Theorem 2. Suppose that (2), (3), and (4) hold; let $X_0 \sim \pi$ have the stationary distribution; and denote conditional expectation given $X_0 = x$ by E^x. Let g and ψ be as in the statement of the Theorem and write $K(x)$ and κ for $K(g, \psi; x)$ and $\kappa(g, \psi)$. The solution to Poisson's equation will be $h = g + Qg + Q^2 g + \ldots$, where Q^n denote the iterates of Q. It is necessary to show that the sum converges in $L^2(\pi)$. To begin, observe that $E^x[g \circ Z_\infty(x)] = 0$ for all x, since $\theta_1, \theta_2, \ldots$ are independent of X_0. So,

$$Q^n g(x) = E^x[g \circ Z_n(x)] = E^x[g \circ Z_n(x) - g \circ Z_\infty(x)], \tag{13}$$

for all n and x. Thus,

$$\begin{aligned} |Q^n g(x)| \le{}& E^x\{K[Z_\infty(x)]\sqrt{\psi \circ \rho[Z_n(x), Z_\infty(x)]}\mathbf{1}_{[0,1]} \circ \rho[Z_n(x), Z_\infty(x)]\} \\ &+ E^x\{|g[Z_n(x)] - g[Z_\infty(x)]|\mathbf{1}_{[0,1]'} \circ \rho[Z_n(x), Z_\infty(x)]\}. \end{aligned}$$

So,

$$|Q^n g(x)|^2 \le I_n(x) + II_n(x),$$

where

$$I_n(x) = 2E^x\{K[Z_\infty(x)]^2\}E^x\{\psi \circ \rho[Z_n(x), Z_\infty(x)]\mathbf{1}_{[0,1]} \circ \rho[Z_n(x), Z_\infty(x)]\}$$

and

$$II_n(x) = 2E^x\{|g[Z_n(x)] - g[Z_\infty(x)]|^2\mathbf{1}_{[0,1]'} \circ \rho[Z_n(x), Z_\infty(x)]\}.$$

In $I_n(x)$, $\psi(\rho) \le \psi(\rho^\alpha)$ for any $0 < \alpha < 1$ when $0 \le \rho \le 1$, and $Z_\infty(x) \sim \pi$ for all x, so that $E^x\{K[Z_\infty(x)]^2\} \equiv \kappa^2$. Let $0 < \alpha < 1$ be so small that (11) and (12) hold. Then $I_n(x) \le 2\kappa^2 E^x\{\psi(\rho[Z_n(x), Z_\infty(x)]^\alpha)\}$ and

$$\begin{aligned}\int_{\mathcal{X}} I_n(x)\pi\{\mathrm{d}x\} &\le 2\kappa^2 E\{\psi(\rho[Y_n, Y_\infty]^\alpha)\}\\ &\le 2\kappa^2 \psi\{E[\rho(Y_n, Y_\infty)^\alpha]\} \le 2\kappa^2 \psi(C_\alpha r_\alpha^n)\end{aligned}$$

for all n, by Jensen's inequality and Lemma 3. For $\mathit{II}_n(x)$, let $p > 1$ be a value for which $g \in L^{2p}(\pi)$ and let q be the conjugate value, $1/p + 1/q = 1$. Then

$$\begin{aligned}\int_{\mathcal{X}} \mathit{II}_n(x)\pi\{\mathrm{d}x\} &\le 2[E|g(Y_n) - g(Y_\infty)|^{2p}]^{1/p} P\{\rho(Y_n, Y_\infty) > 1\}^{1/q}\\ &\le 8\|g\|_{2p}^2 C_\alpha^{1/q} r_\alpha^{n/q}\end{aligned}$$

for all n, where $\|g\|_r$ denotes the norm in $L^r(\pi)$. It follows that

$$\|Q^n g\|_2^2 \le 2\kappa^2 \psi(C_\alpha r_\alpha^n) + 8\|g\|_{2p}^2 C_\alpha^{1/q} r_\alpha^{n/q}$$

and

$$\|Q^n g\|_2 \le 2\kappa \sqrt{\psi(C_\alpha r_\alpha^n)} + 4\|g\|_{2p} C_\alpha^{1/2q} r_\alpha^{n/2q}. \tag{14}$$

The second term on the right side of (14) is clearly summable over $n \ge 1$. For the first, let $t_n = r_\alpha^n$. Then $t_n - t_{n+1} = (1 - r_\alpha)t_n$, and since ψ is non-decreasing,

$$(1 - r_\alpha)\sqrt{\psi(C_\alpha r_\alpha^{n+1})} = \frac{\sqrt{\psi(C_\alpha t_{n+1})}}{t_n}(t_n - t_{n+1}) \le \frac{1}{r_\alpha}\int_{t_{n+1}}^{t_n} \frac{\sqrt{\psi(C_\alpha t)}}{t}\,\mathrm{d}t$$

for $n \ge 0$. So,

$$r_\alpha(1 - r_\alpha)\sum_{n=1}^{\infty} \sqrt{\psi(C_\alpha r_\alpha^n)} \le \int_0^1 \frac{\sqrt{\psi(C_\alpha t)}}{t}\,\mathrm{d}t = \int_0^{C_\alpha} \frac{\sqrt{\psi(t)}}{t}\,\mathrm{d}t,$$

which is finite. It follows that $\|g\|_2 + \|Qg\|_2 + \|Q^2 g\|_2 + \cdots < \infty$ and, therefore, that $h = g + Qg + Q^2 g + \dots$ converges in $L^2(\pi)$.

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