FOURIER TRANSFORMS OF STATIONARY PROCESSES [1]

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ABSTRACT. We consider the asymptotic behavior of Fourier transforms of stationary and ergodic sequences. Under sufficiently mild conditions, central limit theorems are established for almost all frequencies as well as for a given frequency. Applications to the widely used linear processes and iterated random functions are discussed. Our results shed new light on the foundation of spectral analysis in that the asymptotic distribution of periodogram, the fundamental quantity in the frequency-domain analysis, is obtained.

1. INTRODUCTION

Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary and ergodic Markov chain on the state space $\mathcal{X}$; let g be a real-valued function on $\mathcal{X}$ such that $\mathbb{E}[g(X_0)] = 0$ and $\mathbb{E}[|g^2(X_0)|] < \infty$. Define the Fourier transform

$$S_n(\theta) = S_n(\theta; g) = \sum_{k=1}^{n} g(X_k)e^{ik\theta}, \quad \theta \in \mathbb{R}, \tag{1}$$

where $i = \sqrt{-1}$ is the imaginary unit. The setup (1) is general enough to allow one to consider $S_n^Y(\theta) = \sum_{k=1}^{n} Y_k e^{ik\theta}$ for any stationary and ergodic process $(Y_n)_{n\in\mathbb{Z}}$ by constructing a Markov chain $X_n = (\ldots, Y_{n-1}, Y_n)$ and $g(X_n) = Y_n$. The quantity (1) is of fundamental importance in the spectral analysis of stationary processes.

This paper considers asymptotic normality of (1). This problem has a substantial history. Rosenblatt (Theorem 5.3, p 131, 1985) considers mixing processes; Brockwell and Davis (Theorem 10.3.2., p 347, 1991), Walker (1965) and Terrin and Hurvich (1994) discuss linear processes. Other contributions can be found in Olshen (1967), Rootzén (1976), Yajima (1989) and Walker (2000) among others.

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Under surprisingly weak conditions (cf (2)), we show that the real and imaginary parts of $S_n(\theta)/\sqrt{n}$ are asymptotically iid normal for almost all frequencies $\theta \in \mathbb{R}$. Thus the periodogram $n^{-1}|S_n(\theta)|^2$ is asymptotically distributed as a multiple of a $\chi^2(2)$ random variable. For a fixed θ, we present sufficient conditions for the asymptotic normality of $S_n(\theta)/\sqrt{n}$. Our general results go beyond earlier ones by providing mild and verifiable conditions. In our proof, we apply the celebrated Carleson's theorem in Fourier analysis to construct approximating martingales. Applications to special sequences such as linear processes and iterated random functions which are widely used in time series modelling are discussed.

2. Main Results

Let $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ be the probability space on which the sequence X_n is defined, where $\mathcal{B}(\Omega)$ is a sigma algebra on Ω. For a random variable ξ on the space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, we denote its $\mathcal{L}^2(\mathbb{P})$ norm by $\|\xi\| = \sqrt{\mathbb{E}(|\xi|^2)} \le \infty$. Define the projection operator

$$\mathcal{P}_k \xi = \mathbb{E}(\xi | \ldots, X_{k-1}, X_k) - \mathbb{E}(\xi | \ldots, X_{k-1}).$$

Then by the Markovian property, $\mathcal{P}_1 g(X_n) = \mathbb{E}[g(X_n)|X_1] - \mathbb{E}[g(X_n)|X_0]$ for $n \in \mathbb{N}$. Let $\Re z$ and $\Im z$ be the real and the imaginary parts of the complex number z, namely $z = \Re z + i \Im z$, and let $|z| = \sqrt{(\Re z)^2 + (\Im z)^2}$. Denote by $N(\mu, \Sigma)$ the multivariate normal distribution with mean vector μ and covariance matrix Σ; denote by Id_p the $p \times p$ identity matrix. Theorems 1 and 2 concern the asymptotic normality for almost all θ and for a given θ respectively.

Theorem 1. *Assume that*

$$\sum_{n=1}^{\infty} \frac{1}{n} \|\mathbb{E}[g(X_n)|X_0]\|^2 < \infty. \tag{2}$$

Then (i) for almost all $\theta \in \mathbb{R}$ (Lebesgue), there exists $0 \le \sigma(\theta) < \infty$ such that

$$\begin{pmatrix} \Re \\ \Im \end{pmatrix} \frac{S_n(\theta)}{\sqrt{n}} \Rightarrow N[0, \sigma^2(\theta)\mathrm{Id}_2]. \tag{3}$$

(ii) Moreover, for almost all pairs (θ, φ) (Lebesgue), $S_n(\theta)/\sqrt{n}$ and $S_n(\varphi)/\sqrt{n}$ are asymptotically independent.

Theorem 2. *For a given $\theta \in [0, 2\pi)$, suppose there exists $\xi(\theta, X_0, X_1) \in \mathcal{L}^2(\mathbb{P})$ such that*

$$\sum_{n=1}^{N} e^{in\theta} \mathcal{P}_1 g(X_n) \to \xi(\theta, X_0, X_1) \quad (\mathcal{L}^2(\mathbb{P})), \tag{4}$$

as $N \to \infty$, and

$$\|\mathbb{E}[S_n(\theta)|X_0]\|^2 = o(n). \tag{5}$$

Then (i) if $\theta \neq 0, \pi$,

$$\begin{pmatrix} \Re \\ \Im \end{pmatrix} \frac{S_n(\theta)}{\sqrt{n}} \Rightarrow N[0, \sigma^2(\theta)\mathrm{Id}_2], \tag{6}$$

where $\sigma^2(\theta) = \|\xi(\theta, X_0, X_1)\|^2/2$, and (ii) $S_n(\theta)/\sqrt{n} \Rightarrow N(0, \sigma^2)$ if $\theta = 0$ or π, where $\sigma^2(\theta) = \|\xi(\theta, X_0, X_1)\|^2$.

Proposition 1. *Assume that*

$$\sum_{n=1}^{\infty} \|\mathcal{P}_1 g(X_n) - \mathcal{P}_1 g(X_{n+1})\| < \infty. \tag{7}$$

Then (4) and (5), and consequently (6), hold for all $0 < \theta < 2\pi$.

Condition (2) is fairly mild and it imposes very weak decay rate of $\|\mathbb{E}[g(X_n)|X_0]\|$. Rootzén (1976) obtained a central limit theorem under conditions which are not easily verifiable. In comparison, our Conditions (4), (5) and (7) are tractable in many cases. It is easily seen that (2) is equivalent to

$$\sum_{k=1}^{\infty} 2^{-k} \sum_{i=1}^{2^k} \|\mathbb{E}[g(X_i)|X_0]\|^2 < \infty \tag{8}$$

by exchanging the order of summation. Notice that $\|\mathbb{E}[g(X_n)|X_0]\|$ is non-increasing in n in view of $\|\mathcal{P}_{-n} g(X_0)\|^2 = \|\mathbb{E}[g(X_0)|X_{-n}]\|^2 - \|\mathbb{E}[g(X_0)|X_{-n-1}]\|^2$. So another equivalent condition of (2) is $\sum_{k=1}^{\infty} \|\mathbb{E}[g(X_{2^k})|X_0]\|^2 < \infty$. Inequality (8) is needed to establish a connection between $\sigma(\theta)$ in (3) and spectral densities (cf Proposition 2).

Let $r_k = \mathbb{E}[g(X_0)g(X_k)]$ be the covariance function and introduce the spectral distribution function $F(\theta)$, $0 \le \theta \le 2\pi$ via Herglotz's Theorem $r_k = \int_0^{2\pi} \exp(ik\theta) dF(\theta)$. Assume that F is absolutely continuous with the spectral density function f, namely $F(\theta) = \int_0^\theta f(u)du$.

Proposition 2. *Assume (2). Then for almost all $\theta \in [0, 2\pi]$ (Lebesgue), $f(\theta) = \sigma^2(\theta)/\pi$.*

Example 1. Iterated Random Functions. Let $(\mathcal{X}, \rho)$ be a complete and separable metric space and let $X_n = F_{\varepsilon_n}(X_{n-1})$, where $F_\varepsilon(\cdot) = F(\cdot, \varepsilon)$ is the ε-section of a jointly measurable function $F : \mathcal{X} \times \Upsilon \mapsto \mathcal{X}$ and $\varepsilon, \varepsilon_n, n \in \mathbb{Z}$ are iid random variables which take values in a second measurable space Υ. Define $L_\varepsilon = \sup_{x \ne x'} \rho[F_\varepsilon(x), F_\varepsilon(x')]/\rho(x, x')$. Diaconis and Freedman (1999) prove that X_n admits a unique stationary distribution (say Π) if

$$\mathbb{E}(L_\varepsilon^\alpha) < \infty, \quad \mathbb{E}(\log L_\varepsilon) < 0, \text{ and } \mathbb{E}[\rho^\alpha(x_0, F_\varepsilon(x_0))] < \infty \tag{9}$$

for some $\alpha > 0$ and $x_0 \in \mathcal{X}$. Let $\Delta_g(\delta) = \sup\{\|[g(X) - g(X')]\mathbf{1}_{[\rho(X,X') \le \delta]}\| : X, X' \sim \Pi\}$.

Corollary 1. *Assume (9), $\mathbb{E}[g(X_1)] = 0$ and $\mathbb{E}[|g(X_1)|^\ell] < \infty$ for some $\ell > 2$. (a) If*

$$\int_0^{1/2} \frac{\Delta_g^2(t)}{t|\log t|} dt < \infty, \tag{10}$$

then (2) holds. (b) If

$$\int_0^{1/2} \frac{\Delta_g(t)}{t} dt < \infty, \tag{11}$$

Then $\sum_{n=1}^\infty \|\mathbb{E}[g(X_n)|X_1]\| < \infty$ and hence (7) hold.

Proof. Let $X_0' \sim \Pi$ and X_0' be independent of X_0 and $(\varepsilon_k)_{k \in \mathbb{Z}}$. For $n \ge 1$ let $X_n' = F_{\varepsilon_n} \circ F_{\varepsilon_{n-1}} \circ \ldots \circ F_{\varepsilon_1}(X_0')$. Then (9) implies that there exists $\beta, C > 0$ and $0 < r < 1$ such that $\mathbb{E}[\rho^\beta(X_n, X_n')] \le Cr^n$ hold for all $n \ge 0$ (cf Lemma 3 in Wu and Woodroofe (2000)). Let $p = \ell/2$, $q = p/(p-1)$ and $\delta_n = (Cr^n)^{1/(q+\beta)}$. Since $\mathbb{E}[g(X_n')|X_0] = 0$,

$$\begin{aligned}
\|\mathbb{E}[g(X_n)|X_0]\| &\le \|[g(X_n) - g(X_n')]\mathbf{1}_{\rho(X_n,X_n')<\delta_n}\| + \|[g(X_n) - g(X_n')]\mathbf{1}_{\rho(X_n,X_n')\ge\delta_n}\| \\
&\le \Delta_g(\delta_n) + \|[g(X_n) - g(X_n')]^2\|_p^{1/2} \times [\mathbb{P}(\rho(X_n, X_n') \ge \delta_n)]^{\frac{1}{2q}} \le \Delta_g(\delta_n) + C'\delta_n^{1/2}
\end{aligned}$$

So (2) follows since (10) entails $\sum_{n=1}^{\infty} \Delta_g^2(\delta_n)/n < \infty$. (b) similarly follows. ■

Example 2. Linear processes. Let ε_k, $k \in \mathbb{Z}$ be iid random variables with mean 0 and finite variance; let $X_n = (\ldots, \varepsilon_{n-1}, \varepsilon_n)$ and $g(X_n) = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i}$, where a_i are real numbers such that $\sum_{i=0}^{\infty} a_i^2 < \infty$. Then (7) is reduced to $\sum_{i=1}^{\infty} |a_i - a_{i-1}| < \infty$ and the central limit theorem (6) holds for all $0 < \theta < 2\pi$. Notice that this can not be extended to $\theta = 0$ if $|a_n|$ is not summable. For example, $a_n = n^{-\beta}$, $1/2 < \beta < 1$ for $n \geq 1$.

3. Proofs

Let $\mathbb{U}$ be the uniform probability measure on $\Theta = [0, 2\pi)$ and $\mathcal{B}(\Theta)$ be the class of the Borel sets of Θ. Denote by $(\Theta \times \Omega, \mathcal{B}(\Theta) \times \mathcal{B}(\Omega), \mathbb{U} \times \mathbb{P})$ the product probability space of $(\Theta, \mathcal{B}(\Theta), \mathbb{U})$ and $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$.

Lemma 1. *Assume (2). Then for almost all $\theta \in \Theta\,(\mathbb{U})$,*

$$\frac{1}{\sqrt{n}} \mathbb{E}[S_n(\theta)|X_0] \to 0 \textit{ almost surely } (\mathbb{P}).$$

Proof of Lemma 1. Let $A \in \mathcal{B}(\Theta) \times \mathcal{B}(\Omega)$ be the set on which

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} e^{in\theta} \mathbb{E}[g(X_n)|X_0]$$

does not converge as $N \to \infty$. Let A^{ω} and A_{θ} be the $\omega-$section and θ-section of A respectively. Define

$$\Omega_0 = \left\{ \omega \in \Omega : \sum_{n=1}^{\infty} \frac{1}{n} |\mathbb{E}[g(X_n)|X_0]|^2 < \infty \right\}. \tag{12}$$

By (2), $\mathbb{P}(\Omega_0) = 1$. For $\omega \in \Omega_0$, by Carleson's theorem $\mathbb{U}(A^{\omega}) = 0$. Thus, by Fubini's theorem, for almost all $\theta \in \Theta\,(\mathbb{U})$, $\mathbb{P}(A_{\theta}) = 0$, i.e., $\sum_{n=1}^{\infty} e^{in\theta} \mathbb{E}[g(X_n)|X_0]/\sqrt{n}$ converges almost surely $(\mathbb{P})$. Hence by Kronecker's lemma (cf. Lemma 5.1.2, Chow and Teicher 1988), as $N \to \infty$,

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} \mathbb{E}[g(X_n)|X_0] \to 0 \quad a.s. \quad (\mathbb{P}).$$

■

Lemma 2. *For almost all $\theta \in \Theta\,(\mathbb{U})$, there exists $\xi(\theta, X_0, X_1) \in \mathcal{L}^2$ such that*

$$\mathcal{P}_1 S_n(\theta) \to \xi(\theta, X_0, X_1) \textit{ almost surely } (\mathbb{P}). \tag{13}$$

Proof of Lemma 2. Since the sequence $\mathbb{E}[g(X_0)|X_{1-n}] - \mathbb{E}[g(X_0)|X_{-n}], n = 0, 1, \ldots$ forms martingale differences,

$$\sum_{n=1}^{\infty} \|\mathcal{P}_1 g(X_n)\|^2 = \sum_{n=1}^{\infty} \|\mathcal{P}_{1-n} g(X_0)\|^2 = \|g(X_0)\|^2 < \infty.$$

Hence for almost all $\omega \in \Omega\,(\mathbb{P})$, $\sum_{n=1}^{\infty} |\mathcal{P}_1 g(X_n)|^2 < \infty$, which yields (13) by Carleson's theorem and the same argument as in Lemma 1. Next we show that $\xi(\theta, X_0, X_1) \in \mathcal{L}^2$. To this end, since the trigonometric bases $e^{in\cdot}, n \in \mathbb{Z}$ are orthogonal,

$$\sum_{k=1}^{n} \|\mathcal{P}_{1-k} g(X_0)\|^2 = \int_{\Theta} \mathbb{E}|\mathcal{P}_1 S_n(\theta)|^2 \mathbb{U}(d\theta).$$

By Fatou's lemma,

$$\begin{aligned}
\|g(X_0)\|^2 &\geq \liminf_{n\to\infty} \int_{\Theta} \mathbb{E}|\mathcal{P}_1 S_n(\theta)|^2 \mathbb{U}(d\theta) \\
&\geq \int_{\Theta} \mathbb{E} \liminf_{n\to\infty} |\mathcal{P}_1 S_n(\theta)|^2 \mathbb{U}(d\theta) = \int_{\Theta} \mathbb{E}[|\xi(\theta, X_0, X_1)|^2] \mathbb{U}(d\theta).
\end{aligned}$$

Hence for almost all $\theta \in \Theta\,(\mathbb{U})$, $\|\xi(\theta, X_0, X_1)\|^2 < \infty$. ■

Lemma 3. *Assume (2). Then for almost all $\theta \in \Theta\,(\mathbb{U})$,*

$$\mathbb{E}[\xi(\theta, X_0, X_1)|X_0] = 0 \quad a.s. \quad (\mathbb{P}) \tag{14}$$

Proof of Lemma 3. The almost sure convergence (13) in Lemma 2 alone does not guarantee the $\mathcal{L}^2$ convergence. To prove (14), we shall show below that, under (2), the Cesàro average $\sum_{n=1}^{N} \mathcal{P}_1 S_n(\theta)/N$ converges to $\xi(\theta, X_0, X_1)$ in $\mathcal{L}^2$. The relation (14) reveals the martingale structure and we can apply the martingale central limit theorem.

Let the tail

$$R_N(\theta, X_0, X_1) = \sum_{n=N+1}^{\infty} e^{in\theta} \{\mathbb{E}[g(X_n)|X_1] - \mathbb{E}[g(X_n)|X_0]\}.$$

Then $R_0(\theta, X_0, X_1) = \xi(\theta, X_0, X_1)$. Again by the orthogonality of the bases $e^{in\cdot}$,

$$\int_\Theta |R_N(\theta, X_0, X_1)|^2 \mathbb{U}(d\theta) = \sum_{n=1+N}^{\infty} |\mathcal{P}_1 g(X_n)|^2 < \infty.$$

Hence by (2)

$$\begin{aligned}\sum_{N=1}^{\infty} \frac{1}{N} \int_\Theta \mathbb{E}|R_N(\theta, X_0, X_1)|^2 \mathbb{U}(d\theta) &= \sum_{N=1}^{\infty} \frac{1}{N} \sum_{n=1+N}^{\infty} \mathbb{E}|\mathcal{P}_1 g(X_n)|^2 \\ &= \sum_{N=1}^{\infty} \frac{1}{N} \|\mathbb{E}[g(X_0)|X_{-N}]\|^2 < \infty.\end{aligned}$$

So for almost all $\theta \in \Theta\,(\mathbb{U})$,

$$\sum_{N=1}^{\infty} \frac{1}{N} \mathbb{E}|R_N(\theta, X_0, X_1)|^2 < \infty.$$

It follows from Kronecker's lemma that as $N \to \infty$

$$\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}|R_j(\theta, X_0, X_1)|^2 \to 0, \tag{15}$$

which yields

$$\mathbb{E}\left|\frac{1}{N} \sum_{j=1}^{N} R_j(\theta, X_0, X_1)\right|^2 \to 0$$

by Cauchy's inequality. In other words,

$$\bar{s}_N(\theta, X_0, X_1) := \frac{1}{N} \sum_{n=1}^{N} \mathcal{P}_1 S_n(\theta) \to \xi(\theta, X_0, X_1) \quad in\ \mathcal{L}^2.$$

Clearly, $\mathbb{E}[\bar{s}_N(\theta, X_0, X_1)|X_0] = 0$ $a.s.$ ($\mathbb{P}$). Hence as $N \to \infty$,

$$\begin{aligned}\mathbb{E}|\mathbb{E}[\xi(\theta, X_0, X_1)|X_0]| &= \mathbb{E}|\mathbb{E}[\xi(\theta, X_0, X_1) - \bar{s}_N(\theta, X_0, X_1)|X_0]| \\ &\le \mathbb{E}|\xi(\theta, X_0, X_1) - \bar{s}_N(\theta, X_0, X_1)| \\ &\le \|\xi(\theta, X_0, X_1) - \bar{s}_N(\theta, X_0, X_1)\| \to 0,\end{aligned}$$

which proves (14). ■

Lemma 4. *Suppose (15) holds for θ for which $\xi(\theta, X_0, X_1) \in \mathcal{L}^2(\mathbb{P})$. Then*

$$\begin{pmatrix} \Re \\ \Im \end{pmatrix} \frac{S_n(\theta) - \mathbb{E}[S_n(\theta)|X_0]}{\sqrt{n}} \Rightarrow N[0, \Sigma(\theta)], \tag{16}$$

where $\Sigma(\theta) = \frac{1}{2}\|\xi(\theta, X_0, X_1)\|^2 \mathrm{Id}_2$ if $\theta \neq 0, \pi$, and

$$\Sigma(\theta) = \begin{pmatrix} \|\xi(\theta, X_0, X_1)\|^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{if } \theta = 0, \pi.$$

Proof of Lemma 4. We generalize the arguments in Woodroofe (1992) where the asymptotic normality for the case $\theta = 0$ was obtained. Let $\xi_n(\theta, x_0, x_1) = \mathbb{E}[S_n(\theta)|X_1 = x_1] - \mathbb{E}[S_n(\theta)|X_0 = x_0]$ and write

$$\begin{aligned} S_n(\theta) - \mathbb{E}[S_n(\theta)|X_0] &= \sum_{j=1}^{n} e^{i\theta(j-1)} \xi(\theta, X_{j-1}, X_j) \\ &+ \sum_{j=1}^{n} e^{i\theta(j-1)} [\xi_{n-j+1}(\theta, X_{j-1}, X_j) - \xi(\theta, X_{j-1}, X_j)]. \end{aligned} \tag{17}$$

Now we claim that (15) implies that the second part in the proceeding display is $o_{\mathbb{P}}(\sqrt{n})$. Actually, by Lemma 3, $\mathbb{E}[\xi_{n-j+1}(\theta, X_{j-1}, X_j) - \xi(\theta, X_{j-1}, X_j)|X_{j-1}] = 0$ almost surely $\mathbb{P}$. Then it suffices to establish

$$\frac{1}{n} \sum_{j=1}^{n} \|\xi_{n-j+1}(\theta, X_{j-1}, X_j) - \xi(\theta, X_{j-1}, X_j)\|^2 \to 0, \tag{18}$$

which follows from (15) via the stationarity of X_j. By Lemma 3, $\xi(\theta, X_{j-1}, X_j)$, $j \in \mathbb{Z}$ is a martingale difference sequence, hence we can use the martingale central limit theorem to deduce (16) from (17). To this end, let

$$\xi(\theta, X_{j-1}, X_j) = A_j(\theta) + iB_j(\theta),$$

$$\alpha_j(\theta) = A_j(\theta) \cos[(j-1)\theta] - B_j(\theta) \sin[(j-1)\theta],$$

$$\beta_j(\theta) = B_j(\theta) \cos[(j-1)\theta] + A_j(\theta) \sin[(j-1)\theta].$$

Then $A_j(\theta)$, $B_j(\theta)$, $\alpha_j(\theta)$ and $\beta_j(\theta)$ form martingale difference sequences, and

$$\sum_{j=1}^n e^{i\theta(j-1)}\xi(\theta, X_{j-1}, X_j) = \sum_{j=1}^n [\alpha_j(\theta) + i\beta_j(\theta)].$$

For a fixed $0 \le \theta_0 < 2\pi$ let $D_j = \alpha_j(\theta)\cos\theta_0 + \beta_j(\theta)\sin\theta_0$. Then D_j are again martingale differences. By the Cramer-Wold device, it suffices to show that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^n D_j \Rightarrow N[0, \sigma^2(\theta, \theta_0)], \quad \text{where} \quad \sigma^2(\theta, \theta_0) = (\cos\theta_0, \sin\theta_0)\Sigma(\theta)\begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix}.$$

We now apply the martingale central limit theorem to $\sum_{j=1}^n D_j/\sqrt{n}$. The Lindeberg condition is automatically satisfied. Since $\mathbb{E}[\xi^2(\theta, X_{j-1}, X_j)|\mathcal{F}_{j-1}]$, $j \in \mathbb{Z}$ forms a stationary and ergodic sequence in $\mathcal{L}^1(\mathbb{P})$, by Lemma 5, the limit

$$V(\theta) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \mathbb{E}[\xi^2(\theta, X_{j-1}, X_j)|\mathcal{F}_{j-1}]e^{2(j-1)i\theta} \tag{19}$$

exists and equals to 0 almost surely ($\mathbb{P}$) if $\theta \neq 0$, π, and equals to $\|\xi(\theta, X_0, X_1)\|^2$ if $\theta = 0$ or π. After some algebra,

$$\begin{aligned}
\Sigma_{11}(\theta) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \mathbb{E}[\alpha_j^2(\theta)|\mathcal{F}_{j-1}] = \frac{\|\xi(\theta, X_0, X_1)\|^2}{2} + \frac{\Re V(\theta)}{2}, \\
\Sigma_{22}(\theta) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \mathbb{E}[\beta_j^2(\theta)|\mathcal{F}_{j-1}] = \frac{\|\xi(\theta, X_0, X_1)\|^2}{2} - \frac{\Re V(\theta)}{2}, \\
\Sigma_{12}(\theta) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \mathbb{E}[\alpha_j(\theta)\beta_j(\theta)|\mathcal{F}_{j-1}] = \frac{\Im V(\theta)}{2}
\end{aligned}$$

almost surely ($\mathbb{P}$), which completes the proof. ∎

Lemma 5. *Let η_n, $n \in \mathbb{Z}$ be a real-valued, stationary and ergodic sequence with $\mathbb{E}|\eta_1| < \infty$. Then for all $0 < \alpha < 2\pi$,*

$$\frac{1}{n}\sum_{t=1}^n e^{it\alpha}\eta_t \to 0 \tag{20}$$

almost surely and in $\mathcal{L}^1$.

Proof of Lemma 5. This result seems to be implied but not explicitly stated in the literature (see for example Doob (1953, pp 469–470), Rozanov (1967, pp 160–161), Wiener and Wintner (1941)). Since $\sum_{t=1}^{n} e^{it\alpha}\mathbb{E}(\eta_t)/n \to 0$, we can assume without loss of generality that $\mathbb{E}(\eta_t) = 0$. If $\alpha/(2\pi)$ is a rational number, then (20) clearly follows from the classical ergodic theorem. In the case that $\alpha/(2\pi)$ is not a rational number, define the rotation $\rho_\alpha : \Theta \mapsto \Theta$ by $\rho_\alpha(\theta) = \theta + \alpha \bmod 2\pi$. Then ρ_α is measure-preserving and ergodic. Let the random variable $\vartheta \sim \text{Uniform}(\Theta)$ be independent of η_n, $n \in \mathbb{Z}$. Since η_t is stationary and ergodic, the sequence $\varphi_t = e^{it\alpha+\vartheta}\eta_t$, $t \in \mathbb{Z}$ is then stationary (cf. Rozanov, p. 161) as well as ergodic on the product space $(\Theta \times \Omega, \mathcal{B}(\Theta) \times \mathcal{B}(\Omega), \mathbb{U} \times \mathbb{P})$. Therefore the classical ergodic theorem asserts that $n^{-1}\sum_{t=1}^{n} \varphi_t \to 0$ almost surely $(\mathbb{U} \times \mathbb{P})$ and in $\mathcal{L}^1$, which clearly entails (20) since $\mathbb{U}$ is a uniform distribution. ∎

Proof of Theorem 1. (i) It follows from Lemmas 1 and 4. (ii) By Lemmas 1, 3 and 4, for almost all $\theta, \varphi \in \Theta$ $(\mathbb{U})$, martingale differences $\xi(\tau, X_{t-1}, X_t) = A_t(\tau) + iB_t(\tau)$, $\tau = \theta, \varphi$ can be constructed and it suffices to show that $n^{-1/2}\sum_{t=1}^{n} \mathbf{v}_t \Rightarrow N[0, \Sigma(\theta, \varphi)]$, where random vectors $\mathbf{v}_t = (\alpha_t(\theta), \beta_t(\theta), \alpha_t(\varphi), \beta_t(\varphi))$ and the covariance matrix $\Sigma(\theta, \varphi) = \frac{1}{2}\text{Diag}(w(\theta), w(\theta), w(\varphi), w(\varphi))$, $w(\tau) = \|\xi(\tau, X_0, X_1)\|^2$. To this end, by (i) and the Cramer-Wold device, it follows from

$$\frac{1}{n}\sum_{t=1}^{n} \begin{pmatrix} \alpha_t(\theta)\alpha_t(\varphi) & \alpha_t(\theta)\beta_t(\varphi) \\ \beta_t(\theta)\alpha_t(\varphi) & \beta_t(\theta)\beta_t(\varphi) \end{pmatrix} \to 0 \quad a.s. \quad (\mathbb{P}),$$

which in view of Lemma 5 is valid provided that $\varphi \neq \theta$ and $\varphi + \theta \neq 2\pi$. ∎

Proof of Theorem 2. Observe that (16) in Lemma 4 holds for θ for which (4) is satisfied. Therefore (6) follows in view of (5). ∎

Proof of Proposition 1. Let $h_n = \sum_{t=1}^{n} e^{it\theta}$. Since $0 < \theta < 2\pi$, $h := \sup_{n\geq 1}|h_n| < \infty$. By (7), $\sum_{n=1}^{\infty} h_n[\mathcal{P}_1 g(X_n) - \mathcal{P}_1 g(X_{n+1})]$ converges in $\mathcal{L}^2(\mathbb{P})$. Thus (4) holds since $\|\mathcal{P}_1 g(X_n)\| \to 0$. To prove (5), let $\phi_t = \|\mathcal{P}_1 g(X_t) - \mathcal{P}_1 g(X_{t+1})\|$ and $\Phi_k = \sum_{t=k}^{\infty} \phi_t$. Then

(7) entails $\sum_{j=0}^{\infty}[\sum_{t=1}^{n}\phi_{t+j}]^2 \leq \Phi_1 \sum_{j=0}^{\infty}\sum_{t=1}^{n}\phi_{t+j} = o(n)$. So (5) follows from

$$\|\mathcal{P}_{-j}S_n(\theta)\| \leq h\sum_{t=1}^{n}\|\mathcal{P}_{-j}g(X_t) - \mathcal{P}_{-j}g(X_{t+1})\| + h\|\mathcal{P}_{-j}g(X_{n+1})\|,$$

$\|\mathbb{E}[S_n(\theta)|X_0]\|^2 = \sum_{j=0}^{\infty}\|\mathcal{P}_{-j}S_n(\theta)\|^2$ and $\sum_{j=0}^{\infty}\|\mathcal{P}_{-j}g(X_{n+1})\|^2 \leq \|g(X_0)\|^2$ by the orthogonality of $\mathcal{P}_k$. ■

Proof of Proposition 2. Let $K_n(\lambda) = \sin^2(n\lambda/2)/[2\pi n\sin^2(\lambda/2)]$ be the Fejér kernel. It is well known that $\|S_n(\theta)\|^2/(2\pi n) = \int_0^{2\pi} K_n(\lambda-\theta)f(\lambda)d\lambda$ (cf Theorem 8.2.7 in Anderson (1971, p. 454)), which by the Fejér-Lebesgue Theorem (cf Bary (1964, p. 139)) converges almost everywhere to $f(\theta)$. Therefore, by (17) and (18), for almost all θ (Lebesgue),

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\|\mathbb{E}[S_n(\theta)|X_0]\|^2 &= \lim_{n\to\infty}\frac{1}{n}\|S_n(\theta)\|^2 - \lim_{n\to\infty}\frac{1}{n}\|S_n(\theta) - \mathbb{E}[S_n(\theta)|X_0]\|^2 \\ &= 2\pi f(\theta) - \|\xi(\theta, X_0, X_1)\|^2. \end{aligned} \tag{21}$$

Notice that by (8),

$$\sum_{k=1}^{\infty}2^{-k}\int_0^{2\pi}\|\mathbb{E}[S_{2^k}(\theta)|X_0]\|^2\mathbb{U}(d\theta) = \sum_{k=1}^{\infty}2^{-k}\sum_{i=1}^{2^k}\|\mathbb{E}[g(X_i)|X_0]\|^2 < \infty.$$

Hence by the Borel-Cantelli Lemma, for almost all θ (Lebesgue), $2^{-k}\|\mathbb{E}[S_{2^k}(\theta)|X_0]\|^2 \to 0$ as $k\to\infty$, which by (21) entails $2\pi f(\theta) = \|\xi(\theta, X_0, X_1)\|^2 = 2\sigma^2(\theta)$ almost surely. ■

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