

MARTINGALE APPROXIMATIONS FOR SUMS OF STATIONARY PROCESSES¹

BY WEI BIAO WU AND MICHAEL WOODROOFE

University of Chicago and University of Michigan

June 1, 2003

Abstract

Approximations to sums of stationary and ergodic sequences by martingales are investigated. Necessary and sufficient conditions for such sums to be asymptotically normal conditionally given the past up to time 0 are obtained. It is first shown that a martingale approximation is necessary for such normality and then that the sums are asymptotically normal if and only if the approximating martingales satisfy a Lindeberg-Feller Condition. Using the explicit construction of the approximating martingales, a central limit theorem is derived for the sample means of linear processes. The conditions are not sufficient for the functional version of the central limit theorem. This is shown by an example, and a slightly stronger sufficient condition is given.

1. Introduction. The central limit problem for sums of stationary and ergodic processes has attracted continuing interest for over half a century, and two major lines of inquiry have developed. Under conditions of weak dependence such as strong mixing, blocking techniques have proved effective. Ibragimov (1962) provides an early account of this line. See Doukhan (1999) and Peligrad (1996) for more recent ones. An alternative approach, due to Gordin (1969), uses martingale approximation to establish asymptotic normality; see also Gordin and Lifsic (1978). Ho and Hsing (1997), Maxwell and Woodroffe (2000) and Wu and Woodroffe (2000) have followed this line recently. Here we come down

¹*Mathematical Subject Classification (2000)*: Primary 60F17, 60G42, 60F05; secondary 60J10

Key words and phrases. Central limit theorem, invariance principle, linear process, Markov chain, martingale, Poisson equation, stationary process.

on the side of martingale approximations by showing that if the partial sums of a stationary process are asymptotically normal in a suitable sense, then the martingale structure is present and the result could have obtained by using it. In addition, we sharpen the result of Maxwell and Woodroffe (2000), so that the necessary and sufficient conditions meet.

It is convenient to address the problem using the following notation. Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary and ergodic Markov chain with values in the state space \mathcal{X} , and consider additive functionals

$$S_n = S_n(g) = \sum_{i=1}^n g(X_i), \quad (1)$$

where $g : \mathcal{X} \rightarrow \mathbb{R}$ is a measurable function for which $g(X_0)$ has mean 0 and finite variance. The partial sums of any stationary and ergodic process $(\xi_n)_{n \in \mathbb{Z}}$, may be written in this form by letting $X_n = (\dots, \xi_{n-1}, \xi_n)$ and $g(X_n) = \xi_n$. Let π denote the marginal distribution of X_0 ; suppose that there is a regular conditional distribution for X_1 given X_0 , say $Q(x; B) = P(X_1 \in B | X_0 = x)$; and write $Qh(x) = E[h(X_1) | X_0 = x]$ a.e. (π) for $h \in L^1(\pi)$.

Let $\sigma_n^2 = E(S_n^2)$ and suppose throughout the paper that

$$\sigma_n^2 \rightarrow \infty \quad (2)$$

as $n \rightarrow \infty$. This condition is needed to avoid degeneracy since otherwise there exists a stationary sequence Y_n such that $g(X_n) = Y_n - Y_{n-1}$ (Theorem 18.2.2, Ibragimov and Linnik (1971)). It will not be repeated in statements of lemmas and theorems. Consider a doubly indexed sequence D_{nj} of random variables for which D_{nj} , $j = 1, 2, \dots$, are martingale differences with respect to the filter $\mathcal{F}_j = \sigma(\dots, X_{j-1}, X_j)$ for each n ; and let $M_{nk} = D_{n1} + \dots + D_{nk}$, so that M_{nk} , $k = 1, 2, \dots$, is a martingale with respect to \mathcal{F}_k for each n . The D_{nj} or M_{nk} are called a *martingale approximation* (to S_n) if

$$\max_{k \leq n} E[(S_k - M_{nk})^2] = o(\sigma_n^2). \quad (3)$$

A martingale approximation is called *stationary* if D_{nj} , $j = 1, 2, \dots$, is a stationary sequence for each n , and *non-triangular* if $D_{nj} = D_j$ are independent of n . It is shown

below that the existence of a stationary martingale approximation is equivalent to the existence of a non-triangular one. When (3) holds, asymptotic normality of S_n/σ_n is equivalent to asymptotic normality of M_{nn}/σ_n , and this question may be addressed using the Martingale Central Limit Theorem—for example, Billingsley (1995, pp. 475-478).

It is shown in Section 2 that a simple growth condition on $E[E(S_n|X_0)^2]$ is necessary and sufficient for the existence of a martingale approximation. Then, in Section 3, it is shown that S_n/σ_n is asymptotically standard normal, conditionally given X_0 , iff the approximating martingales satisfy the conditions of the Martingale Central Limit Theorem. These conditions are not sufficient for the functional version of the Central Limit Theorem. This is shown by example in Section 4, and a set of sufficient conditions is developed there.

Dedecker and Merlevede (2002) have used blocking techniques to obtain necessary and sufficient conditions for conditional asymptotic normality without assuming that the process is strongly mixing, or even ergodic. One of their conditions is closely related to (4) below, but their conditions do not include the existence of a martingale approximation and their uniform integrability condition for S_n^2/n looks quite different from our Lindeberg-Feller Conditions, (11) and (12). Using the explicit construction of martingales, we are able to obtain novel asymptotic theory for the sample means of linear processes, important and widely-used stationary processes.

2. Martingale Approximations. Below $\|\cdot\|$ denotes the norm in an L^2 space, which may vary from one use to the next. For example, $\|\cdot\|$ denotes the norm in $L^2(P)$ in (4), and the norm in $L^2(\pi)$ in (5).

LEMMA 1. *If*

$$\|E(S_n|X_0)\| = o(\sigma_n), \tag{4}$$

as $n \rightarrow \infty$, then there is a slowly varying function ℓ for which $\sigma_n^2 = n\ell(n)$.

PROOF. If relation (4) holds, then $|E[S_n(S_{n+m} - S_n)]| = |E[S_n E(S_{n+m} - S_n|X_n)]| \leq \|S_n\| \times \|E(S_m|X_0)\| \leq \epsilon_m \sigma_m \sigma_n$, where $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$. The lemma then follows

from Theorem 18.2.3 of Ibragimov and Linnik (1971) and the remark on page 330, after correcting for obvious typographical errors. \diamond

Relation (4) is crucial in what follows. Since $\|E[g(X_k)|X_0]\| = \|Q^k g\|$, it is implied by the condition, $\sum_{k=1}^n \|E[g(X_k)|X_0]\| = \sum_{k=1}^n \|Q^k g\| = o(\sigma_n)$, on the individual summands; but (4) is weaker and not unintuitive.

Recall that the equation $h = Qh + g$ is called *Poisson's Equation*. Below, we will call a sequence $h_n \in L^2(\pi)$ an *approximate solution to Poisson's Equation (for g)* if

$$\|h_n\| + n\|(I - Q)h_n - g\| = o(\sigma_n) \quad (5)$$

as $n \rightarrow \infty$. Also, if a_n and b_n are positive sequences, then $a_n \sim b_n$ iff $\lim_{n \rightarrow \infty} a_n/b_n = 1$.

THEOREM 1. *The following are equivalent:*

- a) *Relation (4) holds.*
- b) *There is an approximate solution to Poisson's Equation (5).*
- c) *There is a stationary martingale approximation (3).*
- d) *There is a non-triangular martingale approximation.*

In this case $E(D_{n1}^2) \sim \ell(n)$ for any stationary martingale approximation; and there is a stationary martingale approximation for which $\max_{k \leq n} \|S_k - M_{nk}\| \leq 3 \max_{k \leq n} \|E(S_k|X_0)\|$.

PROOF. It will be shown first that $a) \Rightarrow b) \Rightarrow c) \Rightarrow a)$ and then that $c) \Rightarrow d) \Rightarrow a)$. The remainder of the proof is placed between the two equivalences.

$a) \Rightarrow b)$: If (4) holds, let $h_n^o = g + Qg + \cdots + Q^{n-1}g$. Then $h_n^o(x) = E(S_n|X_1 = x)$ and $Qh_n^o(x) = E(S_n|X_0 = x)$ for *a.e. x* . Clearly, $h_n^o = g + Qh_n^o - Q^n g$, and $\|h_n^o - Qh_n^o\| \leq 2\|g\|$. So, $\|h_n^o\| \leq \|E(S_n|X_0)\| + 2\|g\| = o(\sigma_n)$, by (4). Next, let

$$h_n = \frac{h_1^o + \cdots + h_n^o}{n}.$$

Then $\|h_n\| \leq \max_{k \leq n} \|h_k^o\| = o(\sigma_n)$, and $h_n = g + Qh_n - Qh_n^o/n$. So, $n\|(I - Q)h_n - g\| \leq \|Qh_n^o\| = o(\sigma_n)$, establishing (5).

$b) \Rightarrow c)$: If (5) holds, let $f_n = g - (I - Q)h_n$,

$$D_{nk} = h_n(X_k) - Qh_n(X_{k-1}), \quad (6)$$

and $M_{nk} = D_{n1} + \cdots + D_{nk}$ for $k = 1, 2, \dots$. Then D_{n1}, D_{n2}, \dots are stationary martingale differences for each n . Next, writing $g(X_k) = h_n(X_k) - Qh_n(X_k) + f_n(X_k)$ in (1) and rearranging terms then leads to $S_k = M_{nk} + S_k(f_n) + Qh_n(X_0) - Qh_n(X_k)$. So,

$$\max_{k \leq n} \|S_k - M_{nk}\| \leq n\|f_n\| + 2\|Qh_n\| = o(\sigma_n),$$

and (3) holds.

$c) \Rightarrow a)$: If (3) holds, then $\|E(S_n|X_0)\| = \|E(S_n - M_{nn}|X_0)\| \leq \|S_n - M_{nn}\| = o(\sigma_n)$.

This establishes the equivalence of $a)$, $b)$, and $c)$.

For any stationary martingale approximation in (3), $nE(D_{n1}^2) = E(M_{nn}^2) = E(S_n^2) + o(\sigma_n^2) \sim \sigma_n^2$, so that $E(D_{n1}^2) \sim \ell(n)$; and for the stationary martingale approximation constructed in the proof of $a) \Rightarrow c)$, $\max_{k \leq n} \|S_k - M_{nk}\| \leq n\|f_n\| + 2\|Qh_n\| \leq 3 \max_{k \leq n} \|Qh_k^o\| \leq 3 \max_{k \leq n} \|E(S_n|X_0)\|$.

$c) \Rightarrow d) \Rightarrow a)$. If there is a stationary martingale approximation, then (4) holds and there is a stationary martingale approximation of the form (6), say $M_{nk} = D_{n1} + \cdots + D_{nk}$. Then $\|M_{nk} - M_{mk}\| \leq \|S_k - M_{nk}\| + \|S_k - M_{mk}\|$, and $m\|D_{n1} - D_{m1}\|^2 = \|M_{nm} - M_{mm}\|^2 \leq 2\|S_m - M_{mm}\|^2 + 2\|S_m - M_{nm}\|^2$. Let $D_k = D_{kk}$ and $M_n = D_1 + \cdots + D_n$. Then M_1, M_2, \dots is a martingale, $\|S_n - M_n\| \leq \|S_n - M_{nn}\| + \|M_{nn} - M_n\|$, and $\|S_n - M_{nn}\| = o(\sigma_n)$, by assumption. Here $\|M_{nn} - M_n\|^2 = \sum_{k=1}^n \|D_{nk} - D_{kk}\|^2 = \sum_{k=1}^n \|D_{n1} - D_{k1}\|^2$. So,

$$\|M_{nn} - M_n\|^2 \leq \sum_{k=1}^n \frac{2\|S_k - M_{kk}\|^2}{k} + \sum_{k=1}^n \frac{2\|S_k - M_{nk}\|^2}{k} = I_n + II_n,$$

say. Karamata's Theorem [see e.g. Theorem 0.6 in Resnick (1987)] implies that for $\alpha > -1$, $\sum_{i=1}^n i^\alpha \ell(i) \sim n^{1+\alpha} \ell(n)/(1+\alpha)$. Hence $I_n = o[\sum_{k=1}^n \ell(k)] = o[n\ell(n)] = o(\sigma_n^2)$. For the second term, notice that $\|M_{nk}\|^2 = k\|D_{n1}\|^2$ and $\|D_{n1}\|^2 \sim \ell(n)$. Then for some positive

C and any positive $\epsilon < 1/2$,

$$\begin{aligned} II_n &\leq 4 \sum_{k \leq n\epsilon} \frac{\|S_k\|^2 + \|M_{nk}\|^2}{k} + 2 \sum_{n\epsilon < k \leq n} \frac{\|S_k - M_{nk}\|^2}{k} \\ &\leq C \sum_{k \leq n\epsilon} [\ell(k) + \ell(n)] + \frac{2}{\epsilon} \max_{k \leq n} \|S_k - M_{nk}\|^2, \end{aligned}$$

which by Karamata's Theorem implies that $\limsup_{n \rightarrow \infty} II_n/\sigma_n^2 \leq 2C\epsilon$ and, therefore, $\limsup_{n \rightarrow \infty} II_n/\sigma_n^2 = 0$. Conversely, if there is a non-triangular martingale approximation, then $\|E(S_n|X_0)\| = \|E(S_n - M_n|X_0)\| \leq \|S_n - M_n\| = o(\sigma_n)$, as above. \diamond

As it is clear from Theorem 1, martingale approximations are not unique. Any two are asymptotically equivalent, however in following sense: If (3) holds, and if $M'_{nk} = D'_{n1} + \dots + D'_{nk}$ is a second martingale approximation, then

$$E \left[\max_{k \leq n} (M'_{nk} - M_{nk})^2 \right] \leq 4 \|M'_{nn} - M_{nn}\|^2 = 4 \sum_{k=1}^n \|D'_{nk} - D_{nk}\|^2, \quad (7)$$

using Doob's (1953, pp. 317) inequality, and $\|M'_{nn} - M_{nn}\| \leq \|S_n - M'_{nn}\| + \|S_n - M_{nn}\| = o(\sigma_n)$.

If $\ell(n) \rightarrow \infty$ in Lemma 1, then it is impossible to have a martingale approximation that is both non-triangular and stationary, but if $\sigma_n^2 \sim \sigma^2 n$, then it is. Maxwell and Woodroffe (2000) show that if $\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \|E(S_n|X_0)\| < \infty$, then there is a martingale M_1, M_2, \dots with stationary increments for which $\|S_n - M_n\|^2 = o(n)$. A simplified proof of a special case of this result is provided in Lemma 5 below, along with an explicit bound on $\|S_n - M_n\|$.

The proof of Theorem 1 contains the explicit construction of $D_{nk} = h_n(X_k) - Qh_n(X_{k-1})$ in terms of any approximate solution h_n to Poisson's Equation and also an explicit construction of h_n . An alternative approximate solution to Poisson's Equation is provided next.

COROLLARY 1. *If (4) holds, then (5) holds with $h_n = f_{\frac{1}{n}}$, where*

$$f_{\epsilon}(x) = \sum_{j=1}^{\infty} (1 + \epsilon)^{-j} Q^{j-1} g$$

for $0 < \epsilon < 1$.

PROOF. From the definition, it is clear that $(1 + \epsilon)f_\epsilon = g + Qf_\epsilon$ and $(I - Q)h_n = g - h_n/n$. So, the corollary would follow from $\|h_n\| = o(\sigma_n)$. To see this first observe that $f_\epsilon = \epsilon \sum_{k=1}^{\infty} (1 + \epsilon)^{-k-1} h_k^o$, by partial summation, where $h_k^o(x) = E(S_k | X_1 = x)$, as above. Let $V(s) = \sum_{k=1}^{\infty} \sigma_k s^{k+1}$. Then $\|h_n\| = o[V(n/(n+1))]/n$ by (4), and $V(s) \sim \frac{1}{2}\sqrt{\pi}(1-s)^{-3/2}\ell^{1/2}(1/(1-s))$ as $s \uparrow 1$ by Tauberian's theorem—for example, Feller (1971, p. 445). \diamond

For some examples, let $\cdots \eta_{-1}, \eta_0, \eta_1, \cdots$ be a stationary sequence of martingale differences with finite variance; and let $\cdots \theta_{-1}, \theta_0, \theta_1 \cdots$ be a sequence of i.i.d. random elements that is independent of $\cdots \eta_{-1}, \eta_0, \eta_1, \cdots$. Then $X_k = [(\cdots \theta_{k-1}, \theta_k), (\cdots, \eta_{k-1}, \eta_k)]$ is a stationary Markov chain with values in $\mathcal{X} = \Theta^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, where Θ is the range of the θ_k and \mathbb{N} is the non-negative integers. Let $a_j : \mathcal{X} \rightarrow R$ be measurable functions for which

$$\sum_{j=0}^{\infty} E[a_j(X_0)^2 \eta_1^2] < \infty.$$

Then

$$\xi_k = \sum_{j=0}^{\infty} a_j(X_{k-j-1}) \eta_{k-j} \tag{8}$$

converges *w.p.1* for each k and is of the form $g(X_k)$. Processes of the form (8) include linear processes with constant a_j and $\theta_k \equiv 0$, and are called *quasi-linear processes* below. They also include many non-linear time series models, like auto-regressive processes with random coefficients. Writing $\xi_k = \sum_{j \leq k} a_{k-j}(X_{j-1}) \eta_j$ and letting $b_n = a_0 + \cdots + a_n$, it is easily seen that

$$E(S_n | X_0) = \sum_{j \leq 0} [b_{n-j}(X_{j-1}) - b_{-j}(X_{j-1})] \eta_j,$$

$$S_n - E(S_n | X_0) = \sum_{j=1}^n b_{n-j}(X_{j-1}) \eta_j.$$

So, $\sigma_n^2 = \sigma_{n,1}^2 + \sigma_{n,2}^2$, with

$$\sigma_{n,1}^2 = \|E(S_n|X_0)\|^2 = \sum_{j=0}^{\infty} E\{[b_{j+n}(X_0) - b_j(X_0)]^2 \eta_1^2\},$$

$$\sigma_{n,2}^2 = \|S_n - E(S_n|X_0)\|^2 = \sum_{j=1}^{n-1} E[b_j(X_0)^2 \eta_1^2],$$

and (4) is equivalent to $\sigma_{n,1}^2 = o(\sigma_{n,2}^2)$. In this case, by (6) in the proof of Theorem 1, $D_{nk} = \bar{b}_n(X_{k-1})\eta_k$, where $\bar{b}_n = (b_0 + \dots + b_{n-1})/n$, by some routine calculations, and $E[\bar{b}_n(X_0)^2 \eta_1^2]$ must be slowly varying. Observe that if $b_j(X_0)$, $j \leq 0$, are independent of η_1 , then $E[\bar{b}_n(X_0)^2 \eta_1^2] = E[\bar{b}_n(X_0)^2]E(\eta_1^2)$ and that $\sigma_{n,1}^2$ and $\sigma_{n,2}^2$ simplify similarly.

EXAMPLE 1. Linear Processes. Suppose that a_n are constants and (without loss of generality) that $E(\eta_k^2) = 1$. Then b_n are also constants, and $\sigma_{n,1}^2 = o(\sigma_{n,2}^2)$ iff

$$\sum_{j=0}^{\infty} (b_{j+n} - b_j)^2 = o\left[\sum_{k=1}^{n-1} b_k^2\right]. \quad (9)$$

If a_n are absolutely summable and $b := \sum_{n=0}^{\infty} a_n \neq 0$, then $\sigma_{n,2}^2 \sim b^2 n$ and there is a C for which $\sigma_{n,1}^2 \leq C \sum_{i=1}^n \sum_{j=i}^{\infty} |a_j| = o(n)$, so that (9) holds. Relation (9) also holds if $b \neq 0$ and $b_n = b + O(1/n)$. If $a_0 = 0$ and $a_n = 1/n$ for $n \geq 1$, then $b_n \sim \log(n)$, and $\sigma_{n,2}^2 \sim n \log^2(n)$. In this case $\sigma_{n,1}^2 = O(n) = o(\sigma_{n,2}^2)$, so that (9) holds. To see this observe that for $j \geq 3$, $1/(j+1) \leq \int_j^{j+1} u^{-1} du$ and $[\log(j+n) - \log(j)]^2 \leq \int_{j-1}^j [\log(u+n) - \log(u)]^2 du$, so that

$$\begin{aligned} \sum_{j=3}^{\infty} (b_{j+n} - b_j)^2 &\leq \sum_{j=3}^{\infty} \left(\int_j^{j+n} \frac{1}{u} du \right)^2 = \sum_{j=3}^{\infty} \log^2 \frac{j+n}{j} \\ &\leq \int_2^{\infty} \log^2 \frac{u+n}{u} du = O(n). \end{aligned}$$

Similarly, if $a_0 = 0$, $a_1 = 1/\log(2)$, and $a_n = 1/\log(n+1) - 1/\log(n)$ for $n \geq 2$, then $\sigma_{n,2}^2 \sim n/\log^2(n)$ and $\sigma_{n,1}^2 = O[n/\log^3(n)] = o(\sigma_{n,2}^2)$, so that (4) holds. On the other hand, if $a_n = n^{-\beta}$, where $1/2 < \beta < 1$, then there are positive constants $c_{1,\beta}$ and $c_{2,\beta}$ for which $\sigma_{n,i}^2 \sim c_{i,\beta} n^{3-2\beta}$ as $n \rightarrow \infty$ for $i = 1, 2$, so that (4) fails. \diamond

3. Asymptotic Normality. The main result of this section is that $S_n^* := S_n/\sigma_n$ is asymptotically standard normal given X_0 , as described below, iff there is a martingale approximation, Equation (3), and the D_{nk} satisfy the conditions of the Martingale Central Limit Theorem, equations (11) and (12) below. In more detail, let P^x and E^x denote the regular conditional probability and conditional expectation for \mathcal{F}_∞ given $X_0 = x$; and let F_n denote the conditional distribution function

$$F_n(x; z) = P^x(S_n^* \leq z).$$

Further let Φ denote the standard normal distribution function; and let Δ denote the Levy distance between two distribution functions. Then by asymptotically normality given X_0 , we mean

$$\lim_{n \rightarrow \infty} \int_{\mathcal{X}} \Delta[\Phi, F_n(x; \cdot)] \pi\{dx\} = 0. \quad (10)$$

Clearly, (10) implies that S_n^* is asymptotically standard normal, but (10) is stronger in general; it implies that S_n^* is asymptotically standard normal for related models in which X_0 has any distribution that is absolutely continuous with respect to the stationary distribution. Such a property is needed, for example, if asymptotic normality is used to set approximate error bounds for Markov Chain Monte Carlo experiments. See, for example, Tierney (1994). Under conditions of weak dependence, (10) can be deduced from (unconditional) asymptotic normality of S_n^* . See Proposition 1 below for the details and the continuation of Example 1 for a case in which S_n^* is (unconditionally) normal, but (10) fails.

LEMMA 2. *If (10) holds, then (4) holds; that is, $\|E(S_n|X_0)\| = o(\sigma_n)$.*

PROOF. The proof follows Maxwell and Woodroffe (2000), who considered the special case $\sigma_n^2 \sim cn$; it is included because the lemma is crucial to what follows. Let \Rightarrow denote convergence in distribution. Notice that if $Z_m \Rightarrow \Phi$, then $\liminf_{m \rightarrow \infty} \text{var}(Z_m) \geq 1$, where $\text{var}(Z_m) = E(Z_m^2) - [E(Z_m)]^2$. To see this, for $J > 0$ let $T_{m,J} = \min[\max(Z_m, -J), J]$.

Then $\lim_{m \rightarrow \infty} \text{var}(T_{m,J}) = \int_{\mathbb{R}} \{\min[\max(u, -J), J]\}^2 d\Phi(u) \rightarrow_{J \rightarrow \infty} 1$. By Corollary 4.3.2 in Chow and Teicher (1978), $\text{var}(Z_m) \geq \text{var}(T_{m,J})$. So $\liminf_{m \rightarrow \infty} \text{var}(Z_m) \geq 1$.

Assume otherwise that there is a $\delta > 0$ such that $\|\mathbb{E}(S_{n'}^*|X_0)\| > \delta$ along a subsequence $\{n'\}$. By (10), there exists a further subsequence $\{n''\} \subset \{n'\}$ such that $\Delta[\Phi, F_{n''}(x; \cdot)] \rightarrow 0$ for almost all x (π). Let $\tau_n^2(x) = \text{var}(S_n^*|X_0 = x)$. By the result in the previous paragraph, $\liminf_{n'' \rightarrow \infty} \tau_{n''}^2(x) \geq 1$ for almost all x (π). Thus $1 \leq \liminf_{n'' \rightarrow \infty} \int_{\mathcal{X}} \tau_{n''}^2(x) \pi(dx)$ by Fatou's lemma. On the other hand, the integral in the previous inequality equals to $\|S_{n''}^*\|^2 - \|\mathbb{E}(S_{n''}^*|X_0)\|^2 \leq 1 - \delta^2$, which is a contradiction. \diamond

LEMMA 3. *Suppose there is a martingale approximation $\{D_{nk}\}$ for which*

$$\frac{1}{\sigma_n^2} \sum_{k=1}^n E(D_{nk}^2 | \mathcal{F}_{k-1}) \rightarrow^p 1 \quad (11)$$

and

$$\frac{1}{\sigma_n^2} \sum_{k=1}^n E(D_{nk}^2 \mathbf{1}_{\{|D_{nk}| \geq \epsilon \sigma_n\}} | \mathcal{F}_{k-1}) \rightarrow^p 0 \quad (12)$$

hold for each $\epsilon > 0$. Then for any martingale approximation $\{D'_{nk}\}$ (say), (11) and (12) are satisfied. In addition,

$$\sup_{0 < t \leq 1} \left| \frac{1}{\sigma_n^2} \sum_{k \leq nt} E(D'_{nk}{}^2 | \mathcal{F}_{k-1}) - t \right| \rightarrow^p 0. \quad (13)$$

PROOF. Observe that $E|E(D'_{nk}{}^2 | \mathcal{F}_{k-1}) - E(D_{nk}^2 | \mathcal{F}_{k-1})| \leq E|D'_{nk}{}^2 - D_{nk}^2|$ and

$$E(D'_{nk}{}^2 \mathbf{1}_{\{|D'_{nk}| \geq 2\epsilon \sigma_n\}} | \mathcal{F}_{k-1}) \leq 2E(D_{nk}^2 \mathbf{1}_{\{|D_{nk}| \geq \epsilon \sigma_n\}} | \mathcal{F}_{k-1}) + 2E(|D'_{nk}{}^2 - D_{nk}^2| | \mathcal{F}_{k-1}).$$

So, if D_{nk} satisfy (11) and (12), then so do D'_{nk} , since

$$E \left(\sum_{k=1}^n |D'_{nk}{}^2 - D_{nk}^2| \right) \leq \sqrt{\sum_{k=1}^n \|D'_{nk} + D_{nk}\|^2} \times \sqrt{\sum_{k=1}^n \|D'_{nk} - D_{nk}\|^2} = o(\sigma_n^2)$$

as in (7). To establish (13), let $m = \lfloor nt \rfloor$, where $\lfloor x \rfloor$ is the greatest integer that does not exceed x ; let $M'_{nk} = D'_{n_1} + \cdots + D'_{nk}$. Observe that $\sigma_m^2 / \sigma_n^2 \rightarrow t$ as $n \rightarrow \infty$, (11) implies

$$\frac{1}{\sigma_n^2} \sum_{k=1}^m E(D'_{mk}{}^2 | \mathcal{F}_{k-1}) \rightarrow^p t.$$

Since $\|M'_{nm} - M'_{mm}\| \leq \|M'_{nm} - S_m\| + \|S_m - M'_{mm}\| = o(\sigma_n)$,

$$E \left(\sum_{k=1}^m |D_{nk}'^2 - D_{mk}'^2| \right) \leq \sqrt{\sum_{k=1}^m \|D'_{nk} + D'_{mk}\|^2} \times \sqrt{\sum_{k=1}^m \|D'_{nk} - D'_{mk}\|^2} = o(\sigma_n^2).$$

Let $V_n(t) = \sigma_n^{-2} \sum_{k=1}^m E(D_{nk}'^2 | \mathcal{F}_{k-1})$. Then $V_n(t) - t \xrightarrow{p} 0$. Let $I \geq 2$ be an integer. Observe that $\sup_{t \leq 1} |V_n(t) - t| \leq \max_{i \leq I} |V_n(i/I) - i/I| + 1/I$. By first letting $n \rightarrow \infty$ and then $I \rightarrow \infty$, (13) follows. \diamond

THEOREM 2. *Relation (10) holds iff there is a martingale approximation for which (11) and (12) hold.*

PROOF. Suppose first that there is a martingale approximation (3) for which (11) and (12) hold. By Lemma 3, assume without loss of generality that the martingale approximation is defined by (6). Then, it suffices to establish (10) for all subsequences n_r , $r \geq 1$, that increase to ∞ sufficiently fast as $r \rightarrow \infty$. Observe that D_{nk} , $k = 1, 2, \dots$ are martingale differences with respect to P^x for *a.e.* $x(\pi)$ by the Markov property. If $n_r \rightarrow \infty$ sufficiently quickly as $r \rightarrow \infty$, then (12) and (13) both hold with convergence in probability replaced by convergence *w.p.1* (P), and $\lim_{n \rightarrow \infty} (S_n - M_{nn})/\sigma_n = 0$ *w.p.1* too. So, these relations hold *w.p.1* (P^x) for *a.e.* $x(\pi)$. Then, for *a.e.* $x(\pi)$, $\lim_{r \rightarrow \infty} F_{n_r}(x; z) = \Phi(z)$ for all z , by the Martingale Central Limit Theorem applied conditionally given $X_0 = x$, and (10) holds (along the subsequence) by the Bounded Convergence Theorem.

The converse will be deduced from Theorem 2 of Gänsler and Häusler (1979), that provide necessary conditions for the functional version of the Martingale Central Limit Theorem. If (10) holds, then so does (4), by Lemma 2; and then there is a stationary martingale approximation, by Theorem 1. So, the issues are (11) and (12). Let \mathcal{B} denote a standard Brownian motion. Then, since the process is stationary and S_n^* is asymptotically normal given X_0 ,

$$\frac{1}{\sigma_n} [S_{[nt_1]}, S_{[nt_2]} - S_{[nt_1]}, \dots, S_{[nt_k]} - S_{[nt_{k-1}]}]$$

$$\Rightarrow [\mathbb{B}_{t_1}, \mathbb{B}_{t_2} - \mathbb{B}_{t_1}, \dots, \mathbb{B}_{t_k} - \mathbb{B}_{t_{k-1}}]$$

for every choice of $0 < t_1 < t_2 < \dots < t_k \leq 1$, where \Rightarrow denotes convergence in distribution.

For example, if $k = 2$, $0 < s < t < 1$, and $m = \lfloor nt \rfloor - \lfloor ns \rfloor$, then

$$\begin{aligned} & \left| P[S_{\lfloor ns \rfloor} \leq \sigma_n y, S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \leq \sigma_n z] - \Phi\left(\frac{y}{\sqrt{s}}\right) \Phi\left(\frac{z}{\sqrt{t-s}}\right) \right| \\ & \leq \int_{\mathcal{X}} \left| F_m\left(x; \frac{\sigma_n z}{\sigma_m}\right) - \Phi\left(\frac{z}{\sqrt{t-s}}\right) \right| \pi\{dx\} \\ & + \Phi\left(\frac{z}{\sqrt{t-s}}\right) \left| P[S_{\lfloor ns \rfloor} \leq \sigma_n y] - \Phi\left(\frac{y}{\sqrt{s}}\right) \right| \end{aligned}$$

which approaches zero as $n \rightarrow \infty$ since $\sigma_m/\sigma_n \rightarrow \sqrt{t-s}$. Next let

$$M_n(t) = \frac{1}{\sigma_n} \sum_{k \leq nt} D_{nk} \quad (14)$$

for $0 \leq t < 1$, and $M_n(1) = M_n(1-)$. Then the finite dimensional distributions of M_n converges to those of \mathbb{B} , since $|S_{\lfloor nt \rfloor} - M_{\lfloor nt \rfloor}|/\sigma_n \rightarrow^p 0$ for each $0 < t < 1$; and since $E[M_n(t)^2] \sim ntE(D_{n1}^2)/\sigma_n^2 \rightarrow t$, it follows that each $M_n(t)^2$, $n \geq 1$, are uniformly integrable for each $0 < t \leq 1$. It then follows from the Martingale Inequality that M_n are tight in $D[0, 1]$. So, M_n converges to \mathbb{B} in $D[0, 1]$; and relations (11) and (12) then follow from Theorem 2 of Gänsler and Häeusler (1979). \diamond

EXAMPLE 1 *Continued.* For linear processes, relations (11) and (12) follow from (4), which implies that $D_{nk} = \bar{b}_n \eta_k$ and that $|\bar{b}_n|$ is slowly varying, for the stationary martingale approximation constructed in the proof of Theorem 1. On the other hand, if $a_n = n^{-\beta}$, where $1/2 < \beta < 1$, then S_n/σ_n is asymptotically standard normal, but (4) and (10) do not hold. \diamond

In the next corollary, let π_1 denote the joint distribution of X_0 and X_1 , so that $\pi_1(B) = P[(X_0, X_1) \in B]$ for measurable $B \subseteq \mathcal{X}^2$; and let $H_n(x_0, x_1) = h_n(x_1) - Qh_n(x_0)$, so that $D_{nk} = H_n(X_{k-1}, X_k)$ in (6).

COROLLARY 2. *If (4) holds and $H_n/\sqrt{\ell(n)} \rightarrow H \in L^2(\pi_1)$, then (10) holds.*

PROOF. Let $D_{nk} = H_n(X_{k-1}, X_k)$ be the martingale approximation (6) and let $D'_{nk} = \sqrt{\ell(n)}H(X_{k-1}, X_k)$ and $M'_{nk} = D'_{n1} + \dots + D'_{nk}$. Then the D'_{nk} provide another stationary martingale approximation, since $\|M_{nn} - M'_{nn}\|^2 = n\|D_{n1} - D'_{n1}\|^2 = n\|H_n - \sqrt{\ell(n)}H\|^2 = o(\sigma_n^2)$. Moreover, the D'_{nk} satisfy (11) and (12). For example,

$$\frac{1}{\sigma_n^2} \sum_{k \leq nt} E(D_{nk}^2 | X_k) = \frac{1}{n} \sum_{k \leq nt} E[H(X_{k-1}, X_k)^2 | X_{k-1}] \rightarrow tE[H(X_0, X_1)^2]$$

by the Ergodic Theorem; and $E[H(X_0, X_1)^2] = \|H\|^2 = 1$, since $\|H_n\|^2 \sim \ell(n)$, by Lemma 1. Condition (12) may be obtained similarly. \diamond

To relate the condition in Corollary 2 to the sums S_n , first observe that $H_n/\sqrt{\ell(n)}$ converges in $L^2(\pi_1)$ iff $D_{n1}/\sqrt{\ell(n)}$ converges in $L^2(P)$ and next that D_{n1} is the average of $E(S_k|X_1) - E(S_k|X_0)$ over $k = 1, \dots, n$. It is not difficult to see that if $[E(S_n|X_1) - E(S_n|X_0)]/\sqrt{\ell(n)}$ converges in $L^2(P)$, then so does $D_{n1}/\sqrt{\ell(n)}$. Woodroffe (1992) shows how the condition of Corollary 2 can be related to the Fourier coefficients of g when X_k is a Bernoulli or Lebesgue shift process.

EXAMPLE 2. For a quasi-linear process (8), $D_{n1} = \bar{b}_n(X_0)\eta_1$. So, if $\dots\eta_{-1}, \eta_0, \eta_1\dots$ are i.i.d., $\sigma_{n,1} = o(\sigma_{n,2})$, and $\bar{b}_n/\sqrt{\ell(n)} \rightarrow b \neq 0$ in $L^2(\pi)$, the $D_{n1}/\sqrt{\ell(n)}$ converges in $L^2(P)$ and, therefore, (11) and (12) both hold. \diamond

3.1. *Strong mixing processes.* Many classical results concerning asymptotic normality for stationary processes require strong mixing conditions; see for example, Peligrad (1986, 1996). Here we show how strong mixing assumption is related to our main condition (4). Let $X_n = (\dots, \xi_{n-1}, \xi_n)$ and $S_n = \xi_1 + \dots + \xi_n$, where $(\xi_i)_{i \in \mathbb{Z}}$ is a stationary sequence that is strong mixing; that is,

$$\alpha_n := \sup_{A \in \mathcal{F}_0, B \in \mathcal{G}_n} |P(A \cap B) - P(A)P(B)| \rightarrow 0$$

as $n \rightarrow \infty$, where $\mathcal{F}_n = \sigma(\dots, \xi_{n-1}, \xi_n)$ and $\mathcal{G}_n = \sigma(\xi_n, \xi_{n+1}, \dots)$.

LEMMA 4. If F and G are two distribution functions and $\epsilon > 0$, then there are continuous functions w_1, \dots, w_m , depending only on ϵ and G , for which $|w_i| \leq 1$ and $\int_{\mathbb{R}} w_i dG = 0$ for all i and

$$\Delta(G, F) \leq \epsilon + \max_{i \leq m} \left| \int_{\mathbb{R}} w_i dF - \int_{\mathbb{R}} w_i dG \right|.$$

PROOF. The proof consists of first finding a and b for which $G(a) + 1 - G(b) \leq \epsilon$, then partitioning $[a, b]$ into $a = x_0 < x_1 < \dots < x_m = b$, where $x_i - x_{i-1} \leq \epsilon/2$, constructing piecewise linear functions u_i for which $u_i(x) = 1$ for $x \leq x_{i-1}$ and $u_i(x) = 0$ for $x \geq x_i$, and the letting $w_i = u_i - \int_{\mathbb{R}} u_i dG$. The details are omitted. \diamond

PROPOSITION 1. Assume that $(\xi_n)_{n \in \mathbb{Z}}$ is a strong mixing process with mean 0 and finite variance. Then $S_n^* \Rightarrow \Phi$ implies (10), and consequently (4).

PROOF. By Lemma 4, it suffices to show that $\int_{\mathcal{X}} | \int_{\mathbb{R}} w(z) F\{x; dz\} | \pi\{dx\} \rightarrow 0$ as $n \rightarrow \infty$ for all continuous $w : \mathbb{R} \rightarrow [-1, 1]$ for which $\int_{\mathbb{R}} w d\Phi = 0$; and since the inner integral is just $E^x[w(S_n^*)]$, it suffices to show that $E|E[w(S_n^*)|X_0]| \rightarrow 0$ as $n \rightarrow \infty$ for all such w . To see this let $m = m_n$ be a sequence for which $m \rightarrow \infty$ and $S_m/\sigma_n \rightarrow^p 0$; and let $\tilde{S}_n = (S_{n+m} - S_m)/\sigma_n$. Further, let $w : \mathbb{R} \rightarrow [-1, 1]$ be a continuous function for which $\int_{\mathbb{R}} w d\Phi = 0$ and let $w_n(x) = E^x[w(S_n^*)]$ and $\tilde{w}_n(x) = E^x[w(\tilde{S}_n)]$. Then $E[w_n(X_0)] = E[w(S_n^*)] \rightarrow 0$, since $S_n^* \Rightarrow \Phi$; $E|w_n(X_0) - \tilde{w}_n(X_0)| \leq E|w(S_n^*) - w(\tilde{S}_n)| \rightarrow 0$, since $\tilde{S}_n - S_n^* \rightarrow^p 0$ as $n \rightarrow \infty$; and

$$E|\tilde{w}_n(X_0)|^2 = \int \tilde{w}_n(X_0) w(\tilde{S}_n) dP \leq E[w(\tilde{S}_n)]^2 + 4\alpha_m \rightarrow 0,$$

by standard mixing inequalities—for example, Hall and Heyde (1980, pp. 277). So, $E|w_n(X_0)| \rightarrow 0$ as $n \rightarrow \infty$ as required. \diamond

4. An Invariance Principle. Let

$$B_n(t) = \frac{1}{\sigma_n} S_{[nt]}$$

for $0 \leq t < 1$, $\mathbb{B}_n(1) = \mathbb{B}_n(1-)$, where $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x . If (10) holds, then the finite dimensional distributions of \mathbb{B}_n converge to those of standard Brownian motion \mathbb{B} , and M_n converges in distribution to \mathbb{B} in the space $D[0, 1]$, both from the proof of Theorem 2. Relations (4) and (10) do not imply that \mathbb{B}_n converges in distribution to \mathbb{B} in $D[0, 1]$, however.

EXAMPLE 3. Let G be a symmetric distribution function for which

$$1 - G(y) \sim \frac{1}{y^2 \log^{\frac{3}{2}}(y)}$$

as $y \rightarrow \infty$. Let $\cdots \eta_{-1}, \eta_0, \eta_1 \cdots \sim \Phi$ and $\cdots Y_{-1}, Y_0, Y_1, \cdots \sim G$ be independent random variables. Let $a_0 = 0, a_1 = 1/\log(2)$, and $a_k = 1/\log(k+1) - 1/\log(k)$ for $k \geq 2$, as in Example 1. Define ξ_k by (8); let $\xi'_k = \xi_k + Y_k - Y_{k-1}$; and let $S_n = \xi_1 + \cdots + \xi_n$ and $S'_n = \xi'_1 + \cdots + \xi'_n$. Then (4), (11), and (12) hold for both S_n and S'_n with $\sigma_n^2 \sim n/\log^2(n)$.

In this example,

$$\frac{1}{\sigma_n} \max_{k \leq 1} |Y_k - Y_0| \rightarrow \infty$$

in probability, so that \mathbb{B}_n and \mathbb{B}'_n cannot both converge to \mathbb{B} . \diamond

In Theorem 3 and Corollary 3 below, we consider the special case in which $\sigma^2 = \lim_{n \rightarrow \infty} \sigma_n^2/n$ exists. These results improve Theorem 2 and Corollary 4 in Maxwell and Woodroffe (2000) by imposing weaker condition as well as by obtaining a stronger result. The heart of the matter is whether there is a martingale approximation for which $\max_{k \leq n} |S_k - M_{nk}|/\sqrt{n} \rightarrow 0$ in probability. This question is addressed first. Two lemmas are needed.

LEMMA 5. *Suppose that for some $q > 1$,*

$$\|E(S_n | X_0)\| = o(\sqrt{n} \log^{-q} n). \quad (15)$$

Then there is a martingale M_1, M_2, \cdots with stationary increments for which $\|S_n - M_n\| = o(\sqrt{n} \log^{1-q} n)$.

PROOF. Recall the construction D_{nk} and $M_{nk} = D_{n1} + \dots + D_{nk}$ from (6) and also that $\max_{k \leq n} \|S_k - M_{nk}\| \leq 3 \max_{k \leq n} \|E(S_k | X_0)\|$. Thus, $\max_{k \leq n} \|S_k - M_{nk}\| = o[\sqrt{n} \log^{-q}(n)]$ in the present context. So, if $m \geq 2$ and $m \leq n \leq 2m$, then $\|M_{nm} - M_{mm}\| = o[\sqrt{m} \log^{-q}(m)]$. Since $\|M_{nm} - M_{mm}\|^2 = m\|D_{n1} - D_{m1}\|^2 = m\|H_n - H_m\|^2$, it then follows that

$$\sum_{k=j}^{\infty} \|H_{2^k} - H_{2^{k-1}}\| \leq \sum_{k=j}^{\infty} o[\log^{-q}(2^k)] = o[\log^{1-q}(2^j)]. \quad (16)$$

It follows that H_{2^k} has a limit H , say, in $L^2(\pi_1)$ and that $\|H - H_m\| = o[\log^{1-q}(m)]$. Letting $D_k = H(X_{k-1}, X_k)$ and $M_n = D_1 + \dots + D_n$, the lemma then follows from $\|S_n - M_n\| \leq \|S_n - M_{nn}\| + \sqrt{n}\|H_n - H\|$. \diamond

LEMMA 6. Let $Y_k, k \in \mathbb{Z}$ be a second order stationary process with mean 0 and let $T_n = Y_1 + \dots + Y_n$. Then

$$E \left[\max_{k \leq n} T_j^2 \right] \leq d \sum_{j=0}^d 2^{d-j} \|T_{2^j}\|^2,$$

where $d = \lceil \log_2(n) \rceil$, the least integer that is greater than or equal to $\log_2(n)$.

PROOF. The proof uses a simple chaining argument and appears in Doob (1953, pp. 156) for uncorrelated random variables. Briefly, any integer $k \leq n$ may be written as $k = 2^{r_1} + \dots + 2^{r_j}$, where $0 \leq r_j < \dots < r_1 \leq d$. So,

$$|T_k|^2 = \left| \sum_{i=1}^j (T_{2^{r_1+\dots+2^{r_i}}} - T_{2^{r_1+\dots+2^{r_{i-1}}}}) \right|^2 \leq j \sum_{i=1}^j |T_{2^{r_1+\dots+2^{r_i}}} - T_{2^{r_1+\dots+2^{r_{i-1}}}}|^2,$$

where an empty sum is to be interpreted as 0, and

$$\max_{k \leq n} |T_k|^2 \leq d \sum_{j=0}^d \sum_{i=1}^{2^{d-j}} |T_{i2^j} - T_{(i-1)2^j}|^2,$$

from which the lemma follows by stationarity. \diamond

THEOREM 3. Let $R_n = S_n - M_n$, where M_n are as in Lemma 5. If $g \in L^p$ for some $p > 2$ and (15) holds for $q \geq 2$, then $\sigma^2 = \lim_{n \rightarrow \infty} \sigma_n^2/n$ exists, and

$$\lim_{n \rightarrow \infty} P \left[\max_{j \leq n} |R_j| \geq \epsilon \sqrt{n} \right] = 0 \quad (17)$$

for each $\epsilon > 0$; and if (15) holds for some $q > 5/2$, then $\lim_{n \rightarrow \infty} P^x[\max_{j \leq n} |R_j| \geq \epsilon \sqrt{n}] = 0$ for a.e. x (π) for each $\epsilon > 0$.

PROOF. Let $\gamma = 1/4 - 1/(2p) > 0$, where p is as in the statement of the theorem, $a = a_m = \lceil 2^{m\gamma} \rceil$, and $b = b_m = \lceil 2^{m(1-\gamma)} \rceil$. Then

$$\max_{j \leq 2^m} |R_j| \leq \max_{1 \leq k \leq b} \left[|R_{ak}| + \max_{0 \leq j \leq a} |R_{ak+j} - R_{ak}| \right].$$

Here

$$\begin{aligned} \max_{0 \leq j \leq a} |R_{ak+j} - R_{ak}| &\leq \max_{0 \leq j \leq a} |M_{ak+j} - M_{ak}| + \max_{0 \leq j \leq a} |S_{ak+j} - S_{ak}| \\ &\leq \max_{0 \leq j \leq a} |M_{ak+j} - M_{ak}| + a \max_{j \leq 2^m} |g(X_j)| \end{aligned}$$

for each k . So,

$$\begin{aligned} P^x \left[\max_{j \leq 2^m} |R_j| \geq 3\epsilon \sqrt{2^m} \right] &\leq P^x \left[\max^* \frac{|M_k - M_j|}{\sqrt{2^m}} \geq \epsilon \right] \\ &+ P^x \left[\max_{j \leq 2^m} \frac{|g(X_j)|}{\sqrt{2^m}} \geq \frac{\epsilon}{a} \right] + P^x \left[\max_{k \leq b} \frac{|R_{ak}|}{\sqrt{2^m}} \geq \epsilon \right], \quad (18) \end{aligned}$$

where \max^* runs over all pairs (j, k) such that $1 \leq j, k \leq 2^m$ and $|k - j| \leq a$. The first term clearly tends to 0 for a.e. x (π), by the functional Martingale Central Limit Theorem. The second term in (18) also converges to 0 for a.e. x (π) by the Borel-Cantelli Lemma, since

$$\int_{\mathcal{X}} P^x \left[\max_{j \leq 2^m} \frac{|g(X_j)|}{\sqrt{2^m}} \geq \frac{\epsilon}{a} \right] \pi\{dx\} \leq \frac{a^p}{\epsilon^p} 2^{m(1-p)} E|g(X_1)|^p,$$

and the right side is summable over m (recalling that $a = \lceil 2^{\gamma m} \rceil$ and observing that $p\gamma + 1 - p < 0$). Similarly, for the third term on the right side of (18),

$$\int_{\mathcal{X}} P^x \left[\max_{k \leq b} \frac{|R_{ak}|}{\sqrt{2^m}} \geq \epsilon \right] \pi\{dx\} = P \left[\max_{k \leq b} \frac{|R_{ak}|}{\sqrt{2^m}} \geq \epsilon \right] \leq \frac{1}{\epsilon^2} E \left[\max_{k \leq b} \frac{|R_{ak}|}{\sqrt{2^m}} \right]^2,$$

and, letting $d = \lceil \log_2(b) \rceil$,

$$E \left[\max_{k \leq b} \frac{|R_{ak}|}{\sqrt{2^m}} \right]^2 \leq \frac{d}{2^m} \sum_{i=0}^d 2^{d-i} \|R_{a2^i}\|^2$$

$$\begin{aligned}
&\leq \frac{d}{2^m} \sum_{i=0}^d 2^{d-i} \frac{o(a2^i)}{\log^{2(q-1)}(a2^i)} \\
&= \frac{abd}{2^m} o\left[\frac{1}{m^{2q-3}}\right] = o(m^{4-2q}).
\end{aligned}$$

by Lemmas 5 and 6. Relation (17) follows immediately, since $ab = O(2^m)$ and $d = O(m)$; and if $q > 5/2$, then $o(m^{4-2q})$ is summable and $P^x[\max_{k \leq b} |R_{ak}| \geq \epsilon\sqrt{2^m}] \rightarrow 0$ for *a.e.* x , by the Borel-Cantelli Lemma. \diamond

Now let G_n and Ψ be the distributions of B_n and Brownian motion in $D[0, 1]$, and let Δ denote the Prokhorov metric for $D[0, 1]$.

COROLLARY 3. *If (15) holds for some $q \geq 2$ and $0 < \sigma^2 < \infty$, then*

$$\lim_{n \rightarrow \infty} \int_{\mathcal{X}} \Delta[\Psi, G_n(x; \cdot)] \pi\{dx\} = 0;$$

and if $q > 5/2$ in (15), then $\lim_{n \rightarrow \infty} \Delta[\Psi, G_n(x; \cdot)] = 0$ for *a.e.* x (π).

PROOF. Let $K_n(x; \cdot)$ be the distribution of M_n in $D[0, 1]$. Then $K_n(x; \cdot) \Rightarrow \Psi$ as $n \rightarrow \infty$ for *a.e.* x (π), by the Functional Central Limit Theorem, and

$$\Delta[\Psi, G_n(x; \cdot)] \leq \Delta[\Psi, K_n(x; \cdot)] + P^x \left[\max_{k \leq n} |R_k| \geq \epsilon\sigma_n \right] + \epsilon$$

for each $\epsilon > 0$. The case $q > 5/2$ follows immediately, and the case $2 \leq q \leq 5/2$ from $\int_{\mathcal{X}} P^x[\max_{k \leq n} |R_k| \geq \epsilon\sigma_n] \pi\{dx\} = P[\max_{k \leq n} |R_k| \geq \epsilon\sigma_n]$. \diamond

COROLLARY 4. *If (15) holds for some $q \geq 2$ and $\sigma^2 = 0$, then $\max_{k \leq n} |S_k|/\sqrt{n} \rightarrow^p 0$; and if $q > 5/2$, then $\lim_{n \rightarrow \infty} P^x[\max_{k \leq n} |S_k| \geq \epsilon\sqrt{n}] = 0$ for *a.e.* x (π) for each $\epsilon > 0$.*

PROOF. In this case $S_k = R_k$. \diamond

REMARK 1. A simple sufficient condition for (15) is

$$\|E[g(X_n)|X_0]\| = \mathcal{O}(n^{-1/2} \log^{-q} n). \quad (19)$$

However, (15) allows processes of the form (8) with $a_n = n^{-\beta}(-1)^n$ for $n \geq 1$, where $1/2 < \beta < 1$. In this case (19) is violated. Wu (2002) derived central limit theorems for

processes of this sort whose covariances are summable but not absolutely summable. A typical example is the Gegenbauer process which exhibits long-range dependence and has oscillatory covariances (Beran, 1994).

Acknowledgement. The authors gratefully acknowledge the helpful and constructive comments provided by the anonymous referee.

REFERENCES

- BERAN, J. (1994). *Statistics for Long-Memory Processes*. Chapman and Hall, New York.
- BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.
- BILLINGSLEY, P. (1995). *Probability and Measure*. Wiley, New York.
- CHOW, Y. S. and TEICHER, H. (1978). *Probability Theory*, 2nd ed. Springer, New York.
- DEDECKER, J. and MERLEVEDE, P. (2002). Necessary and sufficient conditions for the conditional central limit theorem. *Ann. Probab.* **30** 1044–1081.
- DOOB, J. (1953). *Stochastic Processes*. Wiley.
- DOUKHAN, P. (1999). A new weak dependence condition and applications to moment inequalities. *Stoch. Proc. Appl.*, **84** 313-342.
- FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, **2**. John Wiley and Sons.
- GÄNSSLER, P. and HÄEUSLER, E. (1979). Remarks on the functional central limit theorem for martingales. *Z. Wahrsch. Verw. Gebiete* **50** (3): 237-243.
- GORDIN, M. I. (1969). The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR*, **188** 739-741.
- GORDIN, M.I. and LIFSIC, B. (1978). The central limit theorem for stationary Markov processes. *Doklady*, **19** 392-394.
- HALL, P. G. and HEYDE, C. C. (1980). *Martingale limit theory and its applications*. Academic Press, New York.
- HO, H.-C. and HSING, T. (1997). Limit theorems for functionals of moving averages. *Ann. Probab.* **25** 1636-1669.

- IBRAGIMOV, I. A. (1962). Some limit theorems for stationary processes. *Theory Probab. Appl.*, **7** 349-382.
- IBRAGIMOV, I. A. and YU. V. LINNIK. (1971) *Independent and stationary sequences of random variables*. Groningen, Wolters-Noordhoff.
- MAXWELL, M. and WOODROOFE, M. (2000). Central limit theorems for additive functionals of Markov chains. *Ann. Probab.* **28** 713-724.
- PELIGRAD, M. (1986). Recent advances in the central limit theorem and its weak invariance principle for mixing sequences of random variables (A survey). In *Dependence in Probability and Statistics: A Survey of Recent Results* (E. Eberlein and M. S. Taqqu, eds.) 193-224. Birkhauser, Boston.
- PELIGRAD, M. (1996). On the asymptotic normality of sequences of weak dependent random variables. *J. Theor. Probab.*, **9** 703-715.
- RESNICK, S. I. (1987). *Extreme values, Regular Variation and Point Processes*. Springer, New York.
- TIERNEY, L. (1994). Markov chains for exploring posterior distributions. *Ann. Statist.*, **22** 1701-1728.
- WOODROOFE, M. (1992). A central limit theorem for functions of a Markov chain with applications to shifts. *Stoc. Proc. Appl.*, **41** 33-44.
- WU, W. B. (2002). Central limit theorems for functionals of linear processes and their applications. *Statistica Sinica*, **12** 635-649.
- WU, W. B. AND M. WOODROOFE (2000). A central limit theorem for iterated random functions. *Journal of Applied Probability* **37**, 748-755.

DEPARTMENT OF STATISTICS
 THE UNIVERSITY OF CHICAGO
 5734 S. UNIVERSITY AVENUE
 CHICAGO, IL 60637
 E-MAIL: wbwu@galton.uchicago.edu

DEPARTMENT OF STATISTICS
 THE UNIVERSITY OF MICHIGAN
 462 WEST HALL

ANN ARBOR, MICHIGAN 48109-1285
E-MAIL: michaelw@umich.edu