A Statistical Theory of the Kalman Filter

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Outline

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Kalman Filter

Let \( \{\chi_t : t \in \mathbb{N}\} \) be a sequence of states in \( \mathbb{R}^d \) related by the stochastic difference equation

\[
\chi_{t+1} = D_t \chi_t + \eta_t
\]

where \( D_t \in \mathbb{R}^{d \times d} \) and \( \eta_t \sim \mathcal{N}(0, Q_M) \).

Our objective is to estimate \( \chi_t \) given \( D_t \) and a sequence of noisy observations \( \{Z_t : t \in \mathbb{N}\} \) related to \( \chi_t \)

\[
Z_t = O_t \chi_t + \xi_t
\]

where \( O_t \in \mathbb{R}^{d \times d'} \) and \( \xi_t \sim \mathcal{N}(0, Q_O) \)

The Kalman Filter estimates \( \chi_{t+1} \) and its covariance \( C_{t+1} \) from estimates \( \hat{\chi}_t \) and \( C_t \) using:

\[
\hat{\chi}_{t+1} = \arg\min \left\{ \|Z_{t+1} - O_{t+1} \hat{\chi}_t\|^2_{Q_O^{-1}} + \|\chi - D_t \hat{\chi}_t\|^2_{(D_t C_t D_t^T + Q_M)^{-1}} \right\}
\]

\[
C_{t+1}^{-1} = (D_t C_t D_t^T + Q_M)^{-1} + O_{t+1}^T Q_0^{-1} O_{t+1}
\]
Kalman Filter

Observability/Controllability

- Kalman, 1960
- Kalman & Bucy, 1961

Exponential Convergence (Fixed State, Deterministic Observations)

- Johnstone, Johnson, Bitmead & Anderson, 1982
- Bittanti, Bolzern & Campi, 1990
- Parkum, Poulsen & Holst, 1992
- Cao & Schwarz, 2003

Nonexpansive in Specialized Metrics

- Bougerol, 1993
- Atar & Zeitouni, 1997
- Le Gland, et. al., 2004
- Carli & Sepulchre, 2015
Kalman Filter

Questions important to statisticians:

1. **Q:** How many observations are needed (with stochastic dynamics and noisy observations) to estimate a single state "well"?
   **A:** With probability approaching 1, objective decays like $d/k$.

2. **Q:** Does the covariance estimate actually estimate the asymptotic covariance of the parameter estimate?
   **A:** Yes, it does.

3. **Q:** How important are the *apriori* parameter and covariance estimate?
   **A:** ...
Kalman Filter

Questions important to numerical optimizer:

1. **Q:** Can we do "better" than the Kalman Filter?  
   **A:** No.

2. **Q:** What is the rate of convergence of the Kalman Filter?  
   **A:** Sublinear to a single state. Presumably, we cannot do better.

3. **Q:** Do we need to carry around the covariance estimate from state to state?  
   **A:** ...

4. **Q:** Can we parallelize computations over relevant dimensions?  
   **A:** ...
Kalman Filter

Questions important to UQ community (not in other categories):

1. **Q:** How well does the KF "perform" when the dynamics are just stable (e.v. of 1) or unstable (e.v. > 1)?
   **A:** ...

2. **Q:** How well does the KF "perform" when deterministic dynamics are misspecified?
   **A:** ...

3. **Q:** How well does the KF "perform" when model noise is misspecified/correlated from state to state?
   **A:** ...

4. **Q:** How well does the (E)KF "perform" when the observation function is misspecified?

5. **Q:** How well does the KF "perform" when the observation model noise is misspecified/correlated from state to state?
Single State Estimation

To get to statistical notation:

- Initialization:
  \[ \beta_0 = D_t \hat{\chi}_t \quad M_0 = D_tC_tD_T^T + Q_M \]

- Observations, Errors, Covariates:
  - \( Y_k \) is the \( k^{th} \) component of \( Z_t \)
  - \( \epsilon_k \) is the \( k^{th} \) component of \( \xi_t \)
  - \( X_k \) is the \( k^{th} \) row of \( O_t \)

- True Parameter, Data Model:
  \[ \beta^* = \chi_{t+1} \quad Y_k = X_k^T \beta^* + \epsilon_k \]
**Selective Assimilation**

**Question:** Suppose $\hat{\beta}_k$ is the estimator of $\beta^*$ after $k$ observations are assimilated. How quickly does $\hat{\beta}_k$ converge to $\beta^*$?

**General Assumptions:**
- $\epsilon_k$ are independent, $\mathbb{E} [\epsilon_k|X_k] = 0$ and $\sup_k \mathbb{E} [\epsilon_k^2|X_k] < \infty$
- $X_k$ explore the entire space $\mathbb{R}^d$ regularly

**(Iterative) Batch Estimator:**

$$\hat{\beta}_k = \arg \min \left\{ \sum_{j=1}^{k} \frac{1}{\sigma_j^2} (Y_j - X_j^T \beta)^2 + \| \beta - \beta_0 \|^2_{M_0^{-1}} \right\}$$

If $\beta_0 \in \mathbb{R}^d$ and $M_0 \succ 0$ and conditioned on $X_1, \ldots, X_k$ then $\hat{\beta}_k \rightarrow \beta^*$ almost surely. However:
- $\sigma_j^2$ are never known *a priori*. Requires iteration.
- Batch estimator must be recomputed if $k$ is incremented.
Incremental Estimator

Kalman Filter (Single State)

\[ \hat{\beta}_{k+1} = \arg \min \left\{ \frac{1}{\gamma_k^2} (Y_{k+1} - X_{k+1}^T \beta)^2 + \| \beta - \hat{\beta}_k \|^2_{M_k^{-1}} \right\} \]

\[ M_{k+1}^{-1} = M_k^{-1} + \frac{1}{\gamma_k^2} X_{k+1} X_{k+1}^T \]

Assumptions:

- \( \epsilon_k \) are independent, \( \mathbb{E} [\epsilon_k | X_k] = 0 \) and \( \sup_k \mathbb{E} [\epsilon_k^2 | X_k] < \infty \)
- \( X_1, X_2, \ldots \) are independent, identically distributed with finite second moment.
- \( \mathbb{P} [ |X_1^T v| = 0] < 1 \) for all \( v \in \mathbb{R}^d \) s.t. \( \|v\|_2 = 1 \).
Incremental Estimator

Tuning Parameter Requirements:

0 < \delta^2 \leq \inf_k \gamma^2_k \leq \sup_k \gamma^2_k \leq \Delta^2 < \infty

\beta_0 \text{ is arbitrary and } M_0 > 0

Theorem 1: Conditioned on \(X_1, X_2, \ldots, X_k\), \(||\beta_k - \beta^*|| \rightarrow 0}\) almost surely.

Theorem 2: Let

\[ M_k = \mathbb{E}\left[\left(\hat{\beta}_k - \beta^*\right)\left(\hat{\beta}_k - \beta^*\right)^T\right|X_1, \ldots, X_k] \]

If \(\sigma^2_j = \sigma^2 \ \forall j \in \mathbb{N}\) then for all \(\epsilon > 0\) asymptotically almost surely

\[ \frac{1 - \epsilon}{\Delta^2} M_k \preceq \frac{1}{\sigma^2} M_k \preceq \frac{1 + \epsilon}{\delta^2} M_k \]
Numerical Experiment

Data Set
- Source: Public Use File from Center of Medicare and Medicaid Services.
- Described health care expenses, type of visit, patient demographics.
- Contained $N = 2.8$ million records (too big to fit in my computer's 8GB memory).

Linear Model
- Response: Cost of visit.
- Predictors: Patient's gender, Type of Facility, Patient's age.
- Intercept term was included, giving $p = 31$ unknown variables.

Tuning Parameter
- $\gamma_1^2(k) = \frac{1}{k}$
- $\gamma_2 = 37000$
- $\gamma_3 = 0.001$
Numerical Experiment

Convergence of Estimated Covariance.

- Recall: $\frac{1-\epsilon}{\max_k \gamma_k^2} M_k \prec \frac{1}{\sigma^2} M_k \prec \frac{1+\epsilon}{\min_k \gamma_k^2} M_k$.
- Estimated covariance tracks well with mean residuals squared.
Conclusions

1. Q: How many observations are needed (with stochastic dynamics and noisy observations) to estimate a single state "well"?
   A: (The trace of) $M_k$ is sufficient for determining convergence.

2. Q: Does the covariance estimate actually estimate the asymptotic covariance of the parameter estimate?
   A: Yes. $\frac{1-\epsilon}{\max_k \gamma_k^2} M_k \prec \frac{1}{\sigma^2} \mathcal{M}_k \prec \frac{1+\epsilon}{\min_k \gamma_k^2} M_k$

3. Q: Can we converge faster than the Kalman Filter?
   A: For a single state and given that $M_k$ is estimating $\mathcal{M}_k$, no. We will incrementally invert the hessian of the objective. In fact, we show that the conditioning has no impact on the rate of convergence.
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Slides
This slide deck can be found at galton.uchicago.edu/~vpatel.

Reference