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Part I
Fundamentals of Optimization

1 Overview of Numerical Optimization

1.1 Problem and Classification

1. Problem:
\[ \arg \min_{z \in \mathbb{R}^n} f(z) : \begin{cases} c_i(z) = 0 & i \in \mathcal{E} \\ c_i(z) \geq 0 & i \in \mathcal{I} \end{cases} \]

(a) \( f : \mathbb{R}^n \to \mathbb{R} \) is known as the objective function
(b) \( \mathcal{E} \) are equality constraints
(c) \( \mathcal{I} \) are inequality constraints

2. Classifications

(a) Unconstrained vs. Constrained. If \( \mathcal{E} \cup \mathcal{I} = \emptyset \) then it is an unconstrained problem.
(b) Linear Programming vs. Nonlinear Programming. When \( f(z) \) and \( c_i(z) \) are all linear functions of \( z \) then this is a linear programming problem. Otherwise, it is a nonlinear programming problem.
(c) Note: This is not the same as having linear or nonlinear equations.

1.2 Theoretical Properties of Solutions


Definition 1.1. Let \( f : \mathbb{R}^n \to \mathbb{R} \)

(a) \( x^* \) is a global minimizer of \( f \) if \( f(x^*) \leq f(x) \) for all \( x \in \mathbb{R}^n \)
(b) \( x^* \) is a local minimizer of \( f \) if for some neighborhood, \( N \) of \( x^* \), \( f(x^*) \leq f(x) \) for all \( x \in N(x^*) \).
(c) \( x^* \) is a strict local minimizer of \( f \) if for some neighborhood, \( N(x^*) \), \( f(x^*) < f(x) \) for all \( x \in N(x^*) \).
(d) \( x^* \) is an isolated local minimizer of \( f \) if for some neighborhood, \( N(x^*) \), there are no other minimizers of \( f \) in \( N(x^*) \).

Note 1.1. Every isolated local minimizer is a strict minimizer, but it is not true that a strict minimizer is always an isolated local minimizer.

2. Taylor’s Theorem

Theorem 1.1. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable. Let \( p \in \mathbb{R}^n \). Then for some \( t \in [0, 1] \):
\[ f(x + p) = f(x) + \nabla f(x + tp)^T p \quad (1) \]
If $f$ is twice continuously differentiable, then:

$$
\nabla f(x + p) = \nabla f(x) + \int_0^1 \nabla^2 f(x + tp) p dt
$$

(2)

And for some $t \in [0, 1]$:

$$
f(x + p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T \nabla^2 f(x + tp) p
$$

(3)

3. Necessary Conditions

(a) First Order Necessary Conditions

**Lemma 1.1.** Let $f$ be continuously differentiable. If $x^*$ is a minimizer of $f$ then $\nabla f(x^*) = 0$.

*Proof.* Suppose $\nabla f(x^*) \neq 0$. Let $p = -\nabla f(x^*)$. Then, $p^T \nabla f(x^*) < 0$ and by continuity there is some $T > 0$ such that for any $t \in [0, T)$, $p^T \nabla f(x^* + tp) < 0$. We now apply Equation 1 of Taylor’s Theorem to get a contradiction.

Let $t \in [0, T)$ and $q = x + tp$. Then $\exists t' \in [0, t]$ such that:

$$
f(x^* + q) = f(x^*) + t \nabla f(x^* + t'p)^T p < f(x^*)
$$

□

(b) Second Order Necessary Conditions

**Lemma 1.2.** Suppose $f$ is twice continuously differentiable. If $x^*$ is a minimizer of $f$ then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succeq 0$ (i.e. positive semi-definite)

*Proof.* The first conclusion follows from the first order necessary condition. We proceed with the second one by contradiction and Taylor’s Theorem. Suppose $\nabla^2 f(x^*) \prec 0$. Then there is a $p$ such that:

$$
p^T \nabla^2 f(x^*) p < 0
$$

By continuity, there is a $T > 0$ such that $p^T \nabla^2 f(x^* + tp) p < 0$ for $t \in [0, T]$. Fix $t \in [0, T]$ and let $q = tp$. Then, by Equation 3 from Taylor’s Theorem, $\exists t' \in [0, t]$ such that:

$$
f(x^* + q) = f(x^*) + \frac{1}{2} t^2 p^T \nabla^2 f(x^* + t'p) p < f(x^*)
$$

□

4. Second Order Sufficient Condition

**Lemma 1.3.** Let $x^* \in \mathbb{R}^n$. Suppose $\nabla^2 f$ is continuous, $\nabla^2 f(x^*) \succ 0$, and $\nabla f(x^*) = 0$. Then $x^*$ is a strict local minimizer.
Proof. By continuity, there is a ball of radius $T$ about $x^*$, $B$, in which $\nabla^2 f(x) > 0$. Let $\|p\| < T$. Then, there exists a $t \in [0, 1]$ such that:

$$f(x^* + p) = f(x^*) + \frac{1}{2}p^T \nabla^2 f(x^* + tp)p > f(x^*)$$

\[\square\]

5. Convexity and Local Minimizers

**Lemma 1.4.** When $f$ is convex any local minimizer $x^*$ is a global minimizer. If, in addition, $f$ is differentiable then any stationary point is a global minimizer.

**Proof.** Suppose $x^*$ is not a global minimizer. Let $z$ be the global minimizer. Then $f(x^*) > f(z)$. By convexity, for any $\lambda \in [0, 1]$:

$$f(x^*) > \lambda f(z) + (1 - \lambda) f(x^*) \geq f(\lambda z + (1 - \lambda) x^*)$$

So as $\lambda \to 0$, we see that in any neighborhood of $x^*$ there is a point $w$ such that $f(w) < f(x^*)$. A contradiction.

For the second part, suppose $z$ is as above. Then:

$$\nabla f(x^*)^T(z - x^*) = \lim_{\lambda \to 0} \frac{f(x^* + \lambda (z - x^*)) - f(x^*)}{\lambda} \leq \lim_{\lambda \to 0} \frac{\lambda f(z) + (1 - \lambda) f(x^*) - f(x^*)}{\lambda} \leq f(z) - f(x^*) < 0$$

\[\square\]

1.3 Algorithm Overview

1. In general, algorithms begin with a seed point, $x_0$, and locally search for decreases in the objective function, producing iterates $x_k$, until stopping conditions are met.

2. Algorithms typically generate a local model for $f$ near a point $x_k$:

$$f(x_k + p) \approx m_k(p) = f_k + p^T \nabla f_k + \frac{1}{2}p^T B_k p$$

3. Different choices of $B_k$ will lead to different methods with different properties:

   (a) $B_k = 0$ will lead to steepest descent methods
   (b) Letting $B_k$ be the closest positive definite approximation to $\nabla^2 f_k$ leads to Newton’s methods
   (c) Iterative approximations to the Hessian given by $B_k$ lead to Quasi-Newton’s methods
   (d) Conjugate Gradient methods update $p$ without explicitly computing $B_k$
1.4 Fundamental Definitions and Results

1. Q-convergence.

**Definition 1.2.** Let \( x_k \to x \) in \( \mathbb{R}^n \).

(a) \( x_k \) converge \textit{Q-linearly} if \( \exists q \in (0, 1) \) such that for sufficiently large \( k \):
\[
\frac{\|x_{k+1} - x\|}{\|x_k - x\|} \leq q
\]

(b) \( x_k \) converge \textit{Q-superlinearly} if:
\[
\lim_{k \to \infty} \frac{\|x_{k+1} - x\|}{\|x_k - x\|} \to 0
\]

(c) \( x_k \) converges \textit{Q-quadratically} if \( \exists q^2 > 0 \) such that for sufficiently large \( k \):
\[
\frac{\|x_{k+1} - x\|}{\|x_k - x\|^2} \leq q
\]

2. R-convergence.

**Definition 1.3.** Let \( x_k \to x \) in \( \mathbb{R}^n \)

(a) \( x_k \) converges \textit{R-linearly} if \( \exists v_k \in \mathbb{R} \) converging Q-linearly such that
\[
\|x_k - x\| \leq v_k
\]

(b) \( x_k \) converges \textit{R-superlinearly} if \( \exists v_k \in \mathbb{R} \) converging Q-superlinearly such that
\[
\|x_k - x\| \leq v_k
\]

(c) \( x_k \) converges \textit{R-quadratically} if \( \exists v_k \in \mathbb{R} \) converging Q-quadratically such that
\[
\|x_k - x\| \leq v_k
\]

3. Sherman-Morrison-Woodbury

**Lemma 1.5.** If \( A \) and \( \tilde{A} \) are non-singular and related by \( \tilde{A} = A + ab^T \) then:
\[
\tilde{A}^{-1} = A^{-1} - \frac{A^{-1}ab^T A^{-1}}{1 + b^TA^{-1}a}
\]

**Proof.** We can check this by simply plugging in the formula. However, to “derive” this:
\[
\tilde{A}^{-1} = (A + ab^T)^{-1}
\]
\[
= (A [I + A^{-1}ab^T])^{-1}
\]
\[
= [I + A^{-1}ab^T]^{-1} A^{-1}
\]
\[
= [I - A^{-1}ab^T + A^{-1}ab^T A^{-1}ab^T - \cdots] A^{-1}
\]
\[
= [I - A^{-1}a(1 - b^TA^{-1}a + \cdots)b^T] A^{-1}
\]
\[
= A^{-1} - \frac{A^{-1}ab^T A^{-1}}{1 + b^TA^{-1}a}
\]
\[
\Box
\]
2 Line Search Methods

2.1 Step Length

1. Descent Direction

**Definition 2.1.** $p \in \mathbb{R}^n$ is a descent direction if $p^T \nabla f_k < 0$.

2. Problem: Find $\alpha_k \in \text{arg min} \phi(\alpha)$ where $\phi(\alpha) = f(x_k + \alpha p_k)$. However, it is too costly to find the exact minimizer of $\phi$. Instead, we find an $\alpha$ which acceptably reduces $\phi$.

3. Wolfe Condition


**Definition 2.2.** Fix $x$ and $p$ to be a descent direction. Let $\phi(\alpha) = f(x + \alpha p)$ and so $\phi'(\alpha) = f^T(x + \alpha p)$. Fix $0 < c_1 < c_2 < 1$.

i. $\alpha$ satisfies the Armijo Condition if $\phi(\alpha) \leq \phi(0) + c_1 \phi'(0)$. Note $l(\alpha) := \phi(0) + c_1 \phi'(0)$.

ii. $\alpha$ satisfies the Curvature Condition if $\phi'(\alpha) \geq c_2 \phi'(0)$

iii. $\alpha$ satisfies the Strong Curvature Condition if $|\phi'(\alpha)| \leq c_2 |\phi'(0)|$

iv. The Wolfe Condition is the Armijo Condition with the Curvature Condition

v. The Strong Wolfe Condition is the Armijo Condition with the Strong Curvature Condition.

(b) The Armijo Condition: guarantees a decrease at the next iterate by ensuring $\phi(\alpha) < \phi(0)$.

(c) Curvature Condition

i. If $\phi'(\alpha) < c_2 \phi'(0)$, then $\phi$ is still decreasing at $\alpha$ and so we improve the reduction in $f$ by taking a larger $\alpha$

ii. If $\phi'(\alpha) \geq c_2 \phi'(0)$, then either we are closer to a minimum where $\phi' = 0$ or $\phi' > 0$ which means we have surpassed the minimum.

iii. The strong condition guarantees that the choice of $\alpha$ is closer to $\phi' = 0$.

(d) Existence of Step satisfying Wolfe and Strong Wolfe Conditions

**Lemma 2.1.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Let $p$ be a descent direction at $x$ and assume that $f$ is bounded from below along the ray $\{x + \alpha p : \alpha > 0\}$. If $0 < c_1 < c_2 < 1$ there exists an interval satisfying the Wolfe and Strong Wolfe Condition.

**Proof.**

i. Satisfying Armijo Condition: Let $l(\alpha) = f(x) + c_1 \alpha p^T \nabla f(x)$.

Since $l(\alpha)$ is decreasing and $\phi(\alpha)$ is bounded from below, eventually $\phi(\alpha) = l(\alpha)$. Let $\alpha_A > 0$ be the first time this intersection occurs. Then $\phi(\alpha) < l(\alpha)$ for $\alpha \in (0, \alpha_A)$. 

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ii. Curvature Condition: By the mean value theorem, there exists $\beta \in (0, \alpha)$ such that (since $l(\alpha_A) = \phi(\alpha_A)$):
$$\alpha_A \phi'(\beta) = \phi(\alpha_A) - \phi(0) = c_1 \alpha_A \phi'(0) > c_2 \alpha_A \phi'(0)$$
And since $\phi'$ is smooth there is an interval containing $\beta$ for which this inequality holds.

iii. Strong Curvature Condition. Since $\phi'(\beta) = c_1 \phi'(0) < 0$, it follows that:
$$|\phi'(\beta)| < |c_2 \phi'(0)|$$

4. Goldstein Condition

(a) Goldstein Condition

**Definition 2.3.** Take $c \in (0, 1/2)$. The Goldstein Condition is satisfied by $\alpha$ if:
$$\phi(0) + (1 - c)\alpha \phi'(0) \leq \phi(\alpha) \leq \phi(0) + c \alpha \phi'(0)$$

(b) Existence of Step satisfying Goldstein Condition

**Lemma 2.2.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Let $p$ be a descent direction at $x$ and assume that $f$ is bounded from below along the ray $\{x + \alpha p : \alpha > 0\}$. Let $c \in (0, 1/2)$. Then there exists an interval satisfying the Goldstein condition.

**Proof.** Let $l(\alpha) = \phi(0) + (1 - c)\alpha \phi'(0)$ and $u(\alpha) = \phi(0) + c \alpha \phi'(0)$. For $\alpha > 0$, $l(\alpha) < u(\alpha)$ since $c \in (0, 1/2)$. Since $\phi$ is bounded from below let $\alpha_l$ be the smallest point of intersection between $u(\alpha)$ and $\phi(\alpha)$ for $\alpha > 0$. And let $\alpha_u$ be the largest point of intersection, less than $\alpha_u$, of $l(\alpha)$ and $\phi(\alpha)$. Then, for $\alpha \in (\alpha_l, \alpha_u)$, $l(\alpha) < \phi(\alpha) < u(\alpha)$.

5. Backtracking

(a) Backtracking

**Definition 2.4.** Let $\rho \in (0, 1)$ and $\bar{\alpha} > 0$. Backtracking checks over a sequence $\alpha, \rho \alpha, \rho^2 \alpha, \ldots$ until a $\alpha$ is found satisfying the Armijo condition.

(b) Existence of Step satisfying Backtracking

**Lemma 2.3.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Let $p$ be a descent direction at $x$. Then there is a step produced by the backtracking algorithm which satisfies the Armijo condition.

**Proof.** $\exists \epsilon > 0$ such that for $0 < \alpha < \epsilon$, $\phi(\alpha) < \phi(0) + c \alpha \phi'(0)$ by continuity. Since $\rho^n \bar{\alpha} \to 0$, there is an $n$ such that $\rho^n \bar{\alpha} < \epsilon$. 

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2.2 Step Length Selection Algorithms

1. Algorithms which implement the Wolfe or Goldstein Conditions exist and are ideal, but are not discussed herein.

2. Backtracking Algorithm

**Algorithm 1: Backtracking Algorithm**

*input*: Reduction rate $\rho \in (0, 1)$, Initial estimate $\bar{\alpha} > 0$, $c \in (0, 1)$, $\phi(\alpha)$

$\alpha \leftarrow \bar{\alpha}$

**while** $\phi(\alpha) \geq \phi(0) + c\alpha \phi'(0)$ **do**

$\alpha \leftarrow \rho \alpha$

**end**

**return** $\alpha$

3. Interpolation with Expensive First Derivatives

(a) Suppose the Interpolation technique produces a sequence of guesses $\alpha_0, \alpha_1, \ldots$

(b) Interpolation produces $\alpha_k$ by modeling $\phi$ with a polynomial $m$ (quadratic or cubic) and then minimizes $m$ to find a new estimate for $\alpha$. The parameters of the polynomial are given by requiring:

i. $m(0) = \phi(0)$

ii. $m'(0) = \phi'(0)$

iii. $m(\alpha_{k-1}) = \phi(\alpha_{k-1})$

iv. $m(\alpha_{k-2}) = \phi(\alpha_{k-2})$

(c) When $k = 1$, we model using a quadratic polynomial

4. Interpolation with Inexpensive First Derivatives

(a) Suppose interpolation produces a sequence of guesses $\alpha_0, \alpha_1, \ldots$

(b) In this case, $m$ is always a cubic polynomial such that, $\alpha_k$ is computed by:

i. $m(\alpha_{k-1}) = \phi(\alpha_{k-1})$

ii. $m(\alpha_{k-2}) = \phi(\alpha_{k-2})$

iii. $m'(\alpha_{k-1}) = \phi'(\alpha_{k-1})$

iv. $m'(\alpha_{k-2}) = \phi'(\alpha_{k-2})$

5. Interpolation Algorithm

**Algorithm 2: Interpolation Algorithm**

*input*: Feasible Search Region $[\bar{a}, \bar{b}]$, Initial estimate $\alpha_0 > 0$, $\phi$

$\alpha \leftarrow 0$, $\beta \leftarrow \alpha_0$

**while** $\phi(\beta) \geq \phi(0) + c\beta \phi'(0)$ **do**

Approximate $m$

**if** Inexpensive Derivatives **then**

$\alpha \leftarrow \beta$

**end**

Explicitly compute minimizer of $m$ in feasible region. Store as $\beta$.

**end**

**return** $\beta$
2.3 Global Convergence and Zoutendjik


**Definition 2.5.** Suppose we have an algorithm, $\Omega$ which produces iterates $(x_k)$ and denote $\nabla f(x_k) = \nabla f_k$.

(a) $\Omega$ is **globally convergent** if $\|\nabla f_k\| \to 0$. (That is, $x_k$ converge to a stationary point.)

(b) Suppose $\Omega$ produced search directions $p_k$ such that $\|p_k\| = 1$. And let $\theta_k$ be the angle between $\nabla f_k$ and $p_k$. $\Omega$ satisfies the **Zoutendjik Condition** if

$$\sum_{k=1}^{\infty} \cos^2(\theta_k) \|\nabla f_k\|^2 < \infty$$

2. Zoutendjik Condition & Angle Bound implies global convergence

**Lemma 2.4.** Suppose $\Omega$ produces a sequence $(x_k, p_k, \nabla f_k, \theta_k)$ such that $\exists \delta > 0$ and $\forall k \geq 1$, $\cos(\theta_k) \geq \delta$. If $\Omega$ satisfies the Zoutendjik condition then $\Omega$ is globally convergent.

**Proof.** By the Zoutendjik condition:

$$\delta^2 \sum_{k=1}^{\infty} \|\nabla f_k\|^2 < \infty$$

Therefore, $\|\nabla f_k\|^2 \to 0$. \qed 

3. Example of Angle Bound

**Example 2.1.** Suppose $p_k = -B_k^{-1}\nabla f_k$ where $B_k$ are positive definite and $\|B_k\| B_k^{-1} < M < \infty$ for all $k$. Letting $\|\cdot\| = \|\cdot\|_2$, $B_k = X\Lambda X^T$ be the EVD of $B_k$, and $X^T\nabla f_k = z$:

$$|\cos(\theta_k)| = \left| \frac{-\nabla f_k^T B_k^{-1} \nabla f_k}{\|\nabla f_k\| B_k^{-1} \nabla f_k\|} \right|$$

$$= \left| \frac{z^T \Lambda^{-1} z}{\|z\| \|\Lambda^{-1} z\|} \right|$$

$$= \left| \frac{\sum_{i=1}^{n} \frac{z_i^2}{\lambda_i}}{\|z\| \sqrt{\sum_{i=1}^{n} \frac{z_i^2}{\lambda_i}}} \right|$$

$$\geq \frac{1/\lambda_1 \|z\|^2}{1/\lambda_n \|z\|^2}$$

$$\geq \frac{\lambda_n}{\lambda_1}$$

$$\geq \frac{1}{M}$$

Hence, if $\Omega$ satisfies the Zoutendjik Condition, we see that $\lim_n \|\nabla f_n\| \to 0$. 

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**Theorem 2.1.** Suppose we have an objective $f$ satisfying:

(a) $f$ is bounded from below in $\mathbb{R}^n$

(b) Given an initial $x_0$, there is an open set $N$ containing $L = \{x : f(x) \leq f(x_0)\}$

(c) $f$ is continuously differentiable on $N$

(d) $\nabla f$ is Lipschitz continuous on $N$

Suppose we have an algorithm $\Omega$ producing $(x_k, p_k, \nabla f_k, \theta_k, \alpha_k)$ such that:

(a) $p_k$ is a descent direction (with $\|p_k\| = 1$)

(b) $\alpha_k$ satisfies the Wolfe Conditions

Then $\Omega$ satisfies the Zoutendjik Condition.

**Proof.** The general strategy is to lower bound $\alpha_k$ uniformly by $C|\nabla f_k^T p_k|$, where $C > 0$. Then using the descent condition:

$$f_{k+1} \leq f_k - \alpha_k|\nabla f_k^T p_k| \leq f_k - C|\nabla f_k^T p_k|^2 \leq f_k - C \cos^2(\theta_k) \|\nabla f_k\|^2_2 \leq f_0 - C \sum_{j=1}^k \cos^2(\theta_j) \|\nabla f_j\|^2_2$$

And we can conclude that since $f$ is bounded from below by $l$, then $f_0 - f_k \leq f_0 - l < \infty$. The result follows.

So now we set out to show that $\alpha_k \geq C|\nabla f_k^T p_k|$. We do this by leveraging the Curvature Condition.

(a) From the curvature condition, we have that:

$$(\nabla f_{k+1} - \nabla f_k)^T p_k \geq (c_2 - 1)|\nabla f_k^T p_k|$$

(b) From Lipschitz Continuity, with constant $L$:

$$|(|\nabla f_{k+1} - \nabla f_k|^T p_k| \leq \|\nabla f_{k+1} - \nabla f_k\| \|p_k\| \leq L \alpha_k \|p_k\|^2 = L \alpha_k$$

Together these imply that $\frac{1-c_2}{L} |\nabla f_k^T p_k| \leq \alpha_k$. \hfill $\square$

5. Goldstein Condition Line Search satisfies Zoutendjik Condition

**Theorem 2.2.** Suppose $f$ has the same properties as it does in Theorem 2.1. And suppose $\Omega$ produces $(x_k, p_k, \nabla f_k, \theta_k, \alpha_k)$ such that:

(a) $p_k$ is a descent direction with $\|p_k\| = 1$

(b) $\alpha_k$ satisfies the Goldstein conditions.

Then $\Omega$ satisfies the Zoutendjik Condition.
Proof. We use Taylor’s Theorem and then Lipschitz Continuity to find the lower bound. By Taylor’s theorem, for $t_k \in (0, 1)$:

$$(1 - c)\alpha_k \nabla f_k^T p_k \leq f_{k+1} - f_k = \nabla f(x_k + t_k \alpha_k p_k)^T \alpha_k p_k$$

Therefore,

$$c\alpha_k |\nabla f_k^T p_k| \leq |(\nabla f(x_k + t_k \alpha_k p_k)^T - \nabla f_k)(p_k)|$$

$$\leq L t_k \alpha_k \|p_k\|^2$$

The result follows.

6. Backtracking Line Search satisfies Zoutendjik Condition

**Theorem 2.3.** Suppose $f$ has the same properties as it does in Theorem 2.1. And suppose $\Omega$ produces $(x_k, p_k, \nabla f_k, \theta_k, \alpha_k)$ such that:

(a) $p_k$ is a descent direction with $\|p_k\| = 1$

(b) $\alpha_k$ is selected by backtracking with $\bar{\alpha} = 1$

Then $\Omega$ satisfies the Zoutendjik condition.

Proof. Suppose $\Omega$ uses $\rho \in (0, 1)$. There are two cases to consider. When $\alpha_k = 1$ and $\alpha_k < 1$. We denote the first subsequence by $k(j)$ and the second by $k(l)$. For $\alpha_{k(l)}$ we know at least that $\alpha_{k(l)}/\rho$ was rejected, that is:

$$f(x_k + \alpha_{k(l)} p_k/\rho) > f_k + c\alpha_{k(l)} \|\nabla f_k^T p_k/\rho\|

Using the same strategy as in Goldstein, $\exists t_k$ such that:

$$\nabla f(x_k + t_k \alpha_{k(l)} p_k/\rho)^T \alpha_k p_k/\rho > c\alpha_{k(l)} \|\nabla f_k^T p_k/\rho\|

Therefore, by Lipschitz continuity:

$$(1 - c)|\nabla f_k^T p_k| \leq L t_k \alpha_{k(l)}/\rho \leq L \alpha_{k(l)}/\rho$$

We now consider $\alpha_{k(j)} = 1$ since the Armijo condition gives the sequence:

$$f_{k(j)+1} \leq f_{k(j)} \leq f_k + c \nabla f_k^T p_k = f_k - c |\cos(\theta_{k(j)})| \|f_k\|

Therefore:

$$c \sum_j |\cos(\theta_{k(j)})| \|\nabla f_{k(j)}\| < \infty$$

Then $\exists J$ such that $j \geq J$,

$$\cos^2(\theta_{k(j)}) \|\nabla f_{k(j)}\|^2 \leq |\cos(\theta_{k(j)})| \|\nabla f_{k(j)}\| < 1$$

Therefore, the Zoutendjik condition holds.
3 Trust Region Methods

3.1 Trust Region Subproblem and Fidelity

1. The strategy of the trust region problem is as follows: at any point \( x \) we have a model \( m_x(p) \) of \( f(x + p) \) which we trust is a good approximation in a region \( \|p\| \leq \Delta \). Minimizing this, we have a new iterate on which to continue applying the approach.

2. Trust Region Subproblem

**Definition 3.1.** Let \( x \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R} \) be an objective function with model \( m_x(p) \) in a region \( \|p\| \leq \Delta \). The Trust Region Subproblem is to find

\[
\arg\min\left\{ m_x(p) : \|p\| \leq \Delta \right\}
\]

**Note 3.1.** Usually \( m_x \) is taken to be

\[
m_x(p) = f(x) + \nabla f_x^T p + \frac{1}{2} p^T B p
\]

where \( B \) is the model Hessian and is not necessarily positive definite.

3. Fidelity

**Definition 3.2.** The Fidelity of \( m_x \) in a region \( \|p\| \leq \Delta \) at a point \( p + x \) is

\[
\rho(m_x, f, p, \Delta) = \frac{f(x + p) - f(x)}{m_x(p) - m_x(0)}
\]

4. Fidelity and Trust Region. Let \( 0 < c_1 < c_2 < 1 \)

(a) If \( \rho \geq c_2 \), especially if \( p \) is intended for descent, the model is a good approximation to \( f \) and we can expand the trust region \( \Delta \)

(b) If \( \rho \leq c_1 \), especially if \( p \) is intended for descent, the model is a poor approximation to \( f \) and we should shrink the trust region \( \Delta \)

(c) Otherwise, the model is a sufficient approximation to \( f \) and the trust region should not be adjusted.

5. Fidelity and Iterates. Suppose \( p \) causes a descent in \( m_x \). And let \( \eta \in (0, 1/4] \)

(a) If \( \rho < \eta \) then we do not have a sufficient decrease in the function and the process should be repeated with \( x \)

(b) If \( \rho \geq \eta \) then we have a sufficient decrease in \( f \), and we should continue the process with \( x + p \)

6. Notation

(a) Let \( x_0, x_1, \ldots \) be the iterates produces by subsequential solutions to the trust region subproblem for some objective \( f \)

(b) \( m_{x_k}(p) =: m_k(p) \)

(c) \( \Delta \) for a particular \( m_k \) is denoted \( \Delta_k \)

(d) Accepted solutions to the subproblem at \( x_k \) are \( p_k \) so that \( x_{k+1} = x_k + p_k \).
3.2 Fidelity Algorithms

1. Fidelity and Trust Region

**Algorithm 3:** Trust Region Management Algorithm

```
input : Thresholds 0 < c_2 < c_1 < 1, Trust Region \Delta, \Delta_{max}, \rho_x
if \rho_x > c_1 then
    \Delta \leftarrow \min(2\Delta, \Delta_{max})
else if \rho < c_2 then
    \Delta \leftarrow \Delta/2
else
    | Continue
end
return \Delta
```

**Note 3.2.** Usually \( c_1 = 1/4 \) and \( c_2 = 3/4 \)

2. Fidelity and Solution Acceptance

**Algorithm 4:** Solution Acceptance Algorithm

```
input : Threshold \eta \in (0, 1/4], \rho_x, x, p
if \rho_x(p) \geq \eta then
    x \leftarrow x + p
else
    | Continue
end
return x
```

3.3 Approximate Solutions to Subproblem

3.3.1 Cauchy Point

1. Cauchy Point

**Definition 3.3.** Let \( m_x \) be a model of \( f \) within \( \|p\| \leq \Delta \). Let \( p^S \) be the direction of steepest descent at \( m_x(0) \) such that \( \|p^S\| = 1 \), and let \( \tau \) be:

\[
\arg \min \{ m_x(\tau p^S) : \tau \leq \Delta \}
\]

Then, \( p^C = \tau p^S \) is the Cauchy Point of \( m_x \).

**Note 3.3.** The Cauchy point is the line search minimizer of \( m_x \) along the direction of steepest descent subject to \( \|p\| \leq \Delta \).

2. Computing the Cauchy Point

**Lemma 3.1.** Let \( m_x \) be a quadratic model of \( f \) within \( \|p\| \leq \Delta \) such that

\[
m_x(p) = f(x) + g^T p + \frac{1}{2} p^T B p
\]

Then, its Cauchy Point is:

\[
p^C = \begin{cases} 
\frac{-g}{\|g\|} \Delta & g^T B g > 0 \\
\frac{-g}{g^T B g} \left( \Delta, \frac{\|g\|^3}{g^T B g} \right) & g^T B g \leq 0
\end{cases}
\]
Proof. First, we compute the direction of steepest descent:

\[ p^S = \frac{-g}{\|g\|} \]

Then, we have that

\[ m_x(tp^S) = f(x) + tg^T p^S + \frac{t^2}{2} (p^S)^T B p^S. \]

If the quadratic term is negative, then \( t = \Delta \) is the minimizer. And so one option is:

\[ p^C = \frac{-\Delta g}{\|g\|} \]

If the quadratic term is positive then:

\[ \frac{d}{dt} m_x(tp^S) = g^T p^S + tp^S B p^S \]

Setting this to 0 and substituting back in for \( p^S \):

\[ t = \frac{\|g\|^3}{g^T B g} \]

This value could potentially be larger than \( \Delta \), so to safeguard against this, in this case:

\[ \tau = \min \left( \Delta, \frac{\|g\|^3}{g^T B g} \right) \]

3. Using the Cauchy point does not produce the best rate of convergence

3.3.2 Dogleg Method

1. Requirement: \( B \succ 0 \)

2. Intuitively, the dogleg method moves towards the minimizer of \( m_x \) which is \( p^{\text{min}} = -B^{-1} g \) until it reaches this point or hits the boundary of the trust region. It does this by first moving to the Cauchy Point, and then along a linear path to \( p^{\text{min}} \)

3. Dogleg Path

Definition 3.4. Let \( m_x \) be the quadratic model with \( B \succ 0 \) of \( f \) with radius \( \Delta \). Let \( p^C \) be the Cauchy Point and \( p^{\text{min}} \) be the unconstrained minimizer of \( m_x \). The Dogleg Path is \( p^{DL}(t) \) where:

\[ p^{DL}(t) = \begin{cases} p^C t & t \in [0, 1] \\ p^C + (t-1)(p^{\text{min}} - p^C) & t \in [1, 2] \end{cases} \]

4. Dogleg Method

Definition 3.5. The Dogleg Method chooses \( x_{k+1} = x_k + p^{DL}(\tau) \) where \( \|p^{DL}(\tau)\| = \Delta \).
5. Properties of Dogleg Path

**Lemma 3.2.** Suppose \( p^{DL}(t) \) is the Dogleg path of \( m_x \) with \( \Delta = \infty \) and \( B \succ 0 \). Then:

(a) \( m_x(p^{DL}(t)) \) is decreasing

(b) \( \|p^{DL}(t)\| \) is increasing

**Proof.** When \( 0 < t < 1 \), this case is uninteresting since for these values we know \( m_x(p^{DL}(t)) \) is decreasing and \( \|p^{DL}(t)\| \) is increasing. So we consider \( 1 < t < 2 \) and reparameterize using \( t - 1 = z \). We have that:

\[
\frac{d}{dz} m_x(p^{DL}(t)) = (p^{min} - p^C)^T (g - Bp^C) + z(p^{min} - p^C)^T B (p^{min} - p^C)
\]

\[
= -(1 - z)(p^{min} - p^C)^T B (p^{min} - p^C)
\]

Letting \( c = (p^{min} - p^C)^T B (p^{min} - p^C) > 0 \) since \( B \succ 0 \), we have that the derivative is strictly negative for \( z < 1 \). To show that the norm is increasing, first we show that the angle between \( p^C \) and \( p^{min} \) is between \( (-\pi/2, \pi/2) \), then we show its length is longer.

(a) For the angle, \( g \neq 0 \):

\[
\langle p^{min}, p^C \rangle = g^T B^{-1} g \frac{\|g\|^2}{g^T B g} > 0
\]

(b) Using Cauchy-Schwartz, \( \|g\| \|p^{min}\| \geq \|g\| \|p^C\| \) since:

\[
\|B^{-1} g\| \|g\| \geq g^T B^{-1} g = \frac{g^T B^{-1} g g^T B g}{g^T B g} \geq \frac{\|g\|^4}{g^T B g}
\]

\[\blacksquare\]

### 3.3.3 Global Convergence of Cauchy Point Methods

1. Reduction obtained by Cauchy Point

**Lemma 3.3.** The Cauchy Point \( p_k^C \) satisfies

\[
m_k(p_k^C) - m_k(0) \leq -\frac{1}{2} \|g_k\| \min \left( \Delta_k, \frac{\|g_k\|}{\|B_k\|} \right)
\]

**Proof.** We have that

\[
m_k(p_k^C) - m_k(0) = g^T p_k^C + \frac{1}{2} (p_k^C)^T B_k (p_k^C)
\]
This brings in two cases when $g^TBg \leq 0$ then (dropping subscripts)

$$m_k(p^C) - m_k(0) = -\|g\| \Delta + \frac{1}{2\|g\|^2}g^TBg$$

$$\leq -\|g\| \Delta$$

$$\leq -\frac{1}{2} \|g\| \min \left( \Delta, \frac{\|g\|}{\|B\|} \right)$$

When $g^TBg \geq 0$, and $\Delta(g^TBg) \leq \|g\|^3$, we are again in the above case:

$$m_k(p^C) - m_k(0) = -\|g\| \Delta + \frac{1}{2\|g\|^2}g^TBg$$

$$\leq -\|g\| \Delta + \frac{1}{2} \|g\| \Delta$$

$$\leq -\frac{1}{2} \|g\| \min \left( \Delta, \frac{\|g\|}{\|B\|} \right)$$

When $\Delta g^TBg > \|g\|^3$:

$$m_k(p^C) - m_k(0) = -\frac{\|g\|^4}{g^TBg} + \frac{\|g\|^4}{2(g^TBg)^2}g^TBg$$

$$= -\frac{\|g\|^4}{2g^TBg}$$

$$\leq -\frac{1}{2} \|g\|^2$$

$$\leq -\frac{1}{2} \|g\| \min \left( \Delta, \frac{\|g\|}{\|B\|} \right)$$


\[\square\]

2. Reduction achieved by Dogleg

**Corollary 3.1.** The dogleg step $p_k^{DL}$ satisfies:

$$m_k(p_k^{DL}) - m_k(0) \leq -\frac{1}{2} \|g_k\| \min \left( \Delta_k, \frac{\|g_k\|}{\|B_k\|} \right)$$

*Proof. Since $m_k(p_k^{DL}) \leq m_k^{p^C}$ the result follows.*

\[\square\]

3. Reduction achieved by any method based on Cauchy Point

**Corollary 3.2.** Suppose a method uses step $p_k$ where $m_k(p_k) - m_k(0) \leq c(m_k(p_k^{p^C}) - m_k(0))$ for $c \in (0, 1)$. Then $p_k$ satisfies:

$$m_k(p_k) - m_k(0) \leq -\frac{1}{2} c \|g_k\| \min \left( \Delta_k, \frac{\|g_k\|}{\|B_k\|} \right)$$

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4. Convergence when \( \eta = 0 \)

**Theorem 3.1.** Let \( x_0 \in \mathbb{R}^n \) and \( f \) be an objective function. Let \( S = \{ x : f(x) \leq f(x_0) \} \) and \( \mathcal{S}(R_0) = \{ x : \| x - y \| < R_0, y \in S \} \) be a neighborhood of radius \( R_0 \) about \( S \). Suppose the following conditions:

(a) \( f \) is bounded from below on \( S \)

(b) \( f \) is Lipschitz continuously differentiable on \( \mathcal{S}(R_0) \) for some \( R_0 > 0 \) and some constant \( L \).

(c) There is a \( c \) such that for all \( k \):

\[
m_k(p_k) - m_k(0) \leq -c \| g_k \| \min \left( \Delta_k, \frac{\| g_k \|}{\| B_k \|} \right)
\]

(d) There is a \( \gamma \geq 1 \) such that for all \( k \):

\[
\| p_k \| \leq \gamma \Delta_k
\]

(e) \( \eta = 0 \) (fidelity threshold)

(f) \( \| B_k \| \leq \beta \) for all \( k \).

Then:

\[
\liminf \| g_k \| = 0
\]

5. Convergence when \( \eta \in (0, 1/4) \)

**Theorem 3.2.** If \( \eta \in (0, 1/4) \) with all other conditions from Theorem 3.1 holding, then:

\[
\lim \| g_k \| = 0
\]

### 3.4 Iterative Solutions to Subproblem

#### 3.4.1 Exact Solution to Subproblem

1. Conditions for Minimizing Quadratics

**Lemma 3.4.** Let \( m \) be the quadratic function

\[
m(p) = g^T p + \frac{1}{2} p^T B p
\]

where \( B \) is a symmetric matrix.

(a) \( m \) attains a minimum if and only if \( B \geq 0 \) and \( g \in \text{Im}(B) \). If \( B \geq 0 \) then every \( p \) satisfying \( Bp = -g \) is a global minimizer of \( M \).

(b) \( m \) has a unique minimizer if and only if \( B > 0 \).

**Proof.** If \( p^* \) is a minimizer of \( m \) we have from second order conditions that \( 0 = \nabla m(p^*) = Bp^* + g \) and \( B = \nabla^2 m(p^*) \geq 0 \). Now suppose that \( g \in \text{Im}(B) \), then \( \exists p \) such that \( Bp = -g \). Let \( w \in \mathbb{R}^n \) then:

\[
m(p + w) = g^T p + g^T w + \frac{1}{2} \left( w^T B w + 2p^T B w + p^T B p \right)
= m(p) + g^T w - \frac{1}{2} 2g^T w + \frac{1}{2} w^T B w
\geq m(p)
\]
where the inequality follows since $B \succeq 0$. This proves the first part.

Now suppose $m$ has a unique minimizer, $p^*$, and suppose $\exists w$ such that $w^T B w = 0$. Since $p^*$ is a minimizer, we have that $B p^* = -g$ and so from the computation above $m(p + w) = m(p)$ and so $p + w, p$ are minimizers to $m$ which is a contradiction. For the other direction, since $B \succ 0$ and letting $p^* = -B^{-1} g$, by Second Order Sufficient Conditions, $m$ has a unique minimizer.

2. Characterization of Exact Solution

Theorem 3.3. The vector $p^*$ is a solution to the trust region problem:

$$\min_{p \in \mathbb{R}^n} f(x) + g^T p + \frac{1}{2} p^T B p : ||p|| \leq \Delta$$

if and only if $p^*$ is feasible and $\exists \lambda > 0$ such that:

(a) $(B + \lambda I)p^* = g$
(b) $\lambda(||p^*|| - \Delta) = 0$
(c) $B + \lambda I \succeq 0$

Proof. First we prove ($\Leftarrow$) direction. So $\exists \lambda \geq 0$ which satisfies the properties. Consider the problem

$$m^*(p) = f(x) + g^T p + \frac{1}{2} p^T (B + \lambda I)p = m(p) + \frac{1}{2} \lambda ||p||^2$$

Moreover, $m^*(p) \geq m^*(p^*)$ which implies that:

$$m(p) \geq m(p^*) + \lambda(||p^*||^2 - ||p||^2)$$

So if $\lambda = 0$ then $m(p) \geq m(p^*)$ and $p^*$ is minimizer of $m$, or if $\lambda > 0$ then $||p^*|| = \Delta$ and so for a feasible $p$, $||p|| \leq \Delta$, which implies $m(p) \geq m(p^*)$.

Now suppose $p^*$ is the minimizer of $m$. If $||p^*|| < \Delta$ then $p^*$ is the unconstrained minimizer of $m$ and so $\lambda = 0$, and so $\nabla m(p^*) = B p^* + g = 0$ and $\nabla^2 m(p^*) = B \succeq 0$ by the previous lemma. Now suppose $||p^*|| = \Delta$.

Then the second condition is automatically satisfied. We now use the Lagrangian to determine the solution. $L(\lambda, p) = g^T p + \frac{1}{2} p^T B p + \frac{1}{2} (p^T p - \Delta^2)$

Differentiating with respect to $p$ and setting this equal to 0, we see that

$$(B + \lambda I)p^* + g = 0$$

Now to show that $(B + \lambda I)$ is positive semi definite, note that for any $p$ such that $||p||^2 = \Delta$ and since $p^*$ is the minimizer of $m$: $m(p) + \frac{1}{2} ||p||^2 \geq m(p^*) + \frac{1}{2} ||p^*||^2$ Rearranging, we have that

$$(p - p^*)^T (B - \lambda I)(p - p^*) \geq 0$$

And since the set of all normalized $\pm (p - p^*)$ where $||p|| = \Delta$ is dense on the unit sphere, $B - \lambda I$ is positive semi-definite. The last thing to show is that $\lambda \geq 0$ which follows using the fact that if $\lambda < 0$ then $p^*$ must be a global minimizer of $m$. By the previous lemma, this is a contradiction. □
3.4.2 Newton’s Root Finding Iterative Solution

1. **Theorem 3.3** guarantees that a solution exists and we need on find $\lambda$ which satisfies these conditions. Two cases exist from this theorem:

Case 1 $\lambda = 0$. If $\lambda = 0$ then $B \succeq 0$ and we can compute a solution $p^*$ using $QR$ decomposition.

Case 2 $\lambda > 0$. First, letting $QAQ^T$, be the EVD of $B$, and defining:

$$p(\lambda) = -Q(\Lambda + \lambda I)^{-1}Q^T g$$

Hence:

$$\|p(\lambda)\|^2 = g^T Q(\Lambda + \lambda I)^{-2}Q^T g = \sum_{j=1}^{n} \frac{(q_j^T g)^2}{(\lambda_j + \lambda)^2}$$

From **Theorem 3.3**, we want to find $\lambda > 0$ for which:

$$\Delta^2 = \sum_{j=1}^{n} \frac{(q_j^T g)^2}{(\lambda_j + \lambda)^2}$$

2. The second case has two sub-cases which must be considered. Let $Q_1$ be the columns of $Q$ which correspond to the eigenspace of $\lambda_1$.

Case 2a If $Q_1^T g \neq 0$, then we implement Newton’s Root finding algorithm on:

$$f(\lambda) = \frac{1}{\Delta} - \frac{1}{\|p(\lambda)\|}$$

since as $\lambda \to \lambda_1$, $\|p(\lambda)\| \to \infty$ and so a solution exists.

Case 2b If $Q_1^T g = 0$, then applying Newton’s root finding algorithm naively is not useful since $\|p(\lambda)\| \not\to \infty$ as $\lambda \to \lambda_1$. We note that when $\lambda = -\lambda_1$, $(B + \lambda I) \succeq 0$ and so we can find a $\tau$ such that:

$$\Delta^2 = \sum_{j: \lambda_j \neq \lambda_1} \frac{(q_j^T g)^2}{(\lambda_j - \lambda_1)^2} + \tau^2$$

**Computation.** $\exists z \neq 0$ such that $(B - \lambda_1 I)z = 0$ and $\|z\| = 1$. Therefore, $q_j^T z = 0$ for all $j: \lambda_1 \neq \lambda_j$. Setting

$$p(\lambda_1, \tau) = \sum_{j: \lambda_j \neq \lambda_1} \frac{(q_j^T g)^2}{(\lambda_j - \lambda_1)^2} + \tau^2$$

we need only find $\tau$.

3.5 Trust Region Subproblem Algorithms
4 Conjugate Gradients

1. Conjugated Gradients Overview

**Algorithm 5:** Overview of Conjugate Gradient Algorithm

```plaintext
input : x_0 a starting point, \( \epsilon \) tolerance, \( f \) objective

\( x \leftarrow x_0 \)
Compute Gradient: \( r \leftarrow \nabla f(x) \)
Compute Conjugated Descent Direction: \( p \leftarrow -r \)

\textbf{while} \( \|r\| > \epsilon \) \textbf{do}
  \hspace{1em} Compute Optimal Step Length: \( \alpha \)
  \hspace{1em} Compute Step: \( x \leftarrow x + \alpha r \)
  \hspace{1em} Compute Gradient: \( r \leftarrow \nabla f(x) \)
  \hspace{1em} Compute Conjugated Step Direction: \( p \)

\textbf{return Minimizer: } x
```

2. Conjugate Gradients for minimizing convex \((A > 0)\), quadratic function

\( f(x) = \frac{1}{2} x^T Ax - b^T x \)

**Algorithm 6:** Conjugate Gradients Algorithm for Convex Quadratic

```plaintext
input : x_0 a starting point, \( \epsilon = 0 \) tolerance, \( f \) objective

\( x \leftarrow x_0 \)
Compute Gradient: \( r \leftarrow \nabla f(x) = Ax - b \)
Compute Conjugated Descent Direction: \( p \leftarrow -r \)

\textbf{while} \( \|r\| > \epsilon \) \textbf{do}
  \hspace{1em} Compute Optimal Step Length: \( \alpha = \frac{r^T r}{p^T Ap} \)
  \hspace{1em} Compute Step: \( x \leftarrow x + \alpha r \)
  \hspace{1em} Store Previous: \( r_{old} \leftarrow r \)
  \hspace{1em} Compute Gradient: \( r \leftarrow r + \alpha Ap \)
  \hspace{1em} Compute Conjugated Step Direction: \( p \leftarrow -r + \frac{\|r\|^2}{\|r_{old}\|^2} p \)

\textbf{return Minimizer: } x
```

3. Fletcher-Reeves for Nonlinear Functions

**Algorithm 7:** Fletcher Reeves CG Algorithm

```plaintext
input : x_0 a starting point, \( \epsilon \) tolerance, \( f \) objective

\( x \leftarrow x_0 \)
Compute Gradient: \( r \leftarrow \nabla f(x) \)
Compute Conjugated Descent Direction: \( p \leftarrow -r \)

\textbf{while} \( \|r\| > \epsilon \) \textbf{do}
  \hspace{1em} Compute Optimal Step Length: \( \alpha \) (Strong Wolfe Line Search)
  \hspace{1em} Compute Step: \( x \leftarrow x + \alpha r \)
  \hspace{1em} Store Previous: \( r_{old} \leftarrow r \)
  \hspace{1em} Compute Gradient: \( r \leftarrow \nabla f(x) \)
  \hspace{1em} Compute Conjugated Step Direction: \( p \leftarrow -r + \frac{\|r\|^2}{\|r_{old}\|} p \)

\textbf{return Minimizer: } x
```
Part II

Model Hessian Selection

5 Newton’s Method

1. Requires $\nabla^2 f \succ 0$ in some region for the method to work.

2. Line Search
   (a) The search direction:
   
   \[ p_k^N = -\nabla^2 f_k^{-1} \nabla f_k \]
   
   (b) Rate of Convergence

   \textbf{Theorem 5.1.} Suppose $f$ is an objective with minimum $x^*$ satisfying the Second Order Conditions. Moreover:
   
   i. Suppose $f$ is twice continuously differentiable in a neighborhood of $x^*$
   ii. Specifically, suppose $\nabla^2 f$ is Lipschitz continuous in a neighborhood of $x^*$
   iii. Suppose $p_k = p_k^N$ and $x_{k+1} = x_k + p_k^N$
   iv. Suppose $x_k \to x^*$

   Then for $x_0$ sufficiently close to $x^*$:
   
   i. $x_k \to x^*$ quadratically
   ii. $\|f_k\| \to 0$ quadratically.

   \textit{Proof.} Our goal is to get $x_{k+1} - x^*$ in terms of $\nabla^2 f$ and use Lipschitz continuity to provide an upper bound.
   
   i. We have that:
   
   \[
   x_{k+1} - x^* \\
   = x_k + p_k - x^* \\
   = x_k - x^* - \nabla^2 f_k^{-1} \nabla f_k \\
   = \nabla^2 f_k^{-1} \left[ \nabla^2 f_k (x_k - x^*) - (\nabla f_k - \nabla f^*) \right] \\
   = \nabla^2 f_k^{-1} \left[ \nabla^2 f_k (x_k - x^*) - \int_0^1 \nabla^2 f(x_k + t(x - x^*)) (x - x^*) dt \right] \\
   = \nabla^2 f_k^{-1} \int_0^1 (\nabla^2 f_k - \nabla^2 f(x_k + t(x - x^*)) (x - x^*) dt
   \]
ii. Using Lipschitz continuity:
\[
\|x_{k+1} - x^*\| \\
\leq \|\nabla^2 f_k^{-1}\| \int_0^1 \|\nabla^2 f_k - \nabla^2 f(x_k + t(x_k - x^*))\| \|x - x^*\| \, dt \\
\leq \|\nabla^2 f_k^{-1}\| \int_0^1 tL \|x_k - x^*\|^2 \, dt \\
\leq \|\nabla^2 f_k^{-1}\| \frac{L}{2} \|x_k - x^*\|^2
\]

We now need to show that \(\|\nabla^2 f_k^{-1}\|\) is bounded from above. Let \(\eta(2r)\) be a neighborhood of radius \(2r\) about \(x^*\) for which \(\nabla^2 f > 0\). In this neighborhood, \(\nabla^2 f^{-1}\) exists and \(g(x) = \|\nabla^2 f(x)^{-1}\|\) is continuous. Then, on the compact closed ball \(\eta(r)\), \(g(x)\) has a finite maximum. Therefore, we have quadratic convergence of \(x_k \to x^*\).

iii. We now consider the first derivative:
\[
\|\nabla f_{k+1} - \nabla f^*\| = \|\nabla f_{k+1}\| \\
= \|\nabla f_{k+1} - \nabla f_k - \nabla^2 f_k p_k\| \\
= \left\| \int_0^1 \left[\nabla^2 f_k (x_k + tp_k) - \nabla^2 f_k \right] p_k \, dt \right\| \\
\leq \left\| \int_0^1 Lt \|p_k\|^2 \right\| \\
\leq \frac{L}{2} g(x_k)^2 \|\nabla f_k\|^2
\]

So if \(x_0 \in \eta(r)\), the result follows.

3. Trust Region
(a) Asymptotically Similar

**Definition 5.1.** A sequence \(p_k\) is asymptotically similar to a sequence \(q_k\) if
\[
\|p_k - q_k\| = o(\|q_k\|)
\]

(b) The Subproblem
\[
\min_{p \in \mathbb{R}^n} f(x) + \nabla f^T p + \frac{1}{2} p^T \nabla^2 f x p : \|p\| \leq \Delta
\]

(c) Rate of Convergence

**Theorem 5.2.** Suppose \(f\) is an objective with minimum \(x^*\) satisfying the Second Order Conditions. Moreover:

i. Suppose \(f\) is twice continuously differentiable in a neighborhood of \(x^*\)
ii. Specifically, suppose $\nabla^2 f$ is Lipschitz continuous in a neighborhood of $x^*$

iii. Suppose for some $c \in (0, 1)$, $p_k$ satisfy

$$m_k(p_k) - m_k(0) \leq -c \|\nabla f_k\| \min\left(\Delta_k, \frac{\|\nabla f_k\|}{\|\nabla^2 f_k\|}\right)$$

and are asymptotically similar to $p_k^N$ when $\|p_k^N\| \leq \frac{1}{2} \Delta_k$.

iv. Suppose $x_{k+1} = x_k + p_k$ and $x_k \to x^*$

Then:

i. $\Delta_k$ becomes inactive for sufficiently large $k$.

ii. $x_k \to x^*$ superlinearly.

Proof. We need to show that $\Delta_k$ are bounded from below. The second part follows from this quickly since $x_k \to x^*$ implies $\|p_k^N\| \to 0$ and eventually $\|p_k^N\| \leq \frac{1}{2} \Delta_k$. So applying the asymptotic similarity:

$$\|x_k + p_k - x^*\| \leq \|x_k + p_k^N - x^*\| + \|p_k^N - p_k\|
\leq o(\|x_k - x^*\|^2) + o(\|p_k^N\|)
\leq o(\|x_k - x^*\|^2) + o(\|x - x^*\|)
\leq o(\|x - x^*\|)$$

where the penultimate inequality follows from:

$$\|\nabla^2 f_k^{-1}\| \|\nabla f_k - \nabla f^*\| \leq \|\nabla^2 f_k^{-1}\| \|x_k - x^*\| \left(\sup_{x \in \eta(r)} \|\nabla^2 f(x)\|\right)$$

To get that $\Delta_k$ is bounded from below, we frame it in terms of $\rho_k$. The numerator of $\rho_k - 1$ satisfies:

$$|f(x_k + p_k) - f(x_k) - m_k(p_k) - m_k(0)|
\leq \left\|\frac{1}{2} p_k^T \int_0^1 \left(\nabla^2 f(x + tsp_k) - \nabla^2 f_k\right) p_k dt\right\|
\leq \frac{L}{4} \|p_k\|^3$$

And the denominator of $\rho_k - 1$ satisfies:

$$|m_k(p_k) - m_k(0)| \geq c \|\nabla f_k\| \min\left(\Delta_k, \frac{\|\nabla f_k\|}{\|\nabla^2 f_k\|}\right)
\geq c \|\nabla f_k\| \min\left(\|p_k\|, \frac{\|\nabla f_k\|}{\|\nabla^2 f_k\|}\right)$$

So we need only lower bound $\|f_k\|$ by $\|p_k\|$ to translate this to a lower bound for $\Delta_k$. If $\|p_k^N\| > \frac{1}{2} \Delta_k$ then (in some neighborhood of $x^*$)

$$\|p_k\| \leq \Delta_k \leq 2 \|p_k^N\| \leq 2 \|\nabla^2 f_k^{-1}\| \|\nabla f_k\| \leq C \|\nabla f_k\|$$
Else if $\|p^N_k\| \leq \frac{1}{4}\Delta_k$ then

$$\|p_k\| \leq \|p^N_k\| + o(\|p^N_k\|) \leq 2 \|p^N_k\| \leq C \|\nabla f_k\|$$

Therefore:

$$|m_k(p_k) - m_k(0)| \geq \frac{c}{C} \min\left(\|p_k\|, \frac{\|p_k\|}{\|\nabla^2 f_k\| \|\nabla^2 f_k^{-1}\|}\right)$$

Since the second term’s denominator is the condition number of the matrix (which must be greater than or equal to 1), we have that:

$$|\rho_k - 1| \leq C' \|p_k\| \leq C'\Delta_k$$

Hence, if $\Delta_k < \frac{3}{2C'}$, $\rho_k > 3/4$ and $\Delta_k$ would double. So we know that $\Delta_k \geq \frac{1}{2C'} \hat{\Delta}$ for some fixed $M$ such that $\frac{1}{2C'} \hat{\Delta} \leq \frac{3}{2C'}$. So we have that for sufficiently large $k$, $p_k \to 0$ and so $\rho_k \to 1$ and $\Delta_k$ becomes inactive.
6 Newton’s Method with Hessian Modification

1. If $\nabla^2 f_k \neq 0$, we can add a matrix $E_k$ so that $\nabla^2 f_k + E_k =: B_k \succ 0$. As long as this model produces a descent direction, and $\|B_k\| ||B_k^{-1}||$ are bounded uniformly, Zoutendijk guarantees convergence.

2. Newton’s Method with Hessian Modification

   Algorithm 8: Line Search with Modified Hessian
   
   \begin{algorithm}
   \textbf{input} : Starting point $x_0$, Tolerance $\epsilon$
   \begin{algorithmic}
   \State $k \leftarrow 0$
   \While {$\|p_k\| > \epsilon$} \Do
   \State Find $E_k$ so that $\nabla^2 f_k + E_k \succ 0$
   \State $p_k \leftarrow$ Solve $B_k p = -g_k$
   \State Compute $\alpha_k$ (Wolfe, Goldstein, Backtracking, Interpolation)
   \State $x_{k+1} \leftarrow x_k + \alpha_k p_k$
   \State $k \leftarrow k + 1$
   \EndWhile
   \end{algorithmic}
   \end{algorithm}

3. Minimum Frobenius Norm
   
   (a) Problem: find $E$ which satisfies $\min \|E\|_{F}^2 : \lambda_{\min}(A + E) \geq \delta$.
   (b) Solution: If $A = Q\Lambda Q^T$ is the EVD of $A$ then $E = \text{Qdiag}(\tau_i)Q^T$ where
   \[ \tau_i = \begin{cases} 0 & \lambda_i \geq \delta \\ \delta - \lambda_i & \lambda_i < \delta \end{cases} \]

4. Minimum 2-Norm
   
   (a) Problem: find $E$ which satisfies $\min \|E\|_2^2 : \lambda_{\min}(A + E) \geq \delta$
   (b) Solution: If $\lambda_{\min}(A) = \lambda_n$ then $E = \tau I$ where:
   \[ \tau = \begin{cases} 0 & \lambda_n \geq \delta \\ \delta - \lambda_n & \lambda_n < \delta \end{cases} \]

5. Modified Cholesky
   
   (a) Compute LDL Factorization but add constants to ensure positive definiteness of $A$
   (b) May require pivoting to ensure numerical stability

6. Modified Block Cholesky
   
   (a) Compute $LBL$ factorization where $B$ is a block diagonal (1 $\times$ 1 and 2 $\times$ 2 where the number of blocks is the number of positive eigenvalues and the number of 2 $\times$ 2 blocks is the number of negative eigenvalues)
   (b) Compute the EVD of $B$ and modify it to ensure appropriate eigenvalues.
7 Quasi-Newton’s Method

1. Fundamental Idea

Computation. Let $f$ be our objective and suppose it is Lipschitz twice continuously differentiable. From Taylor’s Theorem:

$$
\nabla f(x + p) = \nabla f(x) + \int_0^1 \nabla^2 f(x + tp) \, dp
$$

Therefore, letting $x = x_k$ and $p = x_{k+1} - x_k$

$$
\nabla^2 f_k(x_{k+1} - x_k) \approx \nabla f_{k+1} - \nabla f_k
$$

Quasi-Newton methods choose an approximation to $\nabla^2 f_k$, $B_k$ which satisfies:

$$
B_{k+1}(x_{k+1} - x_k) = g_{k+1} - g_k
$$

2. Secant Equation

**Definition 7.1.** Let $x_i$ be a sequence of iterates when minimizing an objective function $f$. Let $s_k = x_{k+1} - x_k$ and $y_k = g_{k+1} - g_k$ where $g_i = \nabla f_i$. Then the secant equation is

$$
B_{k+1} s_k = y_k
$$

where $B_{k+1}$ is some matrix.

3. Curvature Condition

**Definition 7.2.** The curvature condition is $s_k^T y_k > 0$.

**Note 7.1.** If $B_k$ satisfies the secant equation and $s_k, y_k$ satisfy the curvature condition then $s_k^T B_{k+1} s_k > 0$. So the quadratic along the step direction is convex.

4. Line Search Rate of Convergence

**Theorem 7.1.** Suppose $f$ is an objective with minimum $x^*$ satisfying the Second Order Conditions. Moreover:

(a) Suppose $f$ is twice continuously differentiable in a neighborhood of $x^*$

(b) Suppose $B_k$ is computed using a quasi-Newton method and

$$
B_k p_k = -g_k
$$

(c) Suppose $x_{k+1} = x_k + p_k$
(d) Suppose \( x_k \to x^* \)

When \( x_0 \) is sufficiently close to \( x^* \), \( x_k \to x^* \) superlinearly if and only if:

\[
\lim_{k \to \infty} \frac{\| (B_k - \nabla^2 f(x^*))p_k \|}{\| p_k \|} \to 0
\]

Proof. We show that the condition is equivalent to: \( p_k - p_k^N = o(\| p_k \|) \).

First, if the condition holds:

\[
p_k - p_k^N = \nabla^2 f_k^{-1} (\nabla^2 f_k p_k + \nabla f_k)
= \nabla^2 f_k^{-1} (\nabla^2 f_k - B_k) p_k
= \nabla^2 f_k^{-1} [(\nabla^2 f_k - \nabla^2 f(x^*)) p_k + (\nabla^2 f(x^*) - B_k) p_k]
\]

Taking norms and using that \( x_0 \) is sufficiently close to \( x^* \), and continuity:

\[
\| p_k - p_k^N \| = o(\| p_k \|)
\]

For the other direction:

\[
o(\| p_k \|) = \| p_k - p_k^N \| \geq \| (\nabla^2 f(x^*) - B_k) p_k \|
\]

Now we just apply triangle inequality and properties of the Newton Step:

\[
\| x_k + p_k - x^* \| \leq \| x_k + p_k^N - x^* \| + \| p_k - p_k^N \| = o(\| x_k - x^* \|^2) + o(\| p_k \|)
\]

Note that \( o(\| p_k \|) = o(\| x_k - x^* \|) \).

7.1 Rank-2 Update: DFP & BFGS

1. Davidson-Fletcher-Powell Update

(a) Let:

\[
G_k = \int_0^1 \nabla^2 f(x_k + t\alpha p_k) dt
\]

So Taylor’s Theorem implies:

\[
y_k = G_k s_k
\]

(b) Let:

\[
\| A \|_W = \left\| G_k^{-1/2} A G_k^{-1/2} \right\|_F
\]

(c) Problem. Given \( B_k \) symmetric positive definite, \( s_k \) and \( y_k \), find

\[
\arg \min \| B - B_k \|_W : B = B^T, \ B s_k = y_k
\]

(d) Solution:

\[
B_{k+1} = \left( I - \frac{y_k^T s_k}{y_k^T y_k} \right) B_k \left( I - \frac{s_k^T y_k}{y_k^T s_k} \right) + \frac{y_k y_k^T}{y_k^T y_k}
\]

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(c) Inverse Hessian:
\[ H_{k+1} = H_k - \frac{H_k y_k y_k^T H_k}{y_k^T H_k y_k} + \frac{s_k s_k^T}{y_k^T s_k} \]

2. Broyden, Fletcher, Goldfarb, Shanno Update

(a) Problem: Given \( H_k \) (inverse of \( B_k \)) symmetric positive definite, \( s_k \) and \( y_k \), find
\[
\arg \min \| H - H_k \|_W : H = H^T, H y_k = s_k
\]

(b) Solution:
\[
H_{k+1} = \left( I - \frac{s_k y_k^T}{y_k^T y_k} \right) H_k \left( I - \frac{y_k s_k^T}{y_k^T y_k} \right) + \frac{s_k s_k^T}{y_k^T s_k}
\]

(c) Hessian:
\[
B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}
\]

3. Preservation of Positive Definiteness

Lemma 7.1. If \( B_k \succ 0 \) and \( B_{k+1} \) is updated from the BFGS method, then \( B_{k+1} \succ 0 \).

Proof. We can equivalently show that \( H_{k+1} \succ 0 \). Let \( z \in \mathbb{R}^n \setminus \{0\} \). Then:
\[
z^T H_{k+1} z = w^T H_k w + \rho_k (s_k^T z)^2
\]
where \( w = z - \rho_k (s_k^T z) y_k \). If \( s_k^T z = 0 \) then \( w = z \) and \( z^T H_k z \succ 0 \). If \( s_k^T z \neq 0 \), then \( \rho_k (s_k^T z)^2 > 0 \) (regardless of if \( w = 0 \)) and so \( H_{k+1} \succ 0 \).

7.2 Rank-1 Update: SR1

1. Problem: find \( v \) such that:
\[
B_{k+1} = B_k + \sigma v v^T : B_{k+1} s_k = y_k
\]

2. Solution:
\[
B_{k+1} = B_k + \frac{(y_k - B_k s_k) (y_k - B_k s_k)^T}{s_k^T (y_k - B_k s_k)}
\]

Proof. Multiplying by \( s_k \):
\[
y_k - B_k s_k = \sigma (v^T s_k) v
\]
Hence, \( v \) is a multiple of \( y_k - B_k s_k \).
Part III
Specialized Applications of Optimization

8 Inexact Newton’s Method

1. Requirements for search direction:
   (a) \( p_k \) is inexpensive to compute
   (b) \( p_k \) approximates the newton step closely as measured by:
   \[
   z_k = \nabla^2 f_k p_k - \nabla f_k
   \]
   (c) The error is controlled by the current gradient:
   \[
   \|z_k\| \leq \eta_k \|\nabla f_k\|
   \]

2. Typically, inexact methods keep \( B_k = \nabla^2 f_k \) and improve on computing \( b_k \)

3. CG Based Algorithmic Overview

   Algorithm 9: Overview of Inexact CG Algorithm
   \[
   \textbf{input}: x_0 \text{ a starting point, } f \text{ objective, restrictions} \\
   x \leftarrow x_0 \\
   \text{Define Tolerance: } \epsilon \text{ Compute Gradient: } r \leftarrow \nabla f(x) \\
   \text{Compute Conjugated Descent Direction: } p \leftarrow -r \\
   \textbf{while} \|r\| > \epsilon \textbf{ do} \\
   \quad \text{Check Constrained Polynomial} \\
   \quad \textbf{if} \ p^T B p \leq 0 \textbf{ then} \\
   \quad \quad \text{Compute } \alpha \text{ given restrictions} \\
   \quad \quad \text{Compute Minimizer: } x \leftarrow x + \alpha p \\
   \quad \quad \textbf{Break Loop} \\
   \quad \textbf{else} \\
   \quad \quad \text{Compute Optimal Step Length: } \alpha \\
   \quad \quad \textbf{if} \ Restrictions \text{ on } x \textbf{ then} \\
   \quad \quad \quad \text{Compute } \tau \text{ given restrictions} \\
   \quad \quad \quad \text{Compute Minimizer: } x \leftarrow x + \tau \alpha p \\
   \quad \quad \quad \textbf{Break Loop} \\
   \quad \quad \text{else} \\
   \quad \quad \quad \text{Compute Step: } x \leftarrow x + \omega r \\
   \quad \textbf{end} \\
   \quad \text{Compute Gradient: } r \leftarrow \nabla f(x) \\
   \quad \text{Compute Conjugated Step Direction: } p \\
   \textbf{end} \\
   x^* \leftarrow x \textbf{ return Minimizer: } x^*
   \]

4. As with the usual CG, we can implement methods for computing the gradient and conjugate step direction efficiently
5. Line Search Strategy: the restriction of $\alpha$ is simply that it satisfies the strong Wolfe Conditions

6. Trust-Region Strategy:

(a) The restriction on $\alpha$ is that $\|x + \alpha p\| = \Delta$ because the restricted polynomial in the direction $p$ is decreasing, so we should go all the way to the boundary

(b) The restriction on $x$ is that $\|x + \alpha p\| > \Delta$ because $x + \alpha p$ is no longer in the trust region so we need to find $\tau$ that ensures this happens.

7. An alternative is to use Lanczos’ Method, which generalizes CG Methods.

9 Limited Memory BFGS

1. In normal BFGS:

$$ H_{k+1} = (I - \rho_k s_k y_k^T)H_k(I - \rho_k y_k s_k^T) + \rho_k s_k s_k^T $$

2. Substituting in for $H_k$ until $H_0$ reveals that only $s_k, y_k$ must be stored to compute $H_{k+1}$

3. Limited BFGS capitalizes on this by storing a fixed number of $s_k, y_k$ and re-updating $H_k$ from a (typically diagonal) matrix $H_0^*$

Algorithm 10: L-BFGS Algorithm

| input | $x_0$ a starting point, $f$ objective, $m$ storage size, $\epsilon$ tolerance |
|-------|
| $x \leftarrow x_0$ |
| $k \leftarrow 0$ |
| while $\|\nabla f(x)\| > \epsilon$ do |
| Choose $H_0$ |
| Update $H$ from $\{(s_k, y_k), \ldots, (s_{k-m}, y_{k-m})\}$ |
| Compute $p^* \leftarrow -H\nabla f(x)$ |
| Implement Line Search or Trust Region (dogleg) to find $p$ |
| $x \leftarrow x + p$ |
| $k \leftarrow k + 1$ |
| end |

return Minimizer: $x$
10 Least Squares Regression Methods

1. In least squares problems, the objective function has a specialized form, for example:

   \[ f(x) = \sum_{i=1}^{m} r_j(x)^2 \]

   (a) \( x \in \mathbb{R}^n \) and usually are the parameters of a particular model of interest

   (b) \( r_j \) are usually the residuals evaluated for the model \( \phi \) with parameter \( x \) at point \( t_j \) and output \( y_j \). That is:

   \[ r_j(x) = \phi(x, t_j) - y_j \]

2. Formulation

   (a) Let \( r(x) = \begin{bmatrix} r_1(x) & r_2(x) & \cdots & r_m(x) \end{bmatrix} \). Then:

   \[ f(x) = \frac{1}{2} \| r(x) \|^2 \]

   (b) The first derivative:

   \[ \nabla f(x) = \nabla r(x)^T r(x) = \begin{bmatrix} \nabla r_1(x)^T \\ \nabla r_2(x)^T \\ \vdots \\ \nabla r_m(x)^T \end{bmatrix} r(x) = J(x)^T r(x) \]

   (c) The Second Derivative:

   \[ \nabla^2 f(x) = \nabla r(x)^T \nabla r(x) + \nabla^2 r(x) r(x) = J(x)^T J(x) + \nabla^2 r(x) r(x) \]

3. The main conceit behind Least Squares methods is to approximate \( \nabla^2 f(x) \) by \( J(x)^T J(x) \), thus, only first derivative information is required.

10.1 Linear Least Squares Regression

1. Suppose \( \phi(x, t_j) = t_j^T x \) then:

   \[ r(x) = T x - y \]

   where each row of \( T \) is \( t_j^T \)

2. Objective Function:

   \[ f(x) = \frac{1}{2} \| T x - y \|^2 \]

   **Note 10.1.** The objective function can be solved directly using matrix factorizations if \( m \) is not too large.

3. First Derivative:

   \[ \nabla f(x) = T^T T x - T y \]

   **Note 10.2.** The normal equations may be better to solve if \( m \) is large because by multiplying by the transpose, we need only solve an \( n \times n \) system of equations.
10.2 Line Search: Gauss-Newton

1. The Gauss-Newton Algorithm is a line search for non-linear least squares regression with search direction:

\[ J_k^T J_k p_k = -J_k^T r_k \]

2. Notice that these are the normal equations from:

\[ \frac{1}{2} \| J_k p_k + r_k \|^2 \]

3. Using this search direction, we proceed with line search as per usual

10.3 Trust Region: Levenberg-Marquardt

1. The local model is:

\[ m(p) = f(x) + (J(x)^T r(x))^T p + \frac{1}{2} p^T J(x)^T J(x) p \quad \| p \| \leq \Delta \]

2. Since \( f(x) = \| r(x) \|^2 / 2 \), the solution to minimizing \( m(p) \) is equivalent to:

\[ \min \frac{1}{2} \| J(x) p + r \|^2 \quad \| p \| \leq \Delta \]
Part IV
Constrained Optimization

10.4 Theory of Constrained Optimization

1. In constrained optimization, the problem is:

\[ \min \{ x \in \arg \min f(x) : c_i(x) = 0 \ \forall i \in \mathcal{E}, \ c_j(x) \geq 0 \ \forall j \in \mathcal{I} \} \]

(a) \( f \) is the usual objective function
(b) \( \mathcal{E} \) is the index for equality constraints
(c) \( \mathcal{I} \) is the index for inequality constraints

2. Feasible Set. Active Set.

Definition 10.1. The feasible set, \( \Omega \), is the set of all points which satisfy the constraints:

\[ \Omega = \{ x : c_i(x) = 0 \ \forall i \in \mathcal{E}, \ c_j(x) \geq 0 \ \forall j \in \mathcal{I} \} \]

The active set at point \( x \in \Omega \), \( \mathcal{A}(x) \), is defined by:

\[ \mathcal{A}(x) = \mathcal{E} \cup \{ j \in \mathcal{I} : c_j(x) = 0 \} \]

3. Feasible Sequence. Tangent. Tangent Cone.

Definition 10.2. Let \( x \in \Omega \). A sequence \( z_k \to x \) is a feasible sequence if for sufficiently large \( K \), if \( k \geq K \) then \( z_k \in \Omega \). A vector \( d \) is a tangent to \( x \) if there is a feasible sequence \( z_k \to x \) and a sequence \( t_k \to 0 \) such that:

\[ \lim_{k \to \infty} \frac{z_k - x}{t_k} = d \]

The set of all tangents is the Tangent Cone at \( x \), denoted \( T_\Omega(x) \).

Note 10.3. \( T_\Omega(x) \) contains all step directions which for a sufficiently small step size will remain in the feasible region \( \Omega \).

4. Linearized Feasible Directions.

Definition 10.3. Let \( x \in \Omega \) and \( \mathcal{A}(x) \) be its active set. The set of linearized feasible directions, \( \mathcal{F}(x) \), is defined as:

\[ \mathcal{F}(x) = \{ d : d^T \nabla c_i(x) = 0 \ \forall i \in \mathcal{E}, \ d^T \nabla c_i(x) \geq 0 \ \forall \mathcal{A}(x) \cap \mathcal{I} \} \]

Note 10.4. Let \( x \in \Omega \) and suppose we linearize the constraints near \( x \). The set \( \mathcal{F}(x) \) contains all search directions for which the linearized approximations to \( c_i(x + d) \) satisfy the constraints. For example if \( c_i(x) = 0 \) for \( i \in \mathcal{E} \), then \( 0c_i(x + d) \approx c_i(x) + d^T \nabla c_i(x) = d^T \nabla c_i(x) \) for \( d \) to be a good search direction.

5. LICQ Constrain Qualification
Definition 10.4. Given a point \( x \) and active set \( A(x) \), the linear independence constrain qualification (LICQ) holds if the set of active constraint gradients, \( \{ \nabla c_i(x) : i \in A(x) \} \), is linearly independent.

Note 10.5. Most algorithms use line search or trust region require that \( F(x) \approx T_{10}(x) \), else there is not a good way to find another iterate.

6. First Order Necessary Conditions

Theorem 10.1. Let \( f, c_i \) be continuously differentiable. Suppose \( x^* \) is a local solution to the optimization problem and at \( x^* \), LICQ holds. Then there is a Lagrange multiplier vector \( \lambda \), with components \( \lambda_i \) for \( i \in E \cup I \), such that the Karush-Kuhn-Tucker conditions are satisfied at \( (x^*, \lambda) \):

\( (a) \quad \nabla_x \mathcal{L}(x^*, \lambda) = 0 \)
(\( b) \quad \text{For all } i \in E, \ c_i(x^*) = 0 \)
(\( c) \quad \text{For all } j \in I, \ c_j(x^*) \geq 0 \)
(\( d) \quad \text{For all } j \in I, \ \lambda_j \geq 0 \)
(\( e) \quad \text{For all } j \in I, \ \lambda_j c_j(x^*) = 0 \)