A Lower Bound for Calls on Quadratic Variation

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Assumptions and Notation

- The running assumptions are frictionless markets, no arbitrage, zero interest rates, and a positive continuous underlying futures price F.

- Let $X_t \equiv \ln(F_t/F_0)$ be the returns process and we suppose that its instantaneous (lognormal) variance σ_t^2 is an unspecified stochastic process which can jump.

- We will be interested in finding semi-robust bounds for European calls on F_T and European calls on the terminal quadratic variation of log price, $\langle X \rangle_T \equiv \int_0^T \sigma_t^2 dt$.

- The respective payoffs at T are $(F_T - K)^+$ and $(\langle X \rangle_T - H)^+$, where $K \geq 0$ and $H \geq 0$ are the strike prices.

Back to Black

- Although the futures price actually has unknown stochastic volatility, consider the process $\hat{F}$ arising in the Black model with constant unit volatility:

$$\frac{d\hat{F}_t}{\hat{F}_t} = dW_t, \qquad t \geq 0.$$

- Let $B^u(F,R)$ be the Black model value at time t of a claim with payoff $U(F_T)$ at T given that the underlying futures price $\hat{F}_t = F$ and the remaining time to maturity at t is $R > 0$:

$$B^u(F,R) \equiv \int_0^\infty U(G) \frac{1}{\sqrt{2\pi R G}} \exp\left\{ -\frac{1}{2} \left[\frac{\ln(G) - [\ln(F) - R/2]}{\sqrt{R}} \right]^2 \right\} dG.$$

- Let $B^c(F,R)$ denote the Black model value when $U(F) = (F - K)^+$.

Upper Bound on Call on Price

- Mykland shows that if one further assumes that the terminal quadratic variation of log price is bounded above by H, then no arbitrage places an upper bound on the initial price of a European call on F_T.

- The upper bound is the Black model value $B^c(F_0, R)$ evaluated at $R = H$.

- If the market initially prices the call at $C_0 > B^c(F_0, H)$, then the arbitrage is to sell the call and delta hedge by holding $\frac{\partial}{\partial F} B^c(F_t, H - \langle X \rangle_t)$ futures at each $t \in [0, T]$.

- The terminal profit is:

$$\Pi_T = C_0 - B^c(F_0, H) + B^c(F_T, H - \langle X \rangle_T) - (F_T - K)^+ \geq 0.$$

Lower Bound on Call on QV

- Now suppose that there is no upper bound on terminal quadratic variation.
- Also suppose that there exists a continuum of European options whose initial prices $P_0(K,T)$ and $C_0(K,T)$ are known for all $K > 0$ and are arbitrage-free.
- Under these assumptions, Bruno Dupire derived an observable lower bound for the value of a European call on quadratic variation.
- In these overheads, we present the details of his argument.

Twice Differentiable Convex Payoff

- Since the investor can take static finite or infinitessimal positions in European puts and calls written on the underlying asset, the investor can create any C^1 function $U(F_T)$ of the terminal futures price F_T.

- We will only work with a payoff function U that is C^2 and convex:

$$U''(F) \geq 0.$$

- We note that the requirement that U be C^2 actually requires that the investor be able to take an infinitessimal position in options.

- One can explore relaxing the C^2 requirement by replacing the payoff $U(F_T)$ at T with a convex payoff $V(F'_T)$ at $T' > T$. The hope is that the conditional value of this payoff at T is convex in F_T. As unclear dynamical assumptions are needed to ensure this property, we don't explore this alternative today.

Two Separate P&L's

- Suppose that an investor initially sells a portfolio of European options whose aggregate payoff at T is the nonnegative convex function $U(\cdot)$ evaluated at F_T, i.e. $U(F_T)$. The terminal P&L from just this strategy is $E_0^{\mathbb{Q}}U(F_T) - U(F_T)$.

- For given $H > 0$, suppose that the investor initially deposits $B^u(F_0, H)$ in a riskfree asset and delta hedges as follows. At times $t \in [0, T]$ s.t. $\langle X \rangle_t < H$, the investor is long $\frac{\partial}{\partial F} B^u(F_t, H - \langle X \rangle_t)$ futures.

- Let τ_H be the first passage time of $\langle X \rangle$ to the strike H of the call on quadratic variation. If $\tau_H < T$, then at times $t \in [\tau_H, T]$ s.t. $\langle X \rangle_t \geq H$, the investor is long $U'(F_t)$ futures.

- The terminal $P\&L$ from just the portfolio of cash and futures is:

$$-B^u(F_0,H) + 1(\tau_H < T)\left[U(F_T) - \int_{\tau_H}^{T} U''(F_t)\frac{F_t^2}{2} d\langle X \rangle_t\right] + 1(\tau_H \geq T)B^u(F_T, H - \langle X \rangle_T).$$

Total P&L

- The last slide gave the separate terminal P&L's from:
  1. initially selling a portfolio of European options paying off $U(F_T)$ at T, and:
  2. depositing $B^u(F_0, H)$ in a riskfree asset and delta hedging by holding $\frac{\partial}{\partial F}B^u(F_t, H - \langle X\rangle_t)$ futures for $t \in [0, \tau_H \wedge T]$ and holding $U'(F_t)$ futures for $t \in [\tau_H, T]$ if $\tau_H < T$.

- Summing these P&L's leads to a total terminal $P\&L_T$ of:

$$\begin{aligned}\Pi_T &= E_0^{\mathbb{Q}} U(F_T) - B^u(F_0, H) - 1(\tau_H < T)\int_{\tau_H}^{T} U''(F_t)\frac{F_t^2}{2} d\langle X\rangle_t \\ &+ 1(\tau_H \geq T)[B^u(F_T, H - \langle X\rangle_T) - U(F_T)].\end{aligned}$$

- If $\tau_H \geq T$, then the terminal P&L is nonnegative, but if $\tau_H < T$, then the cash outflows generated by being short gamma without any offsetting time decay can lead to an overall loss.

Lower Bound on P&L

- Recall that selling a C^2 convex claim paying $U(F_T)$ and delta hedgng at the realized vol for $t \in [\tau_H \wedge T]$ produces a terminal P&L of:

$$\Pi_T = E_0^{\mathbb{Q}} U(F_T) - B^u(F_0, H) - \int_{\tau_H}^{T} U''(F_t)\frac{F_t^2}{2} d\langle X\rangle_t$$
$$+ \ 1(\tau_H \geq T)[B^u(F_T, H - \langle X\rangle_T) - U(F_T)].$$

- To find a lower bound on this terminal P&L, suppose that the dollar convexity of U is bounded above:

$$U''(F)F^2 \leq M$$

for some $M > 0$. Also note that the assumed positive convexity of U causes the last term in the P&L to be nonnegative. Hence:

$$\Pi_T \geq E_0^{\mathbb{Q}} U(F_T) - B^u(F_0, H) - \frac{M}{2}(\langle X\rangle_T - H)^+,$$

since $1(\tau_H < T)\int_{\tau_H}^{T} d\langle X\rangle_t = (\langle X\rangle_T - H)^+$.

Hedging with Calls on QV

- Recall our lower bound on the P&L arising from selling a C^2 convex claim paying $U(F_T)$ and delta hedgng at the realized vol for $t \in [\tau_H \wedge T]$:

$$\Pi_T \geq E_0^{\mathbb{Q}} U(F_T) - B^u(F_0, H) - \frac{M}{2}(\langle X \rangle_T - H)^+.$$

- Now suppose that an investor also initially buys $\frac{M}{2}$ calls on quadratic variation with the initial price being $\gamma_0(H,T)$ for each such call. Then the lower bound on terminal P&L changes to:

$$\Pi_T \geq E_0^{\mathbb{Q}} U(F_T) - B^u(F_0, H) - \frac{M}{2}\gamma_0(H,T).$$

If $\gamma_0(H,T) \leq \frac{2}{M}[E_0^{\mathbb{Q}} U(F_T) - B^u(F_0,H)]$, then the terminal P&L is nonnegative with positive probability of being positive and hence arbitrage results.

- Hence, no arbitrage $\Rightarrow \gamma_0(H,T) > \frac{2}{M}[E_0^{\mathbb{Q}} U(F_T) - B^u(F_0,H)]$.

Lower Bound on Calls on QV

- Recall that no arbitrage $\Rightarrow \gamma_0(H,T) > \frac{2}{M}[E_0^{\mathbb{Q}}U(F_T) - B^u(F_0,H)]$, where $\gamma_0(H,T)$ is the price of a call on QV with strike H and M is the maximum dollar convexity of the payoff U.

- One can try to maximize the call's lower bound over choice of the payoff function U. In a typical downward sloping skew, the solution is to sell all puts whose implied $\sigma^2 T$ is above H.

- Can we also get a robust upper bound on the value of a call on QV?