A Lower Bound for Calls on Quadratic Variation

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Assumptions and Notation

- The running assumptions are frictionless markets, no arbitrage, zero interest rates, and a positive continuous underlying futures price $F$.

- Let $X_t \equiv \ln\left(\frac{F_t}{F_0}\right)$ be the returns process and we suppose that its instantaneous (lognormal) variance $\sigma_t^2$ is an unspecified stochastic process which can jump.

- We will be interested in finding semi-robust bounds for European calls on $F_T$ and European calls on the terminal quadratic variation of log price, $\langle X \rangle_T \equiv \int_0^T \sigma_t^2 dt$.

- The respective payoffs at $T$ are $(F_T - K)^+$ and $(\langle X \rangle_T - H)^+$, where $K \geq 0$ and $H \geq 0$ are the strike prices.
• Although the futures price actually has unknown stochastic volatility, consider the process $\hat{F}$ arising in the Black model with constant unit volatility:

$$\frac{d\hat{F}_t}{\hat{F}_t} = dW_t, \quad t \geq 0.$$ 

• Let $B^u(F, R)$ be the Black model value at time $t$ of a claim with payoff $U(F_T)$ at $T$ given that the underlying futures price $\hat{F}_t = F$ and the remaining time to maturity at $t$ is $R > 0$:

$$B^u(F, R) \equiv \int_0^\infty U(G) \frac{1}{\sqrt{2\pi RG}} \exp\left\{-\frac{1}{2} \left[\ln(G) - \left[\ln(F) - R/2\right]\right]^2\right\} dG.$$ 

• Let $B^c(F, R)$ denote the Black model value when $U(F) = (F - K)^+$. 
• Mykland shows that if one further assumes that the terminal quadratic variation of log price is bounded above by $H$, then no arbitrage places an upper bound on the initial price of a European call on $F_T$.

• The upper bound is the Black model value $B^c(F_0, R)$ evaluated at $R = H$.

• If the market initially prices the call at $C_0 > B^c(F_0, H)$, then the arbitrage is to sell the call and delta hedge by holding $\frac{\partial}{\partial F} B^c(F_t, H - \langle X \rangle_t)$ futures at each $t \in [0, T]$.

• The terminal profit is:

$$\Pi_T = C_0 - B^c(F_0, H) + B^c(F_T, H - \langle X \rangle_T) - (F_T - K)^+ \geq 0.$$
Lower Bound on Call on QV

• Now suppose that there is no upper bound on terminal quadratic variation.

• Also suppose that there exists a continuum of European options whose initial prices $P_0(K, T)$ and $C_0(K, T)$ are known for all $K > 0$ and are arbitrage-free.

• Under these assumptions, Bruno Dupire derived an observable lower bound for the value of a European call on quadratic variation.

• In these overheads, we present the details of his argument.
Twice Differentiable Convex Payoff

- Since the investor can take static finite or infinitessimal positions in European puts and calls written on the underlying asset, the investor can create any $C^1$ function $U(F_T)$ of the terminal futures price $F_T$.

- We will only work with a payoff function $U$ that is $C^2$ and convex:
  \[ U''(F) \geq 0. \]

- We note that the requirement that $U$ be $C^2$ actually requires that the investor be able to take an infinitessimal position in options.

- One can explore relaxing the $C^2$ requirement by replacing the payoff $U(F_T)$ at $T$ with a convex payoff $V(F'_T)$ at $T' > T$. The hope is that the conditional value of this payoff at $T$ is convex in $F_T$. As unclear dynamical assumptions are needed to ensure this property, we don’t explore this alternative today.
Two Separate P&L’s

- Suppose that an investor initially sells a portfolio of European options whose aggregate payoff at $T$ is the nonnegative convex function $U(\cdot)$ evaluated at $F_T$, i.e. $U(F_T)$. The terminal P&L from just this strategy is $E_0^\otimes U(F_T) - U(F_T)$.

- For given $H > 0$, suppose that the investor initially deposits $B^u(F_0, H)$ in a riskfree asset and delta hedges as follows. At times $t \in [0, T]$ s.t. $\langle X \rangle_t < H$, the investor is long $\frac{\partial}{\partial F} B^u(F_t, H - \langle X \rangle_t)$ futures.

- Let $\tau_H$ be the first passage time of $\langle X \rangle$ to the strike $H$ of the call on quadratic variation. If $\tau_H < T$, then at times $t \in [\tau_H, T]$ s.t. $\langle X \rangle_t \geq H$, the investor is long $U'(F_t)$ futures.

- The terminal P&L from just the portfolio of cash and futures is:

$$-B^u(F_0, H) + 1(\tau_H < T) \left[ U(F_T) - \int_{\tau_H}^T U''(F_t) \frac{F_t^2}{2} d\langle X \rangle_t \right] + 1(\tau_H \geq T) B^u(F_T, H - \langle X \rangle_T).$$
• The last slide gave the separate terminal P&L’s from:

1. initially selling a portfolio of European options paying off \( U(F_T) \) at \( T \), and:
2. depositing \( B^u(F_0, H) \) in a riskfree asset and delta hedging by holding \( \frac{∂}{∂F} B^u(F_t, H - \langle X \rangle_t) \) futures for \( t \in [0, \tau_H \wedge T] \) and holding \( U'(F_t) \) futures for \( t \in [\tau_H, T] \) if \( \tau_H < T \).

• Summing these P&L’s leads to a total terminal \( P&L_T \) of:

\[
\Pi_T = E_0^Q U(F_T) - B^u(F_0, H) - 1(\tau_H < T) \int_{\tau_H}^{T} U''(F_t) \frac{F_t^2}{2} d\langle X \rangle_t \\
+ 1(\tau_H \geq T)[B^u(F_T, H - \langle X \rangle_T) - U(F_T)].
\]

• If \( \tau_H \geq T \), then the terminal P&L is nonnegative, but if \( \tau_H < T \), then the cash outflows generated by being short gamma without any offsetting time decay can lead to an overall loss.
Lower Bound on P&L

- Recall that selling a $C^2$ convex claim paying $U(F_T)$ and delta hedgng at the realized vol for $t \in [\tau_H \wedge T]$ produces a terminal P&L of:

\[ \Pi_T = E^Q_0 U(F_T) - B''(F_0, H) - \int_{\tau_H}^{T} U''(F_t) \frac{F_t^2}{2} d\langle X \rangle_t \]

\[ + 1(\tau_H \geq T)[B''(F_T, H - \langle X \rangle_T) - U(F_T)]. \]

- To find a lower bound on this terminal P&L, suppose that the dollar convexity of $U$ is bounded above:

\[ U''(F)F^2 \leq M \]

for some $M > 0$. Also note that the assumed positive convexity of $U$ causes the last term in the P&L to be nonnegative. Hence:

\[ \Pi_T \geq E^Q_0 U(F_T) - B''(F_0, H) - \frac{M}{2} (\langle X \rangle_T - H)^+, \]

since $1(\tau_H < T) \int_{\tau_H}^{T} d\langle X \rangle_t = (\langle X \rangle_T - H)^+$. 

9
Hedging with Calls on QV

• Recall our lower bound on the P&L arising from selling a $C^2$ convex claim paying $U(F_T)$ and delta hedging at the realized vol for $t \in [\tau_H \wedge T]$:

$$\Pi_T \geq E^Q_0 U(F_T) - B^u(F_0, H) - \frac{M}{2}(\langle X \rangle_T - H)^+.$$ 

• Now suppose that an investor also initially buys $\frac{M}{2}$ calls on quadratic variation with the initial price being $\gamma_0(H,T)$ for each such call. Then the lower bound on terminal P&L changes to:

$$\Pi_T \geq E^Q_0 U(F_T) - B^u(F_0, H) - \frac{M}{2}\gamma_0(H,T).$$ 

If $\gamma_0(H,T) \leq \frac{2}{M}[E^Q_0 U(F_T) - B^u(F_0, H)]$, then the terminal P&L is nonnegative with positive probability of being positive and hence arbitrage results.

• Hence, no arbitrage $\Rightarrow \gamma_0(H,T) > \frac{2}{M}[E^Q_0 U(F_T) - B^u(F_0, H)].$
Lower Bound on Calls on QV

- Recall that no arbitrage $\Rightarrow \gamma_0(H, T) > \frac{2}{M} [E_0^Q(U(F_T) - B^u(F_0, H))]$, where $\gamma_0(H, T)$ is the price of a call on QV with strike $H$ and $M$ is the maximum dollar convexity of the payoff $U$.

- One can try to maximize the call’s lower bound over choice of the payoff function $U$. In a typical downward sloping skew, the solution is to sell all puts whose implied $\sigma^2 T$ is above $H$.

- Can we also get a robust upper bound on the value of a call on QV?