POISSON SUPERPOSITION PROCESSES

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Abstract

Superposition is a mapping on point configurations that sends the n-tuple $(x_1, \ldots, x_n) \in \mathcal{X}^n$ into the n-point configuration $\{x_1, \ldots, x_n\} \subset \mathcal{X}$, counted with multiplicity. It is an additive set operation such the superposition of a k-point configuration in $\mathcal{X}^n$ is a kn-point configuration in $\mathcal{X}$. A Poisson superposition process is the superposition in $\mathcal{X}$ of a Poisson process in the space of finite-length $\mathcal{X}$-valued sequences. From properties of Poisson processes as well as some algebraic properties of formal power series, we obtain an explicit expression for the Janossy measure of Poisson superposition processes, and we study their law under domain restriction. Examples of well-known Poisson superposition processes include compound Poisson, negative binomial, and permanental (boson) processes.

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1. Introduction

A *Poisson process* (PP) $\mathbf{X}$ on a space $\mathcal{X}$ is a random point configuration for which the counts of points in nonoverlapping subsets are independent Poisson random variables. The law of any Poisson process $\mathbf{X}$ is determined by a mean measure ζ on $\mathcal{X}$ so that $\mathbf{X}(A) := \#(\mathbf{X} \cap A)$, the number of points of $\mathbf{X}$ in $A \subseteq \mathcal{X}$, is a Poisson random variable with mean $\zeta(A)$. The class of *Cox processes* generalizes Poisson processes by allowing the mean measure to arise from a random process. Cox processes are useful models in various applications, including neuroscience, finance, and quantum mechanics; see Cox and Isham (1980). *Boson*, or *permanental*, *point processes* appear primarily in quantum mechanical applications. Unlike the Poisson process, permanental processes incorporate non-trivial dependence between counts in non-overlapping regions of $\mathcal{X}$, and so are more appropriate in settings with interaction between points; see Hough *et al.* (2006) or McCullagh and Møller (2006).

In this paper, we study a class of point processes related to each of the above processes. A *Poisson superposition process* (PSP) is a random point configuration in $\mathcal{X}$ obtained from a Poisson process $\mathbf{X}$ in the space $\mathrm{seq}(\mathcal{X}) := \bigcup_{n \geq 0} \mathcal{X}^n$ of finite-length $\mathcal{X}$-valued sequences by counting each component of each point in $\mathbf{X}$ with its multiplicity. The projected configuration $\mathbf{Z}$ can be regarded as either a random multiset of $\mathcal{X}$ or a random integer-valued measure on $\mathcal{X}$ for which $\mathbf{Z}(\{x\}) = 0, 1, 2, \ldots$. For example, the superposition of $\mathbf{X} = \{(2), (3), (1, 3), (2, 3, 4)\}$ is the multiset $\mathbf{Z} = \{1, 2, 2, 3, 3, 3, 4\}$ that counts each component with its multiplicity. Many well-known point processes correspond to Poisson superposition processes with a suitable mean measure, e.g., compound Poisson, negative binomial, and permanental processes.

Our main theorems characterize certain structural properties of Poisson superpositions. In particular, when the mean measure ζ of the associated Poisson process has finite mass, we obtain an exact expression for the Janossy measure as a function of ζ (Theorem 4.1). Conversely, given the Janossy measure of the Poisson superposition process, we can recover the mean measure of the corresponding Poisson process (Theorem 4.2). If the mean measure has infinite mass, then both the Poisson process and the superposition process are almost surely infinite, in which case the superposition distribution is determined by its restrictions to subsets for which the

number of points is finite with probability one. The restricted process is also a Poisson superposition, but the associated mean measure is not obtained by simple restriction of ζ to $\mathrm{seq}(A) \subset \mathrm{seq}(\mathcal{X})$ (Theorem 4.3).

We organize the paper as follows. In Section 2, we review some preliminary properties of scalar Poisson superposition and Poisson point processes. In Section 3, we discuss measures on finite-length sequences and prepare our discussion of Poisson superposition processes in Section 4. In Section 5, we conclude with some familiar special cases of Poisson superposition processes.

2. Preliminaries: Poisson point processes

Throughout the paper, $\mathcal{X}$ is a Polish space with sufficient furnishings to accommodate a Poisson point process, i.e., $\mathcal{X}$ has a Borel σ-field $\mathcal{B}_{\mathcal{X}}$ containing singletons on which a non-negative Kingman-countable measure is defined (Kingman, 1993, Section 2.2). Foreshadowing our study of Poisson processes and their superpositions, we record some facts about Poisson random variables and scalar superpositions.

2.1. Scalar Poisson superposition

A random variable X taking values on the non-negative integers $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ has the Poisson distribution with parameter $\lambda > 0$, denoted $X \sim \mathrm{Po}(\lambda)$, if

$$P\{X = k\} = \lambda^k e^{-\lambda}/k!, \qquad k = 0, 1, 2, \ldots.$$

The moment generating function of $X \sim \mathrm{Po}(\lambda)$ is $M_\lambda(t) := \exp\{\lambda(e^t - 1)\}$, from which *infinite divisibility* and the *convolution property* are evident, i.e., the sum $X_\bullet := \sum_r X_r$ of independent Poisson random variables $X_r \sim \mathrm{Po}(\lambda_r)$ is distributed as $X_\bullet \sim \mathrm{Po}\left(\sum_r \lambda_r\right)$.

Definition 2.1. (*Scalar Poisson superposition.*) For a family $\{X_r\}_{r \geq 1}$ of independent Poisson random variables, with $X_r \sim \mathrm{Po}(\lambda_r)$ for each $r \geq 1$, we call

$$Z := \sum_{r \geq 1} r X_r \tag{1}$$

a *scalar Poisson superposition* with parameter $\lambda = \{\lambda_r\}_{r \geq 1}$.

Note that Z lives on the extended non-negative integers $\{0, 1, \ldots, \infty\}$, but, unlike the Poisson distribution, $E(Z) = \infty$ does not imply $Z = \infty$ with probability one. For example, if $\lambda_r = 1/r^2$, then $E(X_r) = 1/r^2$ and $E(Z) = \sum_{r \geq 1} r \cdot 1/r^2 = \infty$, but $P(X_r > 0) = 1 - e^{-1/r^2}$ is summable. The Borel–Cantelli Lemma implies $P(X_r > 0$ infinitely often$) = 0$ and $Z < \infty$ with probability one.

In the next theorem, we write $\lambda(t) := \sum_{r \geq 1} t^r \lambda_r$ for the ordinary generating function of $\{\lambda_r\}_{r \geq 1}$.

Theorem 2.1. *Scalar Poisson superpositions satisfy the following.*

- *Convolution: Let Z, Z' be independent scalar Poisson superpositions with parameters $\{\lambda_r\}$ and $\{\lambda'_r\}$, respectively. Then $Z + Z'$ is a scalar Poisson superposition with parameter $\{\lambda_r + \lambda'_r\}$.*

- *Infinite divisibility: For each $n \geq 1$, a scalar Poisson superposition with parameter $\{\lambda_r\}$ can be expressed as the sum of n independent, identically distributed scalar Poisson superpositions with parameter $\{\lambda_r/n\}$.*

- *Non-negative integer multiplication: for $m \geq 0$, mZ has probability generating function $\exp\{-\lambda(1) + \lambda(t^m)\}$.*

- *Superposition of scalar Poisson superpositions: If $Z_1, Z_2, \ldots$ are independent scalar Poisson superpositions, Z_j with parameter $\{\lambda_r^j\}_{r \geq 1}$, for each $j = 1, 2, \ldots$, then $\sum_{r \geq 1} r Z_r$ is a scalar Poisson superposition with parameter $\{\lambda_r^*\}_{r \geq 1}$, where*

$$\lambda_r^* := \sum_{s \mid r} \lambda_s^{r/s},$$

  where $s \mid r$ denotes that $r = sk$ for some $k = 1, 2, \ldots$.

- *Thinning: Let $X_1, X_2, \ldots$ be independent Poisson random variables with $X_r \sim \mathrm{Po}(\lambda_r)$ and, given $\{X_r\}_{r \geq 1}$, $B_1, B_2, \ldots$ are conditionally independent Binomial random variables with $B_r \sim \mathrm{Bin}(X_r, p)$, for some $0 < p < 1$. Then the thinned superposition $Z_p := \sum_{r \geq 0} r B_r$ is a scalar Poisson superposition with parameter $\{p\lambda_r\}_{r \geq 1}$.*

Proof. Let $\{X_r\}_{r \geq 1}$ be independent Poisson random variables with $X_r \sim \mathrm{Po}(\lambda_r)$, for each $r \geq 1$. By the convolution property, the sum $X_\bullet := \sum_{r \geq 1} X_r$ is a Poisson random variable with parameter $\lambda(1) = \sum_{r \geq 1} \lambda_r$. If $\lambda(1) = \infty$, then $X_\bullet = Z = \infty$

with probability one. Otherwise, if $\lambda(1) < \infty$, then the joint probability generating function of $(X_\bullet, Z)$ is

$$
\begin{aligned}
G(s,t) &:= \sum_{i,j \geq 0} P\{X_\bullet = i,\, Z = j\} s^i t^j \\
&= \prod_{r \geq 1} \sum_{j \geq 0} e^{-\lambda_r} \lambda_r^j s^j t^{rj}/j! \\
&= \prod_{r \geq 1} e^{-\lambda_r} e^{st^r \lambda_r} \\
&= \exp\{-\lambda(1) + s\lambda(t)\}.
\end{aligned}
\tag{2}
$$

From $G(s,t)$, we can immediately recover the marginal generating functions of $X_\bullet$ and Z: $G_{X_\bullet}(s) = \exp\{-\lambda(1) + s\lambda(1)\}$ and $G_Z(t) = \exp\{-\lambda(1) + \lambda(t)\}$.

The above properties are now immediate by standard properties of Poisson random variables. For Z, Z' independent scalar Poisson superpositions with parameters $\{\lambda_r\}, \{\lambda_r'\}$, respectively, we have $G_{Z+Z'}(t) = G_Z(t) G_{Z'}(t) = \exp\{-\lambda(1) + \lambda(t) - \lambda'(1) + \lambda'(t)\}$. To observe infinite divisibility of Z with parameter λ, we simply take $Z_1, \ldots, Z_n$ independent, identically distributed Poisson superpositions with parameter $\{\lambda_r/n\}$ and appeal to the convolution property. The transformation of $G_Z(t)$ under non-negative integer scaling of Z follows by substituting t^m for t in $G_Z(t)$. Finally, let $Z_1, Z_2, \ldots$ be independent scalar Poisson superpositions, where Z_j has parameter $\{\lambda_r^j\}_{r \geq 1}$. Then $\sum_{r \geq 1} r Z_r$ is also a scalar Poisson superposition with parameter $\{\lambda_r^*\}_{r \geq 1}$, where

$$
\lambda_r^* := \sum_{s \mid r} \lambda_s^{r/s}.
$$

The thinning property follows because the unconditional distribution of B_r is $\mathrm{Po}(\lambda_r)$, for each $r = 1, 2, \ldots$, and $X_1, X_2, \ldots$ are independent by assumption. This completes the proof.

Example 1. (*Compound Poisson distribution.*) The compound Poisson distribution is defined as a convolution of a Poisson random number of integer-valued random variables. Let $N \sim \mathrm{Po}(\mu)$ and $X_1, X_2, \ldots$ be independent, identically distributed according to $p_k := P\{X_1 = k\}$, $k = 0, 1, 2, \ldots$. The random variable $W = X_1 + \cdots + X_N$ has the *compound Poisson distribution* with probability generating function

$$
G_W(t) = \exp\{\mu(\phi_X(t) - 1)\},
$$

where

$$\phi_X(t) := \sum_{k \geq 0} p_k t^k$$

is the probability generating function of each X_i, $i = 1, 2, \ldots$. With $\lambda(t) := \mu \phi_X(t)$, we observe

$$G_W(t) = \exp\{-\lambda(1) + \lambda(t)\},$$

which is also the probability generating function of a scalar Poisson superposition with parameter $\{\lambda_r\}_{r \geq 1}$, as in (2).

2.1.1. *Relation to the negative binomial distribution* A special case of scalar Poisson superposition that occurs in applications takes $\lambda_r = \theta \pi^r / r$, with $0 < \pi < 1$ and $\theta > 0$, for each $r \geq 1$. In this instance, we obtain $\lambda(t) = -\theta \log(1 - \pi t)$ and $G_Z(t) = (1-\pi)^\theta (1-\pi t)^{-\theta}$, the generating function of the negative binomial distribution with parameter (θ, π). This elementary form of univariate Poisson superposition arises in diverse areas, including ecology (Fisher, Corbet and Williams, 1943), population genetics (Ewens, 1972), probabilistic number theory (Donnelly and Grimmett, 1993), and theory of exchangeable random partitions (Kingman, 1978).

For example, if the initial Poisson sequence is written in integer partition style, i.e., $X = 1^{X_1} 2^{X_2} \cdots$ where X_r counts the number of parts of size r in a partition of the random integer $Z = \sum_r r X_r$, then X partitions Z into a random number $X_{\textbf{.}}$ of parts with joint distribution in (2). In particular, the conditional distribution of X, given Z, is

$$P\{X = (x_1, x_2, \ldots) \mid Z = n\} \propto \prod_{1 \leq r \leq n} \frac{\lambda_r^{x_r}}{x_r!}, \quad \sum_{r \geq 1} r x_r = n.$$

When the intensity sequence has the negative binomial pattern, $\lambda_r = \theta \pi^r / r$, then the conditional distribution is independent of π,

$$P\{X = (x_1, x_2, \ldots) \mid Z = n\} = \frac{n! \, \theta^{\sum x_r}}{\theta^{\uparrow n} \prod_j j^{x_j} x_j!}, \tag{3}$$

where $\theta^{\uparrow n} := \theta(\theta + 1) \cdots (\theta + n - 1)$ is the ascending factorial. Equation (3) is the Ewens sampling formula with parameter $\theta > 0$ on the space of partitions of the integer n (Ewens, 1972; Arratia, Barbour and Tavaré 1992).

2.1.2. *Composition of formal power series* The statistical properties in Theorem 2.1 are related to the algebraic properties of the space of formal power series generated by the monomials $t, t^2, t^3, \ldots$. Consider functions f and g with $f(0) = g(0) = 0$ and nth Taylor coefficients f_n and g_n, respectively, so that $f(t) = \sum_{n \geq 1} f_n t^n / n!$. Then the nth Taylor coefficient of the composition $(fg)(t) := f(g(t))$ is

$$(fg)_n = \sum_{\pi \in \mathcal{P}_n} f_{\#\pi} \prod_{b \in \pi} g_{\#b}, \tag{4}$$

where $\pi \in \mathcal{P}_n$ is a set partition of $[n] := \{1, \ldots, n\}$ and $\#\pi$ equals the number of blocks of π.

The exponential formula (4) is related to the lifting of (3) from integer partitions to set partitions by sampling uniformly among set partitions whose block sizes correspond to the parts of a random integer partition drawn from (3). Taking $f_n := \theta^n$ and $g_n := (n-1)!$, so that $f(t) = e^{\theta t} - 1$ and $g(t) = -\log(1-t)$, the coefficient in (4) gives the normalizing constant $\theta^{\uparrow n}$ in the Ewens distribution on set partitions,

$$P_n^\theta(\pi) := \frac{\theta^{\#\pi}}{\theta^{\uparrow n}} \prod_{b \in \pi} (\#b - 1)!, \quad \pi \in \mathcal{P}_n. \tag{5}$$

More generally, the two-parameter Ewens–Pitman(α, θ) distribution on $\mathcal{P}_n$ is defined by

$$P_n^{\alpha,\theta}(\pi) := \frac{(\theta/\alpha)^{\uparrow \#\pi}}{\theta^{\uparrow n}} \prod_{b \in \pi} -(-\alpha)^{\uparrow \#b}, \quad \pi \in \mathcal{P}_n, \tag{6}$$

where (α, θ) satisfies either

- $\alpha < 0$ and $\theta = -\alpha\kappa$ for $\kappa = 1, 2, \ldots$ or

- $0 \leq \alpha \leq 1$ and $\theta > -\alpha$.

By putting $f_n := (\theta/\alpha)^{\uparrow n}$ and $g_n := -(-\alpha)^{\uparrow n}$, the composition (4) again gives the normalizing constant $\theta^{\uparrow n}$ in (6). The Ewens distribution (5) with parameter θ coincides with the Ewens–Pitman distribution with parameter $(0, \theta)$.

The connection between (4) and (6) is curious in a few respects. On the one hand, (4) is an elementary algebraic property of exponential generating functions. On the other hand, the Ewens–Pitman distribution arises in various contexts and, in fact, characterizes the class of canonical Gibbs ensembles on partitions of $[n]$ (Pitman, 2006).

2.1.3. *Relation to the α-permanent* The connection between the Ewens distribution and coefficients of formal power series is also related to the algebraic properties of the α-permanent, which arises in theoretical computer science (Valiant, 1979 ($\alpha = 1$)), statistics (Vere-Jones, 1997, 1998; Rubak, Møller and McCullagh, 2010), and stochastic processes (Crane, 2013; Macchi, 1975; McCullagh and Møller, 2006). For $\alpha \in \mathbb{R}$ and an $\mathbb{R}$-valued matrix $M := (M_{ij})_{1 \leq i,j \leq n}$, the α-*permanent* of M is defined by

$$\mathrm{per}_\alpha(M) := \sum_{\sigma \in \mathcal{S}_n} \alpha^{\#\sigma} \prod_{j=1}^n M_{j,\sigma(j)}, \tag{7}$$

where the sum is over the symmetric group $\mathcal{S}_n$ of permutations of $[n]$ and $\#\sigma$ denotes the number of cycles of $\sigma \in \mathcal{S}_n$. To show the connection to (4), we appeal to the identity

$$\mathrm{per}_{\alpha\beta}(M) = \sum_{\pi \in \mathcal{P}_n} \beta^{\downarrow \#\pi} \prod_{b \in \pi} \mathrm{per}_\alpha(M[b]),$$

where $\beta^{\downarrow j} := \beta(\beta - 1) \cdots (\beta - j + 1)$ and $M[b] := (M_{ij})_{i,j \in b}$ is the submatrix of M indexed by $b \subseteq [n]$ (Crane, 2013). When $M := J_n$ is the $n \times n$ matrix of all ones, this identity simplifies to

$$(\alpha\beta)^{\uparrow n} = \sum_{\pi \in \mathcal{P}_n} \beta^{\downarrow \#\pi} \prod_{b \in \pi} \alpha^{\uparrow \#b},$$

from which the two-parameter distribution (6) is apparent by the transformation $\beta \mapsto \theta/\alpha$.

2.2. Janossy measures

The law of any finite random point configuration $\mathbf{X} \subseteq \mathcal{X}$ is determined by the *Janossy measure* $J = \sum_{n \geq 1} J^{(n)}$, where each $J^{(n)}$ is a non-negative symmetric measure on $\mathcal{X}^n$ such that $P(\#\mathbf{X} = n) = J^{(n)}(\mathcal{X}^n)$ and $\sum_{n \geq 1} J^{(n)}(\mathcal{X}^n) = 1$. Each orbit of the symmetric group acting on $\mathcal{X}^n$ corresponds to an n-point configuration in $\mathcal{X}$, so we can identify J with a probability measure on the space of finite point configurations. Equivalently, each *symmetric* subset $A \subseteq \mathcal{X}^n$ determines a set of n-point configurations and, in some cases, it is convenient to define the *configuration measures* $\{J_n\}_{n \geq 1}$ by

$$J_n(A) = \sum_{\sigma \in \mathcal{S}_n} J^{(n)}(\sigma A) = n! J^{(n)}(A), \tag{8}$$

where $\sigma A = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : x \in A\}$ denotes the image of A by relabeling coordinates according to σ.

The configuration measures of a Poisson process with finite mean measure ζ are

$$J_n(dx_1 \cdots dx_n) = e^{-\zeta_.} \prod_{j=1}^{n} \zeta(dx_j), \quad n = 0, 1, \ldots, \tag{9}$$

where $\zeta_. = \zeta(\mathcal{X})$. Thus, $J^{(n)}(\mathcal{X}^n) = e^{-\zeta_.} \zeta_.^n / n!$ is the probability that $\#\mathbf{X} = n$.

Remark 1. (*Notation.*) Consistent with our notation for Janossy measures, we adopt the following convention for measures on $\mathrm{seq}(\mathcal{X})$. Throughout the paper, ζ denotes the mean measure for a Poisson point process on $\mathrm{seq}(\mathcal{X})$, whose restriction $\zeta^{(n)}$ to $\mathcal{X}^n$ need not be symmetric. The symmetrized version is

$$\zeta_n(A) := \sum_{\sigma \in \mathcal{S}_n} \zeta^{(n)}(\sigma A), \quad A \subseteq \mathcal{X}^n.$$

Each Janossy component measure $J^{(n)}$ is automatically symmetric on $\mathcal{X}^n$, but the configuration measures $\{J_n\}_{n \geq 1}$ in (8) are more natural for Poisson superposition processes. If $A \subseteq \mathcal{X}^n$ is symmetric, i.e., $A = \sigma A$ for all $\sigma \in \mathcal{S}_n$, then $J^{(n)}(A) = J_n(A)/n!$.

3. Finite sequences

Superposition sends each point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n$ to the n-point configuration $\mathrm{Spp}(\mathbf{x}) = \{x_1, \ldots, x_n\} \subset \mathcal{X}$, that is, the multiset whose elements are the components of $\mathbf{x}$ counted with multiplicity. More generally, the superposition of the k-point configuration $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathrm{seq}(\mathcal{X})$ is the additive multiset operation defined as the union of the superpositions, counted with multiplicity.

Definition 3.1. (*Poisson superposition process, Version 1.*) If $\mathbf{X} \sim \mathrm{PP}(\zeta)$ is a Poisson point process in $\mathrm{seq}(\mathcal{X})$, then $\mathbf{Z} = \mathrm{Spp}(\mathbf{X})$ is a Poisson superposition process in $\mathcal{X}$. In words, $\mathbf{X} = (\mathbf{X}_0, \mathbf{X}_1, \ldots)$ is a sequence of independent Poisson processes $\mathbf{X}_r \sim \mathrm{PP}(\zeta^{(r)})$ on $\mathcal{X}^r$ and $\mathbf{Z}$ records the components of every point in $\mathbf{X}$ with multiplicity.

For the remainder of the paper, we equip $\mathrm{seq}(\mathcal{X})$ with the product σ-field $\bigotimes_{n \geq 0} \mathcal{B}_{\mathcal{X}}^n$, where $\mathcal{B}_{\mathcal{X}}^n := \mathcal{B}_{\mathcal{X}} \otimes \cdots \otimes \mathcal{B}_{\mathcal{X}}$ is the n-fold product of Borel σ-fields on $\mathcal{X}$.

We formalize Definition 3.1 through the following discussion of embedded subsequences.

3.1. Embedded subsequences

For $n \geq 1$, let $\mathbf{x} := (x_1, \ldots, x_n) \in \mathcal{X}^n$ be a length-n sequence with components in $\mathcal{X}$. Each ordered subset of integers $i := (i_1, \ldots, i_m)$ for which $1 \leq i_1 < \cdots < i_m \leq n$ determines an embedded length-m subsequence

$$\mathbf{x}[i] := (x_{i_1}, \ldots, x_{i_m}).$$

There may be multiple ways to embed a length-m sequence into a length-n sequence, $m \leq n$, and so we define the *embedding number*

$$L(\mathbf{x}', \mathbf{x}) := \#\{i = (i_1, \ldots, i_m) : 1 \leq i_1 < \cdots < i_m \leq n, \mathbf{x}[i] = \mathbf{x}'\} \qquad (10)$$

as the number of ways to embed $\mathbf{x}'$ in $\mathbf{x}$. From (10), we define the *embedding measure*

$$L(A, \mathbf{x}) := \sum_{\mathbf{x}' \in A} L(\mathbf{x}', \mathbf{x}), \quad A \subseteq \mathrm{seq}(\mathcal{X}),$$

which counts the number of embedded subsequences of $\mathbf{x}$ contained in A, with the convention $L(\mathcal{X}^0, \mathbf{x}) = 1$ for all $\mathbf{x} \in \mathrm{seq}(\mathcal{X})$. For fixed $\mathbf{x} \in \mathrm{seq}(\mathcal{X})$, $L(\cdot, \mathbf{x})$ is a finite measure on $\mathrm{seq}(\mathcal{X})$ with total mass

$$L(\mathrm{seq}(\mathcal{X}), \mathbf{x}) := \sum_{m \geq 0} \binom{\#\mathbf{x}}{m} = 2^{\#\mathbf{x}},$$

where $\#\mathbf{x}$ denotes the length of $\mathbf{x}$.

For $\mathbf{x} \in \mathrm{seq}(\mathcal{X})$, we define the *first-order superposition* $\mathrm{Spp}(\mathbf{x})$ of $\mathbf{x}$ to be the multiset whose elements are all length-1 subsequences (with multiplicity) embedded in $\mathbf{x}$. Formally, $\mathrm{Spp}(\mathbf{x})$ is determined by the restriction of $L(\cdot, \mathbf{x})$ to Borel subsets of $\mathcal{X}$, where $L(y, \mathbf{x})$ counts the number of occurrences of $y \in \mathcal{X}$ in $\mathbf{x}$. For $m \geq 1$, we write $L^{(m)}(\cdot, \mathbf{x})$ to denote the restriction of the embedding measure $L(\cdot, \mathbf{x})$ to Borel subsets of $\mathcal{X}^m$.

Definition 3.2. (*Poisson superposition process, Version 2.*) Let $\mathbf{X}$ be a Poisson process with mean measure ζ on $\mathrm{seq}(\mathcal{X})$. The *Poisson superposition process* $\mathbf{Z} \sim \mathrm{PSP}(\zeta)$ is the random measure obtained by first-order superposition of $\mathbf{X}$,

$$\mathbf{Z}(A) := L^{(1)}(A, \mathbf{X}),$$

for $A \subset \mathcal{X}$. The mean measure or first-order intensity is $\kappa(A) = E(\mathbf{Z}(A))$.

The following example shows that the preceding definition by embedding coincides with the description from the beginning of Section 3.

Example 2. Let $\mathcal{X} = \mathbb{Z}$ and suppose

$$\mathbf{X}_1 = \{(1), (3)\}, \quad \mathbf{X}_2 = \{(2, 0), (1, 2), (3, 4)\}, \quad \mathbf{X}_3 = \{(1, 2, 1)\},$$

and $\mathbf{X}_n = \emptyset$, $n \geq 4$, are realizations of independent Poisson processes on $\mathcal{X}^1, \mathcal{X}^2, \ldots$, respectively. Then the superposition on $\mathcal{X}$ is the multiset $\mathbf{Z} = \{0^1, 1^4, 2^3, 3^2, 4^1\}$, i.e., $\mathbf{Z}(0) = 1$, $\mathbf{Z}(1) = 4$, $\mathbf{Z}(2) = 3$, $\mathbf{Z}(3) = 2$, $\mathbf{Z}(4) = 1$, and $\mathbf{Z}(x) = 0$ otherwise.

By our phrasing in terms of the embedding measure, we can exploit properties of measures on finite sequences to study Poisson superposition processes. The following sections, though technical, are necessary to prove our main theorems.

3.2. Measures on finite sequences

For any measure ζ on $\mathrm{seq}(\mathcal{X})$, we define the measure $L\zeta$ on $\mathrm{seq}(\mathcal{X})$ as the mixture of $L(A, \cdot)$ with respect to ζ,

$$(L\zeta)(A) := \int_{\mathrm{seq}(\mathcal{X})} L(A, \mathbf{x}) \, d\zeta(\mathbf{x}), \quad A \subseteq \mathrm{seq}(\mathcal{X}). \tag{11}$$

As $\mathrm{seq}(\mathcal{X})$ is a disjoint union of $\mathcal{X}^0, \mathcal{X}^1, \mathcal{X}^2, \ldots$, each measure ζ on $\mathrm{seq}(\mathcal{X})$ decomposes as the sum of its disjoint component measures $\zeta = \sum_{r \geq 0} \zeta^{(r)}$. Moreover, when restricted to the subset of length-m sequences, we define

$$L\zeta^{(m)}(A) := \int_{\mathrm{seq}(\mathcal{X})} L(A, \mathbf{x}) d\zeta^{(m)}(\mathbf{x}) = \int_{\mathcal{X}^m} L(A, \mathbf{x}) d\zeta(\mathbf{x}).$$

Therefore, we can also write $L\zeta = \sum_{r \geq 0} L\zeta^{(r)}$, but the components $\{L\zeta^{(r)}\}_{r \geq 1}$ need not be mutually singular.

3.2.1. *Superposition and disaggregation measures* A measurable subset $A \subseteq \mathcal{X}$ acts linearly on $\zeta = \sum_{r \geq 0} \zeta^{(r)}$ by transforming each component measure $\zeta^{(r)} \mapsto A\zeta^{(r)}$ in the obvious way: for each $n = 1, 2, \ldots$,

$$(A\zeta^{(n)})(B_1 \cdots B_n) = \sum_{r=0}^{n} \zeta^{(n+1)}(B_1 \cdots B_r \cdot A \cdot B_{r+1} \cdots B_n), \tag{12}$$

where $B_1, \ldots, B_n$ are measurable subsets of $\mathcal{X}$, and $B_1 \cdots B_n$ denotes the Cartesian product. For example, $n = 0, 1$ correspond to $(A\zeta^{(0)})(\mathcal{X}^0) = \zeta^{(1)}(A)$ and $(A\zeta^{(1)})(B) =$

$\zeta^{(2)}(A \times B) + \zeta^{(2)}(B \times A)$, respectively. Iterating the above action for $A_1, \ldots, A_k \subseteq \mathcal{X}$ gives the k-fold action

$$(A_1 \cdots A_k \zeta^{(n)})(B_1 \cdots B_n) = \sum_{\sigma \in \mathcal{S}_k} \sum_{1 \leq i_1 < \cdots < i_k \leq n+k} \zeta^{(n+k)}(C_1 \cdots C_{n+k}), \qquad (13)$$

where $C_1, \ldots, C_{n+k}$ are subsets of $\mathcal{X}$ such that $C_{i_r} = A_{\sigma(r)}$ and $C_{\bar{i}_r} = B_r$, where $1 \leq \bar{i}_1 < \cdots < \bar{i}_n \leq n+k$ is the complement of $1 \leq i_1 < \cdots < i_k \leq n+k$. It follows that the action (13) on ζ is commutative, $A_{\sigma(1)} \cdots A_{\sigma(k)} \zeta^{(n)} = A_1 \cdots A_k \zeta^{(n)}$ for all permutations $\sigma : [k] \to [k]$, but the resulting measure on $\mathcal{X}^{n+k}$ need not be symmetric.

Choosing $A_1 = \cdots = A_k = \mathcal{X}$ in (13), yields

$$(\mathcal{X}^k \zeta^{(n)})(B^{[1:n]}) = k! \sum_{1 \leq i_1 < \cdots < i_n \leq n+k} \zeta^{(n+k)}(C^{[1:n+k]}),$$

where $B^{[1:n]} := B_1 \times \cdots \times B_n$ is the Cartesian product, $C_{i_j} = B_j$, and $C_{\bar{i}_j} = \mathcal{X}$. For a pair of measurable sets $A_1, A_2 \subset \mathcal{X}$, we extend the action in (12) to formal linear combinations

$$(A_1 + A_2)\zeta := A_1 \zeta + A_2 \zeta,$$

from which we define the *superposition measure*

$$\exp(\mathcal{X})\zeta := \sum_{k \geq 0} \frac{\mathcal{X}^k \zeta}{k!} = L\zeta. \qquad (14)$$

In the reverse direction, we define the *disaggregation measure* of $\kappa := L\zeta$ on $\mathrm{seq}(\mathcal{X})$ by

$$\exp(-\mathcal{X})\kappa := \sum_{k \geq 0} \frac{(-1)^k (\mathcal{X}^k \kappa)}{k!} = \zeta. \qquad (15)$$

The transformation $\zeta \mapsto \kappa$ is invertible. But, unlike the relation between moments and cumulants, the first n components of ζ do not determine the first n components of κ and vice versa. In particular, the first-order superposition measure $\kappa^{(1)} = L\zeta^{(1)}$ does not determine $\zeta^{(1)}$.

3.3. Subset restriction

Janossy measures only determine the law of point processes that are almost surely finite. For infinite, but locally finite, point processes, the law is determined by restriction to certain test sets $\mathcal{V} \subset \mathcal{X}$.

On the one hand, $\mathcal{V}^n \subset \mathcal{X}^n$ implies that each $\mathcal{V}$-sequence is also an $\mathcal{X}$-sequence, so the natural embedding $\mathrm{seq}(\mathcal{V}) \hookrightarrow \mathrm{seq}(\mathcal{X})$ preserves sequence length and component order. On the other hand, subset restriction also determines a natural deletion map $T \colon \mathrm{seq}(\mathcal{X}) \to \mathrm{seq}(\mathcal{V})$ in the reverse direction, by which each component of the $\mathcal{X}$-sequence $\mathbf{x} = (x_1, \ldots, x_n)$ that does not belong to $\mathcal{V}$ is deleted, leaving the remaining components $T\mathbf{x} = (x_{i_1}, \ldots, x_{i_k})$ in their original relative order $i_1 < \cdots < i_k$. In other words, $T\mathbf{x}$ is the maximal embedded subsequence belonging to $\mathrm{seq}(\mathcal{V})$.

If ζ is a measure on $\mathrm{seq}(\mathcal{X})$, then the induced measure $T\zeta$ on $\mathrm{seq}(\mathcal{V})$ is defined as the usual image measure

$$(T\zeta)(A) = \zeta(T^{-1}A), \quad A \subseteq \mathrm{seq}(\mathcal{V}),$$

where $T^{-1}A$ is the inverse image of A in $\mathrm{seq}(\mathcal{X})$. The inverse image $T^{-1}(\{\mathbf{v}\})$ of $\mathbf{v} \in \mathcal{V}^n$ is the set of $\mathcal{X}$-sequences that include $\mathbf{v}$ as an embedded subsequence and have no other components belonging to $\mathcal{V}$, i.e., sets of the form $\{\mathbf{v}\} \times (\mathcal{X} \setminus \mathcal{V})^k$, $k \geq 0$, with any of $\binom{n+k}{k}$ embeddings. Evidently, $T\zeta = \exp(\mathcal{X} \setminus \mathcal{V})\zeta = \exp(-\mathcal{V})\exp(\mathcal{X})\zeta$ and $\exp(\mathcal{V})(T\zeta) = \kappa$.

In words, if ζ is a measure on $\mathrm{seq}(\mathcal{X})$ with superposition $\kappa = \exp(\mathcal{X})\zeta$, then the induced measure on $\mathcal{V}$-restricted sequences is obtained by subset restriction of κ to $\mathrm{seq}(\mathcal{V}) \subset \mathrm{seq}(\mathcal{X})$, followed by disaggregation $(T\zeta)(A) = \exp(-\mathcal{V})\kappa(A)$. The state of affairs is best described by a commutative diagram in which the projection T and the pointwise restriction R are maps on measures associated with the subset $\mathcal{V} \subset \mathcal{X}$.

$$
\begin{array}{ccccc}
\mathcal{V} & \mathrm{meas}(\mathrm{seq}(\mathcal{V})) & \xrightarrow{e^{\mathcal{V}}} & \mathrm{meas}(\mathrm{seq}(\mathcal{V})) \\
{\scriptstyle i}\downarrow & \uparrow{\scriptstyle T} & & \uparrow{\scriptstyle R} \\
\mathcal{X} & \mathrm{meas}(\mathrm{seq}(\mathcal{X})) & \xrightarrow{e^{\mathcal{X}}} & \mathrm{meas}(\mathrm{seq}(\mathcal{X}))
\end{array}
$$

Commutativity asserts that, for each measure ζ on $\mathrm{seq}(\mathcal{X})$ and $\mathcal{V} \subset \mathcal{X}$, the superposition of the projection $e^{\mathcal{V}}T\zeta$ is a measure on $\mathrm{seq}(\mathcal{V})$ equal to the restriction of the superposition $Re^{\mathcal{X}}\zeta$.

4. Poisson superposition processes

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent Poisson processes for which $\mathbf{X}_r \subseteq \mathcal{X}^r$ has finite mean measure $\zeta^{(r)}$, for each $r = 1, 2, \ldots$. We denote the union of $\mathbf{X}_1, \mathbf{X}_2, \ldots$ by

$\mathbf{X} := \bigcup_{r \geq 1} \mathbf{X}_r \subseteq \mathrm{seq}(\mathcal{X})$ and their *superposition* in $\mathcal{X}$ by $\mathbf{Z}(\cdot) := L^{(1)}(\cdot, \mathbf{X})$, as in Definition 3.2. We further assume $\zeta_{\bullet} := \sum_r \zeta^{(r)}(\mathcal{X}^r) < \infty$; whence $n_r := \#\mathbf{X}_r < \infty$, for each $r = 1, 2, \ldots$, and $n_{\bullet} := \sum_r n_r < \infty$ with probability one. In this case, we let $\{J_r^{(n)}\}_{n=1,2,\ldots}$ denote the component Janossy measures of $\{\mathbf{X}_r\}_{r \geq 1}$ and $\{J_{r,n}\}_{n=1,2,\ldots}$ be the associated configuration measures, as in (9). It follows that

$$J_{r,n}(d\mathbf{x}_1, \ldots, d\mathbf{x}_n) = e^{-\zeta_{\bullet}^{(r)}} \prod_{j=1}^{n} \zeta^{(r)}(d\mathbf{x}_j), \quad \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{X}^r,$$

from which the component Janossy measures are given by

$$J_r^{(n)}(d\mathbf{x}_1, \ldots, d\mathbf{x}_n) = (n!)^{-1} e^{-\zeta_{\bullet}^{(r)}} \prod_{j=1}^{n} \zeta^{(r)}(d\mathbf{x}_j), \quad r \geq 1.$$

Theorem 4.1. *Let $\mathbf{Z}$ be the superposition of independent Poisson processes $\mathbf{X}_1, \mathbf{X}_2, \ldots$, where each $\mathbf{X}_r$ has mean measure $\zeta^{(r)} < \infty$ on $\mathcal{X}^r$ such that $\zeta_{\bullet} := \sum_{r \geq 1} \zeta^{(r)}(\mathcal{X}^r) < \infty$. Then the Janossy measures $\{J^{(n)}\}_{n \geq 0}$ of $\mathbf{Z}$ are given by $J^{(n)} = J_n/n!$, where*

$$J_n(d\mathbf{x}) = e^{-\zeta_{\bullet}} \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \zeta_{\#b}(d\mathbf{x}[b]), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n, \, n \geq 0. \qquad (16)$$

Proof. Similar to (2), we obtain the joint probability generating functional for the Janossy measures of $\mathbf{X}_{\bullet} = \bigcup_{r \geq 1} \mathbf{X}_r$ and $\mathbf{Z}$ by

$$
\begin{aligned}
J(s,t) &= \prod_{r \geq 1} \sum_{j \geq 0} J_r^{(j)} s^j t^{rj} \\
&= \prod_{r \geq 1} \sum_{j \geq 0} s^j t^{rj}/j! \times e^{-\zeta_{\bullet}^{(r)}} \prod_{i=1}^{j} \zeta^{(r)} \\
&= e^{-\zeta_{\bullet}} \prod_{r \geq 1} \sum_{j \geq 0} (s t^r \zeta^{(r)})^j/j! \\
&= \exp\left\{-\zeta_{\bullet} + s\zeta(t)\right\},
\end{aligned}
$$

where $\zeta(t) = \sum_{r \geq 1} t^r \zeta^{(r)}$ is the generating functional for the mean measures $\{\zeta^{(r)}\}_{r \geq 1}$ and $\zeta_{\bullet}^{(r)} = \zeta^{(r)}(\mathcal{X}^r)$, for each $r = 1, 2, \ldots$. The generating functional $J(t)$ of the Janossy measures for $\mathbf{Z}$ is thus

$$J(t) = \exp\left(-\zeta_{\bullet} + \sum_{n \geq 1} t^n \zeta^{(n)}\right),$$

from which (16) is apparent by differentiation.

Theorem 4.2. *Let* $\mathbf{Z}$ *be a finite point process in* $\mathcal{X}$ *with Janossy measure* $\{J_n/n!\}_{n\geq 0}$ *and* $J_0 > 0$, *and let*

$$\zeta_n(d\mathbf{x}) = \sum_{\pi \in \mathcal{P}_n} (-1)^{\#\pi - 1} (\#\pi - 1)! \, J_0^{-\#\pi} \prod_{b \in \pi} J_{\#b}(d\mathbf{x}[b]), \tag{17}$$

for $n \geq 1$ *and* $\mathbf{x} \in \mathcal{X}^n$. *Then* $\mathbf{Z}$ *is a Poisson superposition process if and only if* ζ *is non-negative.*

Proof. The relationship (17) is apparent from the relationship between moments and cumulants, see McCullagh (1987), Section 2.3, or Streitberg (1990). Therefore, given a finite total measure $\zeta. < \infty$ and a family of Janossy measures for a Poisson superposition process, the mean measure of the Poisson process on $\mathrm{seq}(\mathcal{X})$ can be recovered uniquely up to permutation of component measures.

Corollary 4.1. *Let* $\{\mathbf{Z}_r\}_{r\geq 1}$ *be independent Poisson superposition processes, where* $\mathbf{Z}_r \sim \mathrm{PSP}(\zeta^r)$, *for each* $r = 1, 2, \ldots$. *The convolution of* $\{\mathbf{Z}_r\}_{r\geq 1}$, *defined by*

$$\mathbf{Z}.(\cdot) := \sum_{r \geq 1} \mathbf{Z}_r(\cdot),$$

is a Poisson superposition process with generating measure $\zeta^. = \sum_r \zeta^r$. *Moreover, every Poisson superposition process is infinitely divisible.*

Proof. The proof follows from the convolution property of Poisson processes and is analogous to the proof of the corresponding property for scalar Poisson superpositions. Infinite divisibility follows immediately from the convolution property.

4.1. Subset restriction

Theorem 4.3. *Let* $\mathbf{Z} \sim \mathrm{PSP}(\zeta)$ *be a Poisson superposition process with parameter* ζ *on* $\mathrm{seq}(\mathcal{X})$ *and denote by* $\mathbf{Z}[\mathcal{V}]$ *its restriction to* $\mathcal{V} \subseteq \mathcal{X}$. *Then* $\mathbf{Z}[\mathcal{V}] \sim \mathrm{PSP}(T\zeta)$ *is a Poisson superposition process on* $\mathcal{V}$ *with parameter* $\zeta^{\mathcal{V}} := T\zeta$ *and configuration measures*

$$J_n^{\mathcal{V}}(d\mathbf{x}) = e^{-\zeta_.^{\mathcal{V}}} \sum_{\pi \in \mathcal{P}_n} \prod_{b \in \pi} \zeta_{\#b}^{\mathcal{V}}(d\mathbf{x}[b]), \quad \mathbf{x} \in \mathcal{X}^n, \tag{18}$$

where $T : \mathrm{seq}(\mathcal{X}) \to \mathrm{seq}(\mathcal{V})$ *is the projection defined in Section 3.3.*

Proof. To observe (18), we first project $\mathbf{X}$ to $\mathrm{seq}(\mathcal{V})$ and then apply superposition. The projection into $\mathrm{seq}(\mathcal{V})$ remains a Poisson process since non-overlapping subsets of

$\mathcal{V}$ correspond to non-overlapping Borel sets of $\mathrm{seq}(\mathcal{X})$ under projection; hence, counts in non-overlapping subsets of $\mathrm{seq}(\mathcal{V})$ are independent Poisson random variables with updated intensity $\zeta^{\mathcal{V}} := T\zeta$. This completes the proof.

5. Special cases

To conclude, we highlight some well-known point processes that are also Poisson superposition processes.

5.1. Self-similar measures

Let $\mathrm{seq}^+(\mathcal{X})$ be the set of non-null finite sequences, $\zeta := \sum_{n\geq 0} \zeta^{(n)}$ be a measure on $\mathrm{seq}(\mathcal{X})$, and $\mathcal{V}' \subset \mathcal{X}$ be any subset such that

$$\zeta(\mathrm{seq}^+(\mathcal{V}')) = \sum_{k=1}^{\infty} \zeta^{(k)}(\mathcal{V}'^k) < \infty.$$

In other words, the restriction $\zeta[\mathcal{V}']$ of ζ to $\mathrm{seq}(\mathcal{V}') \subset \mathrm{seq}(\mathcal{X})$ is finite on non-null sequences. We call ζ *self-similar* if, for each $\mathcal{V} \subset \mathcal{V}' \subset \mathcal{X}$, the projection $T(\zeta[\mathcal{V}'])$ of the $\mathcal{V}'$-restricted measure is proportional to the $\mathcal{V}$-restriction $\zeta[\mathcal{V}]$. Here, $\zeta[\mathcal{V}]$ is the restriction of ζ to $\mathrm{seq}(\mathcal{V}) \subset \mathrm{seq}(\mathcal{X})$, while $T\colon \mathrm{seq}(\mathcal{X}) \to \mathrm{seq}(\mathcal{V})$ is the projection described previously.

For a simple example, consider the weighted product measures $\zeta^{(k)} = \zeta_1^{\otimes k}/k!$ on $\mathcal{X}^k$ so that $\zeta(\mathrm{seq}(\mathcal{V}')) = \exp(\zeta_1(\mathcal{V}')) < \infty$ holds whenever $\zeta_1(\mathcal{V}') < \infty$. For $\mathcal{V} \subset \mathcal{V}' \subset \mathcal{X}$ such that $\zeta_1(\mathcal{V}') < \infty$, restriction of ζ to $\mathrm{seq}(\mathcal{V}')$ followed by projection onto $\mathrm{seq}(\mathcal{V})$ gives

$$(T\zeta[\mathcal{V}'])(A) = \zeta(A)\exp(\zeta_1(\mathcal{V}' \setminus \mathcal{V}))$$

for $A \subset \mathrm{seq}(\mathcal{V})$. Thus, the projected measure exceeds the restricted measure by the constant factor $\exp(\zeta_1(\mathcal{V}' \setminus \mathcal{V})) < \infty$.

For a general self-similar measure ζ on $\mathrm{seq}(\mathcal{X})$, we denote the projection $\mathrm{seq}(\mathcal{V}') \to \mathrm{seq}(\mathcal{V})$, $\mathcal{V} \subseteq \mathcal{V}' \subseteq \mathcal{X}$, by $T_{\mathcal{V},\mathcal{V}'}$. Then $T_{\mathcal{V},\mathcal{V}'}\zeta[\mathcal{V}'] \propto \zeta[\mathcal{V}]$ and, in particular, $T_{\mathcal{V}}\zeta \propto \zeta[\mathcal{V}]$, where $T_{\mathcal{V}} := T_{\mathcal{V},\mathcal{X}}$. The Janossy measures for the Poisson superposition process restricted to $\mathcal{V}$ are determined by (18), where $\zeta^{\mathcal{V}} := T_{\mathcal{V}}\zeta$. In the self-similar case, these Janossy measures satisfy

$$J_n^{\mathcal{V}}(d\mathbf{x}) = e^{-\zeta \cdot} \sum_{\pi \in \mathcal{P}_n} \alpha_{\mathcal{V}}^{\#\pi} \prod_{b \in \pi} \zeta_{\#b}(d\mathbf{x}[b]), \quad \mathbf{x} \in \mathcal{X}^n, \tag{19}$$

where $\alpha_\mathcal{V} = T_\mathcal{V}\zeta(A)/\zeta(A) \geq 1$ is the proportionality constant. Some special cases arise in this setting.

5.1.1. *Product measures* As in the above example, suppose $\zeta^{(k)} := \zeta_1^{\otimes k}/k!$ is a weighted product measure for each k. In this case, the symmetrized measures $\zeta_k = k!\zeta^{(k)} = \zeta_1^{\otimes k}$ are product measures on symmetric subsets $A \subseteq \mathcal{X}^k$ and (19) becomes

$$J_n^\mathcal{V}(d\mathbf{x}) = e^{-\zeta\cdot} \prod_{j=1}^n \zeta_1(dx_j) \sum_{\pi \in \mathcal{P}_n} \alpha_\mathcal{V}^{\#\pi} = e^{-\zeta\cdot} B_n(\alpha_\mathcal{V}) \prod_{j=1}^n \zeta_1(dx_j),$$

where $B_n(\alpha) := \sum_k \mathcal{P}(n,k)\alpha^k$ is the generating function for the number of partitions of $[n]$ with k blocks (see Wilf, 2006, p. 18). Taking $\mathcal{V} = \mathcal{X}$, we get

$$J_n(d\mathbf{x}) = e^{-\zeta\cdot}\mathcal{B}_n \prod_{j=1}^n \zeta_1(dx_j),$$

where $\mathcal{B}_n := \#\mathcal{P}_n$ is the nth Bell number.

5.2. Completely random superpositions

For each $n \geq 1$, let $\zeta^{(n)}$ be *diagonal*, i.e., supported on the diagonal of $\mathcal{X}^n$,

$$\mathrm{diag}(\mathcal{X}^n) := \{(x,\ldots,x) \in \mathcal{X}^n : x \in \mathcal{X}\}.$$

Since $\zeta^{(n)}$ vanishes off the diagonal, it is determined by a measure $\xi^{(n)}$ on $\mathcal{X}$ in the obvious way:

$$\zeta^{(n)}(A_1 \times \cdots \times A_n) = \xi^{(n)}(A_1 \cap \cdots \cap A_n), \quad A_i \subset \mathcal{X}. \tag{20}$$

The first-order superposition $\mathbf{Z}$ is a scalar Poisson superposition in the sense that $\mathbf{Z}(A) = \sum_{r \geq 1} r\mathbf{X}_r(A)$ is a linear combination of independent Poisson random variables with integer coefficients, as in Section 2.1. Moreover, the counts $\mathbf{Z}(A)$ and $\mathbf{Z}(A')$ are independent whenever A and A' are disjoint, so each Poisson superposition $\mathbf{Z} \sim \mathrm{PSP}(\zeta)$ is a *completely random measure*; see Chapter 8 of Kingman (1993).

From (20), the measures $\xi^{(n)}$ on $\mathcal{X}$ determine a measure Λ on the product space $\mathbb{N} \times \mathcal{X}$ by

$$\Lambda(\{n\} \times A) = \xi^{(n)}(A).$$

If $\mathbf{X}$ is a Poisson process in the product space with mean measure Λ, projection into $\mathcal{X}$ by summation of marks gives $\mathbf{Z}(A) = \sum_{n \geq 1} n\mathbf{X}(\{n\} \times A)$, which coincides with the

diagonal Poisson superposition $\mathbf{Z} \sim \mathrm{PSP}(\zeta)$. Since every measure in the product space $\mathbb{N} \times \mathcal{X}$ also determines a diagonal measure on $\mathrm{seq}(\mathcal{X})$, the argument in Kingman (1994, Section 8.2) immediately leads to the following theorem.

Theorem 5.1. *A completely random integer-valued process $\mathbf{Z}(\cdot)$ on $\mathcal{X}$ with no fixed atoms is necessarily a Poisson superposition with diagonal measures.*

5.2.1. *Compound Poisson process* Let $N^{(1)}, N^{(2)}, \ldots$ be independent Poisson processes on $\mathbb{R}$, where $N^{(n)}$ has intensity λp_n and $\sum_{n=1}^{\infty} p_n = 1$. Given $N^{(n)}$, we define the *increment* of $N^{(n)}$ by $I^{(n)}(t) := N^{(n)}(t) - N^{(n)}(t-)$ and $\mathbf{X}_n$ on $\mathrm{diag}(\mathbb{R}^n)$ by putting $(x, \ldots, x) \in \mathbf{X}_n$ if and only if $I^{(n)}(x) = 1$. Thus, $\mathbf{X}_n$ is obtained by embedding a Poisson process in $\mathbb{R}$ with intensity $\zeta^{(n)}(dx) = \lambda p_n dx$ into $\mathrm{diag}(\mathbb{R}^n)$. Defining $\mathbf{Z} := \mathrm{Spp}(\mathbf{X})$, where $\mathbf{X} := \{\mathbf{X}_r\}_{r \geq 1}$, gives a Poisson superposition process with diagonal measures $\{\zeta^{(r)}\}_{r \geq 1}$. On the other hand, $N(t) := \sum_{n=1}^{\infty} N^{(n)}(t)$ is a Poisson process with intensity λ and

$$W(t) := \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

determines the paths of a *compound Poisson process* with intensity λ and increment distribution $\{p_k\}_{k \geq 0}$, where $X_1, X_2, \ldots$ are independent, identically distributed with $P\{X_1 = k\} = p_k$.

5.2.2. *Negative binomial processes* Suppose the nth component measure is

$$\zeta^{(n)}(A_1 \times \cdots A_n) = \theta \zeta_1(A_1 \cap \cdots \cap A_n)/n,$$

for some finite measure ζ_1 on $\mathcal{X}$. Then, each $\zeta^{(n)}$ is diagonal and, from the above discussion, the number of points in non-overlapping subsets of $\mathcal{X}$ are independent and, for each $A \subseteq \mathcal{X}$, $\mathbf{Z}(A)$ is a negative binomial random variable with parameter $(\theta, \zeta_1(A))$. Note that this definition of negative binomial process differs from Barndorff-Nielsen and Yeo's (1969).

5.3. Boson processes

A *cyclic permutation* of $[n]$ is any permutation σ with a single cycle. Given a function $M : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, we define the sum of cyclic products of M at $\mathbf{x} := (x_1, \ldots, x_n)$ by

$$\mathrm{cyp}(M[\mathbf{x}]) := \sum_{\sigma : \#\sigma = 1} \prod_{j=1}^{n} M(x_j, x_{\sigma(j)}).$$

Suppose now that μ is a measure on $\mathcal{X}$, and that μ^n is the product measure on $\mathcal{X}^n$. Define $\mathrm{tr}(M) = \int M(x, x)\, d\mu(x)$,

$$M^2(x, x') = \int_{\mathcal{X}} M(x, t) M(t, x')\, d\mu(t),$$

and M^k by extension. Let $\zeta = \{\zeta^{(n)}\}_{n \geq 1}$ be a measure on $\mathrm{seq}(\mathcal{X})$ for which the density in $\mathcal{X}^n$ with respect to μ^n is

$$\zeta^{(n)}(d\mathbf{x}) = \zeta_n(d\mathbf{x})/n! = \alpha\, \mathrm{cyp}(M[\mathbf{x}])/n!$$

for some $\alpha > 0$. It follows that $\zeta^{(n)}(\mathcal{X}^n) = \alpha\, \mathrm{tr}(M^n)/n$, so the total measure is

$$\zeta_{\boldsymbol{\cdot}} = \zeta(\mathrm{seq}(\mathcal{X})) = \alpha \sum_{n \geq 1} \mathrm{tr}(M^n)/n = -\alpha \log \det(I - M)$$

provided that the spectral norm of M is strictly less than one. Under positivity conditions, the Janossy measure of the Poisson superposition $\mathbf{Z} \sim \mathrm{PSP}(\zeta)$ has a density

$$\frac{dJ_n}{d\mu^n}(\mathbf{x}) = e^{-\zeta_{\boldsymbol{\cdot}}} \sum_{\pi \in \mathcal{P}_n} \alpha^{\#\pi} \prod_{b \in \pi} \mathrm{cyp}\, M(\mathbf{x}[b]) = e^{-\zeta_{\boldsymbol{\cdot}}} \mathrm{per}_{\alpha}(M[\mathbf{x}]), \tag{21}$$

which is proportional to the α-permanent of M at $\mathbf{x}$. Provided that M is symmetric, positive definite, and 2α is a positive integer, this is the Janossy measure of a boson, or permanental, point process.

Acknowledgment

References

[1] R. Arratia, A. Barbour, and S. Tavaré. Poisson process approximations for the Ewens sampling formula. *Ann. Appl. Prob.,* 519–535, 1992.

[2] O. Barndorff-Nielsen and G. Yeo. Negative Binomial Processes. *J. Appl. Prob.*, 6:633–647, 1969.

[3] H. Crane. Permanental partition measures and Markovian Gibbs partitions. *Journal of Statistical Physics*, 153(4):698–726, 2013.

[4] H. Crane. Some algebraic identities for the α-permanent. *Linear Algebra and Its Applications*, 439(11):3445–3459, 2013.

[5] D. R. Cox and V. Isham. *Point Processes*. University Press, Cambridge, 1980.

[6] P. Donnelly and G. Grimmett. On the asymptotic distribution of large prime factors. *J. London Math. Soc.*, 47:395–404, 1993.

[7] W. J. Ewens. The sampling theory of selectively neutral alleles. *Theoret. Population Biology*, 3:87–112, 1972.

[8] B. Efron and R. Thisted. Estimating the number of unseen species: How many words did Shakespeare know? *Biometrika*, 63:435–447, 1976.

[9] B. Efron and R. Thisted. Did Shakespeare write a newly discovered poem? *Biometrika*, 74:445–455, 1987.

[10] R. Fisher, A. Corbet, and C. Williams. The relation between the number of species and the number of individuals in a random sample of an animal population. *J. Animal Ecology*, 12:42–58, 1943.

[11] J. Ben Hough, M. Krishnapur, Y. Peres and B. Virág. Determinantal Processes and Independence. *Probability Surveys*, 3:206–229, 2006.

[12] J. F. C. Kingman. Random partitions in population genetics. *Proc. Roy. Soc. London Ser. A*, 361(1704):1–20, 1978.

[13] J. F. C. Kingman. The representation of partition structures. *J. London Math. Soc. (2)*, 18(2):374–380, 1978.

[14] J. F. C. Kingman. *Poisson Processes*. Oxford University Press, New York, 1993.

[15] O. Macchi. The coincidence approach to stochastic point processes. *Adv. Appl. Prob.*, 7:83–122, 1975.

[16] P. McCullagh. *Tensor Methods in Statistics*. Chapman and Hall, London, 1987.

[17] P. McCullagh and J. Møller. The permanental process. *Adv. in Appl. Probab.*, 38(4):873–888, 2006.

[18] J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard.

[19] E. Rubak, J. Møller, and P. McCullagh. Statistical inference for a class of multivariate negative binomial distributions. Department of Mathematical Sciences, Aalborg University, 2010. 25 pages (Research Report Series; No. R-2010-10).

[20] B. Streitberg. Lancaster interactions revisited. *Ann. Stat.*, 18(4):1878–1885, 1990.

[21] L. G. Valiant. The complexity of computing the permanent. *Theoretical Computer Science*, 8:189–201, 1979.

[22] D. Vere-Jones. Alpha-permanents and their applications to mulivariate gamma, negative binomial and ordinary binomial distributions. *New Zealand J. Math.*, 26:125–149, 1997.

[23] D. Vere-Jones. A generalization of permanents and determinants. *Linear Alg. Appl.*, 111:119–124, 1998.

[24] H. S. Wilf. *Generatingfunctionology, Third Edition*. AK Peters/CRC Press, 2005.