Rank Tests at Jump Events*

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Abstract

We propose a test for the rank of a cross-section of processes at a set of jump events. The jump events are either specific known times or are random and associated with jumps of some process. The test is formed from discretely sampled data on a fixed time interval with asymptotically shrinking mesh. In the first step, we form nonparametric estimates of the jump events via thresholding techniques. We then compute the eigenvalues of the outer product of the cross-section of increments at the identified jump events. The test for rank r is based on the asymptotic behavior of the sum of the squared eigenvalues excluding the largest r. A simple resampling method is proposed for feasible testing.

The test is applied to financial data spanning the period 2007-2015 at the times of stock market jumps. We find support for a one-factor model of both industry portfolio and Dow 30 stock returns at market jump times. This stands in contrast with earlier evidence for higher-dimensional factor structure of stock returns during "normal" (non-jump) times. We identify the latent factor driving the stocks and portfolios as the size of the market jump.

Keywords: factor model, high-frequency data, jumps, rank test, semimartingale.

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1 Introduction

Stock prices are commonly modeled as a jump diffusion, where the jumps reflect occasional abrupt large price moves relative to the usual continuous dynamics. The jump moves are typically generated either by news related to a particular stock's growth prospects or by major economic events such as the release of key macroeconomic data. With the availability of high-frequency data, a very large empirical and theoretical literature has emerged on detecting and estimating jump sizes along with determining jump times. For essentially all estimation contexts, the time span is fixed and the statistical accuracy level is determined by the width of the sampling interval which shrinks asymptotically, see e.g., [1] and the many references therein.

The extensive empirical evidence on stock price jumps naturally leads to the question as to how a cross-section of stock returns behaves at various jump event times. Earlier empirical work, e.g., [21] and [5] (see also [9] for more recent evidence based on high-frequency data), has documented an increased dependence between assets during periods of large negative moves on the market. The increased dependence is also evident in each of the two panels of Figure 1. The left panel is based on the returns across 5-minute increments of the nine large well-diversified industry portfolios comprising the S&P 500 index over the period 2007–2015. The panel shows the eigenvalue decomposition of the jump covariance matrix of returns across only those intervals with market jumps (dots) along with a similar decomposition for returns across the remaining non-jump intervals (triangles). At market jump times, the first common factor accounts for about 90% of the total variation of the portfolios' returns while it accounts for a much smaller 70% of the variation at the non-jump times. Furthermore, the eigenvalues dampen rather quickly for the covariance from the increments at the market jump times compared with those of the covariance formed from non-jump returns. The right panel shows similar eigenvalue decompositions of the covariance matrices of 5-minute returns on the Dow 30 stocks across market jump times (same jump events as for the left panel) and across non-jump times. For the market jump times the first factor accounts for almost 80% of total Dow 30 stock return variation, while it accounts for only 40% in the non-jump case with the eigenvalues decaying much more slowly. The evidence in Figure 1 regarding the apparent increased dependence of asset returns at times of market turmoil motivates our formal statistical investigation of lower-dimensional factor structures at jump events such as the times at which the market jumps.

Below we develop formal statistical inference procedures for studying the factor structure of discretely-observed processes at any specific set of jump events. High-frequency data is well suited for this because it allows an analysis of the local behavior of the cross-section of processes around the events of interest. To develop formal inference theory for the factor structure at the jump events, we need to separate the measurement error due to the discrete sampling from the true (latent) reaction of the cross-section of processes to these events. Indeed, the evidence for the stronger asset dependence at the periods of market stress, discussed in the previous paragraph, might be merely due to the higher signal-to-noise at these events. That is, at these "extreme" events the measurement error is much smaller in relative terms than the signal (the asset response to the event) when compared with "normal" periods in the data. Our high-frequency asymptotics allows us to separate such an explanation for the observed empirical phenomenon from one in which the latent factor structure at the extreme events is truly low-dimensional.

The inference procedure developed in the paper can be described as follows. The user first identifies from the data the jump events of interest. There is considerable flexibility at this stage and the choice of jump events depends on the particular application. For

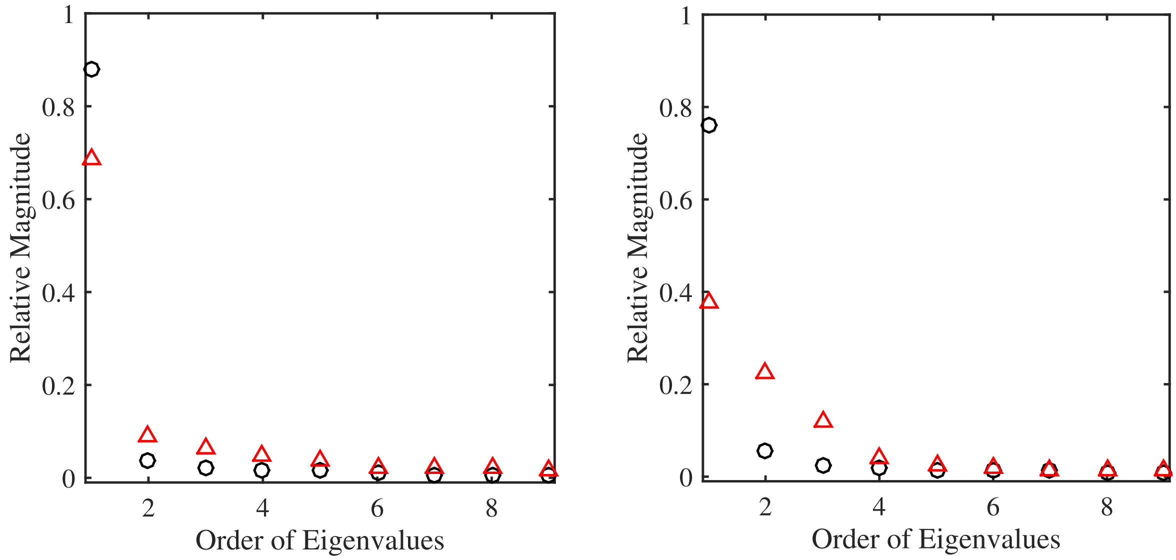

Figure 1: Eigenvalue decompositions of the covariance matrix of 5-minute returns across market jump intervals (dots) and non-jump intervals (triangles), 2007–2015. The left panel pertains to the nine industry portfolios comprising the S&P 500 index and the right panel to the Dow 30 stocks; only the nine largest eigenvalues are plotted. Jump times are identified using short-term (i.e., 1-minute) moves of the S&P 500 index futures price exceeding 7 local standard deviations in magnitude.

example, one possible choice is jump events associated with fixed (known) times such as the times of pre-scheduled releases of key macroeconomic data. Another possible choice of jump events is the set of jumps of an observable process (e.g., risk factor) such as the stock market index or an industry portfolio. One can also further condition on the sign and/or magnitude of such jumps. The jump events, in general, are not directly observable and can happen at random times. Therefore, they need to be inferred from the data via a thresholding technique (see e.g., [22]) that identifies the jump times in the sample as the times at which the high-frequency returns of the process of interest are large in magnitude relative to the local level of volatility.

Once the jump events are selected, our testing problem reduces to testing the rank of the matrix of high-frequency returns of the cross-section of processes at the identified jump events. For this we adapt to the fixed-span infill setting a test proposed by [30] in the classical long span setting concerning non-random population quantities; see also [12] and [10, 11] for alternative tests in such a setting. We first compute the eigenvalues of the jump covariance matrix of high-frequency returns. Provided the d-dimensional cross-section of processes is driven by r separate factors at the jump events of interest ($1 \leq r \leq d$), then asymptotically only the r largest eigenvalues should be non-zero. The test is then based on checking whether the remaining eigenvalues are statistically different from zero.

The limit distribution of our test depends on the asymptotic behavior of the matrix of high-frequency increments at the jump events. Apart from the jumps, these high-frequency increments include also other components of the processes which govern their behavior in the "normal" times outside of the jump events. These residual components shrink asymptotically as we sample more frequently around the jump times and they determine the asymptotic behavior of the test. Our rank test in the high-frequency setting departs from the one in [30], as the limit distribution of the matrix of high-frequency increments at the jump events is in general not (conditionally) Gaussian, which is the case in the classical setting of [30]. Another major difference from the classical setting is that the limit distribution of the rank test in our case depends on the observed path, and in particular on the levels of volatility of the processes before and after the jump events. To evaluate the critical values of the test, we develop a simple resampling technique which involves drawing uniformly from a local block of high-frequency increments from the left and right of the detected jump events. This local block resampling technique accounts for the heteroskedasticity of the measurement error around the different jump events in the sample (which is reflected in the dependence of the limiting distribution on the volatility trajectory).

We can compare our test with some recent work on factor structures in high frequency setting. [2] and [27, 28] propose consistent estimators for the number of factors underlying the quadratic covariation matrices of the diffusive and the jump components of an asymptotically increasing cross-section of processes observed at high frequency. Our focus differs from these papers in three ways. First, instead of estimation, we develop tests for the number of factors, for which we need to characterize the asymptotic distribution of the test statistic. Second, we focus on the factor structure at a specific set of jump events and we are not interested in the diffusive moves that capture the behavior of the processes at "normal" times; as well known, the techniques needed for studying diffusions and jumps are very different. Third, unlike [2] and [27, 28], the dimension of the cross-section in this paper is fixed and this necessitates different asymptotic arguments. Finally, there is a large literature on estimating factor loadings and testing the validity of factor models (primarily bivariate) from high-frequency data, see e.g., [3], [7], [13], [16], [18, 19], [25, 26], [29] and [31]. The main difference between the current paper and this body of work is that in our case the factors are latent and their number is not known.

We apply our test to high-frequency financial data covering the period 2007–2015. Our focus in the empirical application is the factor structure of financial assets at the times of big market jump events. We identify the latter using high-frequency data on a futures contract written on the S&P 500 index. We then apply the rank test at the market jump times for various financial assets. Our results point to low-dimensional factor structure of the cross-sections of industry portfolios comprising the S&P 500 index as well as the Dow 30 stocks at the market jump times.

The rest of the paper is organized as follows. In Section 2 we introduce our setup, develop the test statistic and derive its asymptotic behavior, and present resampling method for feasible testing. Section 3 contains a Monte Carlo evaluation of the test performance. Section 4 applies the test to study the factor structure of various sets of financial assets at market jump times. Section 5 concludes. The proofs are given in Section 6.

2 The testing procedure

2.1 The testing problem

We consider a d-dimensional multivariate Itô semimartingale process $\boldsymbol{X}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the form

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t \boldsymbol{b}_s ds + \int_0^t \boldsymbol{\sigma}_s d\boldsymbol{W}_s + \sum_{s\leq t} \Delta \boldsymbol{X}_s, \tag{1}$$

where $\boldsymbol{b}$ is the drift process, $\boldsymbol{\sigma}$ is the stochastic (co)volatility process, $\boldsymbol{W}$ is a multivariate Brownian motion and $\Delta\boldsymbol{X}_t \equiv \boldsymbol{X}_t - \boldsymbol{X}_{t-}$ denotes the jump of $\boldsymbol{X}$ at time t. This setting accommodates most models used in continuous-time economics and finance (see e.g., [24]), and is the standard model in high-frequency statistics and econometrics; see [15] and [1]. We assume that the process $\boldsymbol{X}$ is sampled regularly at discrete times $i\Delta_n$, $i = 0, \ldots, \lfloor T/\Delta_n \rfloor$, for some fixed time span T, where $\lfloor \cdot \rfloor$ denotes the floor function. In our empirical applications, we take $\Delta_n = 1$ minute.

We consider a list of random times $\tau_1 < \tau_2 < \cdots < \tau_p$ at which a univariate process Z jumps. The process Z does not need to be part of the vector $\boldsymbol{X}$. We only need to have observations of it at the discrete times $i\Delta_n$. We assume Z is an Itô semimartingale with a form like (1). In a typical finance application, such as the one considered here, Z will be an aggregate risk factor (e.g., the S&P 500 index). The jumps of Z can be due to pre-scheduled macroeconomic announcements or due to the arrival of random events such as natural disasters or geopolitical conflicts. For brevity, henceforth, we will treat all jump times as random times, while noting that the procedure can be easily adjusted to accommodate known deterministic jump times exactly as in [18]. More generally, the testing procedure developed here can be applied for any set of random times $\tau_1 < \tau_2 < \cdots < \tau_p$ which can be inferred from the available discrete data.

We collect the jump vectors of $\boldsymbol{X}$ at these jump times using the *jump matrix* denoted by $\boldsymbol{J} = \left[\Delta \boldsymbol{X}_{\tau_1}, \ldots, \Delta \boldsymbol{X}_{\tau_p}\right]$. Our goal in this paper is to design a test for the rank of the $d \times p$ jump matrix $\boldsymbol{J}$. We note that, in general, the matrix $\boldsymbol{J}$ contains only a *subset* of all jumps of the vector $\boldsymbol{X}$ over the interval $[0, T]$. Typically, there will be a lot of jumps in the components of $\boldsymbol{X}$ which are outside the set of events $\tau_1 < \tau_2 < \cdots < \tau_p$ that we study.

For economic applications, the test that we develop here can be used to uncover the factor structure of jump risk for a cross-section of assets during important market-wide jump events. The focus on market-wide events is natural because, from economic theory, factor pricing models only concern aggregate risk. Therefore, for such applications we allow the components of $\boldsymbol{X}$ to contain so-called idiosyncratic jumps and impose essentially no restrictions on them; the idiosyncratic jumps are defined similar to [23] as jumps that do not occur at the same time as aggregate factor jumps.

The testing problem can be formally stated as follows. For some constant $r < \min(d, p)$, we aim to decide in which of the following events the observed path falls:

$$\begin{cases} \Omega_{0,r} \equiv \{\operatorname{rank}(\boldsymbol{J}) = r\}, \\ \Omega_{a,r} \equiv \{\operatorname{rank}(\boldsymbol{J}) > r\}, \end{cases} \tag{2}$$

where we remind the reader that the jump matrix $\boldsymbol{J}$ is random. The events $\Omega_{0,r}$ and $\Omega_{a,r}$ play the role of the null and the alternative hypotheses in our analysis, respectively. Specifying hypotheses in terms of random events is unlike the classical setting of hypothesis testing (e.g., [17]), but is standard in the study of high frequency data; see [15] and references therein.

We consider a variant of the test of [30] which itself is a generalization of the one proposed in [4]. To describe the idea, we start by noting that the rank of $\boldsymbol{J}$ is the same as that of the $d \times d$ jump covariation matrix $\boldsymbol{Q}$ given by

$$\boldsymbol{Q} \equiv \boldsymbol{J}\boldsymbol{J}^{\top} = \sum_{q=1}^{p} \Delta \boldsymbol{X}_{\tau_q} \Delta \boldsymbol{X}_{\tau_q}^{\top}.$$

Let $\lambda_1^2 \geq \cdots \geq \lambda_d^2$ be the ordered eigenvalues of $\boldsymbol{Q}$ and set

$$S_r \equiv \sum_{j=r+1}^{d} \lambda_j^2. \tag{3}$$

Clearly, S_r is a sum of zero eigenvalues under the null hypothesis and it is strictly positive under the alternative. The test for discriminating between the events in (2) can be carried out equivalently by testing whether S_r is zero or strictly positive.

Our approach differs in important ways from previous work using cumulative sums of eigenvalues. First, the estimation of the jump matrix $\boldsymbol{J}$ is nonstandard (cf. [30]). We use a jump detection method to nonparametrically estimate the jump matrix, and the resultant estimator has a (doubly) mixed Gaussian asymptotic distribution. In particular, the sampling variability of the estimator generally depends on the random realization of jump sizes and stochastic volatility before and after jump times. It is important to note that the asymptotic distribution of the jump estimator is generally not $\mathcal{F}$-conditionally Gaussian and, hence, violates a key assumption in [30] (see Assumption 2.2 there). Consequently, the procedure of [30] cannot be directly applied in our setting. We need to take into account the nonstandard asymptotic distribution in the jump estimation when analyzing the asymptotic distribution of our test statistic. Second, we provide an easy-to-implement local i.i.d. resampling method for computing the critical values for our test. The resampling scheme may be extended to other inference problems in high-frequency applications.

2.2 The test statistic

We now describe the construction of the test statistic. The first step is to detect the jumps using a thresholding method (see [22]). To this end, we pick a sequence u_n of truncation threshold that satisfies $u_n \asymp \Delta_n^{\varpi}$ for some $\varpi \in (0, 1/2)$. Since the diffusive moves driven by the Brownian motion are of order $O_p(\Delta_n^{1/2})$, they do not exceed the truncation threshold u_n asymptotically. The detected jump times of Z on the sampling grid are collected by

$$\mathcal{I}_n \equiv \{i : |\Delta_i^n Z| > u_n\}. \tag{4}$$

In applications, it is important to set u_n in an adaptive way which takes into account the fact that the diffusive volatility changes over time. Intuitively, what constitutes a big or small in magnitude move for the diffusive component of Z depends on the level of its volatility. We recommend scaling the threshold u_n according to a local estimate of volatility. We refer to Section 4 for implementation details and the online supplement of this paper for example code in MATLAB.

In developing our test, we further take into account a feature in financial markets where the vector $\boldsymbol{X}$ can be prone to more microstructure effects than Z (due to lower liquidity). In particular, we note that individual assets may have "gradual jumps" as noted by [6]. In the same vein, [20] document that during market-wide events, jumps of individual assets (i.e., $\boldsymbol{X}$) sometimes take a longer time to realize than the highly liquid market index. For this reason, we adopt the mixed-scale strategy of [20] by detecting the market jumps at a fine scale Δ_n as in (4), but estimate the jump matrix at a coarse scale $k\Delta_n$, for some $k \geq 1$. By doing so, we can detect jumps in Z with high precision while being conservative about the time window used to capture the jumps of $\boldsymbol{X}$. More precisely, we denote returns at the coarse scale $k\Delta_n$ using $\Delta_{i,k}^n \boldsymbol{X} = \boldsymbol{X}_{(i-1+k)\Delta_n} - \boldsymbol{X}_{(i-1)\Delta_n}$. Our estimator for the jump matrix is then given by $\hat{\boldsymbol{J}}_n = \left[\Delta_{i,k}^n \boldsymbol{X}\right]_{i \in \mathcal{I}_n}$.

We consider a test statistic given by the sample analogue of (3). Let $\hat{\lambda}_{n,1}^2 \geq \hat{\lambda}_{n,2}^2 \geq \cdots \geq \hat{\lambda}_{n,d}^2$ be the ordered eigenvalues of the matrix $\hat{\boldsymbol{J}}_n \hat{\boldsymbol{J}}_n^\top$. We then set

$$\hat{S}_{n,r} \equiv \sum_{j=r+1}^{d} \hat{\lambda}_{n,j}^2. \tag{5}$$

We reject the null hypothesis of rank r when the test statistic $\hat{S}_{n,r}$ is greater than some critical value that is described in Section 2.3 below. We note that asymptotically equivalent variants of this test statistic can be constructed by applying certain transformations (e.g., the Box–Cox transformation) on the eigenvalues $\hat{\lambda}_{n,j}^2$ like in [4] and [30]; such an extension is straightforward and is omitted for brevity.

In order to determine the critical value, we first characterize the asymptotic distribution of $\hat{S}_{n,r}$ under the null hypothesis. We need some notation to do this. Consider the singular value decomposition for $\boldsymbol{J}$:

$$\boldsymbol{J} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\top, \tag{6}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are respectively $d \times d$ and $p \times p$ orthogonal matrices, $\boldsymbol{D}$ is a $d \times p$ (rectangular) diagonal matrix with its diagonal elements $\lambda_1, \ldots, \lambda_{d \wedge p}$. We further partition $\boldsymbol{U} = [\boldsymbol{U}_1 \vdots \boldsymbol{U}_2]$ and $\boldsymbol{V} = [\boldsymbol{V}_1 \vdots \boldsymbol{V}_2]$, where both $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ contain r columns.

We next introduce i.i.d. random variables κ_q, ξ_{q-} and ξ_{q+} for $q \geq 1$, such that κ_q is uniformly distributed on $[0, 1]$ and $\boldsymbol{\xi}_{q-}$ and $\boldsymbol{\xi}_{q+}$ are independent d-dimensional standard Gaussian random vectors. We then set $\boldsymbol{\zeta}_q = \sqrt{\kappa_q}\boldsymbol{\sigma}_{\tau_q -}\boldsymbol{\xi}_{q-} + \sqrt{k - \kappa_q}\boldsymbol{\sigma}_{\tau_q}\boldsymbol{\xi}_q$ and $\boldsymbol{\zeta} \equiv [\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_p]$. Below, for any matrix $\boldsymbol{A}$, we denote $\|\boldsymbol{A}\| = \sqrt{\mathrm{Trace}\left(\boldsymbol{A}^\top \boldsymbol{A}\right)}$.

Theorem 1, below, characterizes the asymptotic behavior of $\hat{S}_{n,r}$ under the null and the alternative hypotheses. In it we use the notion of stable convergence in law which refers to convergence in law that holds jointly with any bounded random variable defined on the original probability space, see e.g., [15] for further details.

Theorem 1. *Suppose that Assumption 1 in the Appendix holds. Under the null hypothesis, i.e., in restriction to $\Omega_{0,r}$, $\Delta_n^{-1}\hat{S}_{n,r}$ converges stably in law towards $\left\|\boldsymbol{U}_2^\top \boldsymbol{\zeta} \boldsymbol{V}_2\right\|^2$. Under the alternative hypothesis, i.e., in restriction to $\Omega_{a,r}$, $\Delta_n^{-1}\hat{S}_{n,r}$ diverges in probability to $+\infty$.*

The limit behavior of our test is determined by the behavior of the diffusive component of $\boldsymbol{X}$ around the jump events. Importantly, the error in recovering the timing of the jump events is of higher asymptotic order and does not impact the limit distribution. The latter depends both on the level of volatility $\boldsymbol{\sigma}$ before and after the jump events as well as on the jumps of $\boldsymbol{X}$. Even after conditioning on these (random) quantities, the limit distribution of our test remains non-standard. However, there is an easy resampling method that allows for conducting feasible testing which we describe next.

2.3 Computing critical values

Our test rejects the null hypothesis of rank r at significance level $\alpha \in (0,1)$ when the test statistic $\hat{S}_{n,r}$ is greater than a critical value $cv_{n,\alpha}$ which we construct as follows. Analogously to (6), we decompose $\hat{\boldsymbol{J}}_n$ as

$$\hat{\boldsymbol{J}}_n = \hat{\boldsymbol{U}}_n \hat{\boldsymbol{D}}_n \hat{\boldsymbol{V}}_n^\top.$$

We partition $\hat{\boldsymbol{U}}_n = [\hat{\boldsymbol{U}}_{1n} \vdots \hat{\boldsymbol{U}}_{2n}]$ and $\hat{\boldsymbol{V}}_n = [\hat{\boldsymbol{V}}_{1n} \vdots \hat{\boldsymbol{V}}_{2n}]$, where $\hat{\boldsymbol{U}}_{1n}$ and $\hat{\boldsymbol{V}}_{1n}$ contain r columns. The column spaces of $\hat{\boldsymbol{U}}_{2n}$ and $\hat{\boldsymbol{V}}_{2n}$ are used to estimate those of $\boldsymbol{U}_2$ and $\boldsymbol{V}_2$, respectively. We approximate the $\mathcal{F}$-conditional distribution of $\boldsymbol{\zeta}$ by resampling the diffusive increments of $\boldsymbol{X}$ around the jump times from local windows of size k_n. We also pick a sequence $\boldsymbol{v}_n = (v_{j,n})_{1\leq j\leq d}$ such that $v_{j,n} \asymp \Delta_n^{\varpi}$ for $\varpi \in (0, 1/2)$ which is used for trimming increments containing jumps. We construct the critical value using the following algorithm, for which a MATLAB package is available in the online supplement.

Algorithm 1. (Computation of Critical Values)

Step 1. For each $i \in \mathcal{I}_n$, draw $\kappa_i^* \sim \text{Uniform}[0,1]$ and draw

$$\begin{cases} \boldsymbol{\xi}_{n,i-}^* \text{ uniformly from } \{\min\{\max\{\Delta_{i-j}^n \boldsymbol{X}, -\boldsymbol{v}_n\}, \boldsymbol{v}_n\} : 1 \leq j \leq k_n\}, \\ \boldsymbol{\xi}_{n,i+}^* \text{ uniformly from } \{\min\{\max\{\Delta_{i+j}^n \boldsymbol{X}, -\boldsymbol{v}_n\}, \boldsymbol{v}_n\} : 1 \leq j \leq k_n\}, \end{cases}$$

and set $\boldsymbol{\zeta}_{n,i}^* = \sqrt{\kappa_i^*}\boldsymbol{\xi}_{n,i-}^* + \sqrt{k - \kappa_i^*}\boldsymbol{\xi}_{n,i+}^*$ and $\boldsymbol{\zeta}_n^* = [\boldsymbol{\zeta}_{n,i}^*]_{i \in \mathcal{I}_n}$.

Step 2. Repeat step 1 for a large number of times. Set $cv_{n,\alpha}$ as the $1-\alpha$ quantile of $\|\hat{\boldsymbol{U}}_{2n}^\top \boldsymbol{\zeta}_n^* \hat{\boldsymbol{V}}_{2n}\|^2$ in the simulated sample. □

Theorem 2, below, justifies the asymptotic validity of this critical value construction and summarizes the asymptotic size and power properties of the proposed feasible test.

Theorem 2. *Suppose that Assumption 1 in the Appendix holds, $k_n \to \infty$ and $k_n\Delta_n \to 0$. Let $cv_{n,\alpha}$ be defined by Algorithm 1. Then $\Delta_n^{-1} cv_{n,\alpha}$ converges in probability to the $1-\alpha$ $\mathcal{F}$-conditional quantile of $\left\|\boldsymbol{U}_2^\top \boldsymbol{\zeta} \boldsymbol{V}_2\right\|^2$. Consequently, the test associated with the confidence region $\{\hat{S}_{n,r} > cv_{n,\alpha}\}$ has asymptotic level α under the null hypothesis and asymptotic power one under the alternative hypothesis. That is,*

$$\mathbb{P}\left(\hat{S}_{n,r} > cv_{n,\alpha} \middle| \Omega_{0,r}\right) \to \alpha, \quad \mathbb{P}\left(\hat{S}_{n,r} > cv_{n,\alpha} \middle| \Omega_{a,r}\right) \to 1.$$

We note that the requirement on the asymptotic order of the local block length k_n is rather minimal. We only need k_n to increase to infinity at a rate that is slower than the sampling frequency. This suggests that our testing procedure is not very sensitive to this tuning parameter and we confirm this later on in our Monte Carlo study.

3 Monte Carlo study

We now examine the asymptotic theory above on simulated data that mimic the one used in our empirical application in Section 4.

3.1 The setting

We set the sample span $T = 1$ year, or equivalently, 250 trading days. Each day contains 390 high-frequency returns, corresponding to 1-minute sampling, which are generated using an Euler scheme with a 5-second mesh. As a result, each Monte Carlo realization contains $n = 97,500$ returns which are expressed in annualized percentage terms. We set the fine scale $\Delta_n = 1/n$ and implement the mixed-scale jump rank test at the coarse scale $k\Delta_n$, for $k = 5$ and 10. Throughout, we fix the dimension $d = 30$. There are 2,000 Monte Carlo trials in each experiment.

The processes of interest are simulated from the following model. We consider mutually independent standard Brownian motions B, $\boldsymbol{W}$ and $\widetilde{\boldsymbol{W}}$ taking values in $\mathbb{R}$, $\mathbb{R}^4$ and $\mathbb{R}^{30}$, respectively. The jump factors are driven by a Poisson process N_t with intensity $\lambda_N = 40$ and the independent Poisson processes $\{\widetilde{N}_{j,t}\}_{1\leq j\leq d}$ (also independent from N_t), each with intensity of $\tilde{\lambda}_N = 10$, capture so-called idiosyncratic jump risk in the components of $\boldsymbol{X}$. The data generating process in our simulation is then given by

$$\begin{cases} d\log(V_t) = -0.1\lambda_N dt + dB_t + \phi_t dN_t, \quad V_0 = 18^2, \\ dZ_t = \sqrt{V_t} d\boldsymbol{W}_{1,t} + \varphi_{1,t} dN_t, \\ d\boldsymbol{X}_t = \sqrt{V_t}\boldsymbol{\Xi} d\boldsymbol{W}_t + 0.548\sqrt{V_t} d\widetilde{\boldsymbol{W}}_t + \boldsymbol{\beta}\, d\boldsymbol{J}_t + d\widetilde{\boldsymbol{J}}_t, \\ d\boldsymbol{J}_t = [\varphi_{1,t} dN_t, \varphi_{2,t} dN_t]^\top, \quad d\widetilde{\boldsymbol{J}}_t = [\widetilde{\varphi}_{1,t} d\widetilde{N}_{1,t}, \ldots, \widetilde{\varphi}_{d,t} d\widetilde{N}_{d,t}]^\top, \end{cases}$$

and the jump sizes are drawn according to

$$\begin{cases} \phi_t \overset{i.i.d.}{\sim} \text{Exponential}(0.1), \\ \varphi_{1,t}, \ \varphi_{2,t}, \ \{\widetilde{\varphi}_{j,t}\}_{1\leq j\leq d} \overset{i.i.d.}{\sim} \text{Exponential}(\bar{\varphi}). \end{cases}$$

We calibrate the mean parameter $\bar{\varphi}$ so that the number of detected jumps in the simulated data is close to the average number of jumps we find later on in our empirical analysis. We consider $\bar{\varphi} = 0.25$, 0.3 or 0.35, which correspond to approximately 8, 10 or 12, respectively, detected jumps per year.

The loading matrices Ξ and $\boldsymbol{\beta}$ are specified as follows. Below, let $\boldsymbol{\iota}_{l:m}$ denote a 30-vector with zeros and ones, with the jth element being one for $l \leq j \leq m$, and the rest of the elements of the vector being zero. We set the 30×4 matrix $\Xi = [0.5 \cdot \boldsymbol{\iota}_{1:30}, \boldsymbol{\iota}_{15:22}, \boldsymbol{\iota}_{23:28}, \boldsymbol{\iota}_{29:30}]$, which is calibrated so as to match the structure of the eigenvalues of the integrated diffusive covariance matrix seen in our empirical analysis (see right panel of Figure 1). This choice of Ξ implies the diffusive component of $\boldsymbol{X}$ loads on four systematic factors (driven by $\boldsymbol{W}$) and in addition contains an idiosyncratic piece (driven by $\widetilde{\boldsymbol{W}}$). This factor structure for the diffusive returns is in line with existing empirical evidence, see e.g., [2].

The loading matrix $\boldsymbol{\beta}$ for the jump factors is given by

$$\boldsymbol{\beta} = \begin{cases} [\boldsymbol{\iota}_{1:30}, 0 \cdot \boldsymbol{\iota}_{1:30}] & r = 1 \text{ (one jump factor in } \boldsymbol{X}\text{)}, \\ [\boldsymbol{\iota}_{1:30}, \varsigma \cdot \boldsymbol{\iota}_{1:15} - \varsigma \cdot \boldsymbol{\iota}_{16:30}] & r = 2 \text{ (two jump factors in } \boldsymbol{X}\text{)}. \end{cases}$$

Below, we test for $\operatorname{rank}(\boldsymbol{J}) = 1$, which holds true when $r = 1$ but is violated when $r = 2$. These two cases assess the size and the power properties of our test, respectively. In particular, ς controls the extent to which the alternative deviates from the null; we conduct experiments with $\varsigma \in \{0.6, 0.7, 0.8\}$.

In our model, there are two types of jumps in $\boldsymbol{X}$: idiosyncratic (captured by $\widetilde{\boldsymbol{J}}$ and representing diversifiable jump risk in the sense of [23]) and systematic (captured by $\boldsymbol{J}$). These two groups of jumps arrive at different times almost surely, and our focus is testing the factor structure of the latter. Our first specification for $\boldsymbol{\beta}$ implies a one-factor model for the systematic jumps, with the jump size of Z playing the role of the latent factor. Our second specification for $\boldsymbol{\beta}$ implies a two-factor structure, with one factor being the jump size of Z and another one that is not spanned by the latter.

In the implementation of the test we consider the local window size k_n within a range of values $\{30, 60\}$, so as to examine the robustness of the test procedure to this tuning parameter. Finally, the truncation threshold u_n (resp. $\boldsymbol{v}_n$) is set to be 7 (resp. 4) times of local standard deviations of the diffusive component, formed using the bipower variation estimator of [8]; see the example code in the online supplement for details.

3.2 Results

We report the Monte Carlo rejection rates of our tests in Table 1. As shown in this table, under the null hypothesis, the finite-sample rejection rates are very close to the associated nominal levels. The results are robust with respect to the average jump size $\bar{\varphi}$ as well as to the choice of the scale parameter k and the local window k_n. Under the alternative hypothesis, the power is adequate in various settings. We observe that, when the jump size $\bar{\varphi}$ and the factor loading ς increase, the rejection rate approaches 1; intuitively, the signal (i.e., the jump component) is stronger relative to the noise (i.e., the diffusive component) when these parameters are larger. The power in the 5-minute mixed-scale setting is greater than that in the 10-minute setting, because of the higher sampling variability in the latter. The power results are also robust to the choice of the local window parameter k_n. Overall, the simulation results are consistent with our asymptotic theory and demonstrate good size and power properties of the proposed test in an empirically relevant simulation design.

4 Empirical application

We use the developed test to study the factor structure of various financial assets at the times of market jumps. Our proxy for the market is the E-mini S&P 500 index futures, a leading equity index futures for the U.S. stock market. We study the response of three groups of assets to market jump events. The first group consists of the exchange traded

	$k=5$ Mixed Scale						$k=10$ Mixed Scale					
	$k_n=30$			$k_n=60$			$k_n=30$			$k_n=60$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
	Under the Null Hypothesis											
	One Jump Factor						One Jump Factor					
$\bar{\varphi}=0.25$	10.5	4.7	1.1	10.7	5.6	1.3	8.9	4.3	1.1	9.8	5.0	1.2
$\bar{\varphi}=0.30$	11.1	5.8	1.3	10.6	4.9	1.5	10.8	6.3	1.6	11.2	6.2	1.9
$\bar{\varphi}=0.35$	10.0	5.7	1.5	12.2	7.2	1.8	12.3	6.6	1.8	12.1	6.3	1.5
	Under the Alternative Hypothesis											
	Two Jump Factors ($\varsigma=0.6$)						Two Jump Factors ($\varsigma=0.6$)					
$\bar{\varphi}=0.25$	75.1	71.7	63.0	77.3	73.0	63.6	65.5	59.2	48.9	64.6	57.8	47.2
$\bar{\varphi}=0.30$	88.0	84.7	78.5	87.5	84.2	77.8	77.5	72.5	62.8	77.9	72.0	61.4
$\bar{\varphi}=0.35$	91.5	90.0	85.2	93.7	91.8	85.9	85.8	82.7	75.0	85.0	81.3	72.7
	Two Jump Factors ($\varsigma=0.7$)						Two Jump Factors ($\varsigma=0.7$)					
$\bar{\varphi}=0.25$	78.1	74.1	67.8	78.8	74.2	66.9	70.9	66.5	57.5	72.5	66.8	55.8
$\bar{\varphi}=0.30$	88.4	85.7	80.9	88.3	85.7	80.5	82.0	78.6	70.0	82.0	77.6	67.7
$\bar{\varphi}=0.35$	94.2	92.3	88.7	93.8	91.9	87.0	88.3	85.2	78.5	90.0	87.2	79.8
	Two Jump Factors ($\varsigma=0.8$)						Two Jump Factors ($\varsigma=0.8$)					
$\bar{\varphi}=0.25$	80.8	77.9	71.9	82.2	79.8	72.7	73.0	68.1	59.4	74.1	69.5	59.5
$\bar{\varphi}=0.30$	91.8	89.8	85.1	89.2	87.4	82.9	85.2	81.6	73.6	86.2	82.2	73.7
$\bar{\varphi}=0.35$	95.0	93.6	91.0	95.0	93.5	90.5	91.0	88.5	82.4	90.8	87.8	81.5

Table 1: Monte Carlo rejection rates (%) under the null and the alternative hypotheses.

funds (ETFs) on the nine industry portfolios comprising the stocks in the S&P 500 index. The second group consists of the thirty stocks in the Dow Jones Industrial Average index as of the end of 2015, except that Visa Inc. (NYSE: V) is replaced with Bank of America (NYSE: BAC) so as to maintain a balanced panel in our sample. Finally, the third group of assets is formed from the futures on the S&P 500 index, 30-year US Treasury bond, Dollar-Euro and Dollar-Yen exchange rates. The data covers the period from the beginning of 2007 till the end of 2015. We exclude half-trading days as well as May 6, 2010 and April 23, 2013 when there were periods of malfunctioning (i.e., "Flash Crashes") on the financial markets. Altogether we have 2223 trading days in our sample, and in each of them we have 391 one-minute price records of each of the assets included in our analysis. Our sample period includes the rather turbulent events surrounding the global financial crisis of 2008, the two recent sovereign debt crises in Europe as well as other major political and economic events that have caused big moves on the financial markets. As a result, the data set contains a lot of market jump events with a rather diverse set of economic forces behind them.

We set the threshold level for identifying the market jump events as in the Monte Carlo study: we identify as jumps one-minute increments which exceed in absolute value 7 local volatility estimates. This results in a total of 83 market jump events in our sample. As in the Monte Carlo, we use aggregation levels of $k = 5$ and $k = 10$ to guard against the gradual jump phenomenon in the cross-section of assets, with the latter being more conservative. Finally, we set the local window for determining the critical value of the test to $k_n = 60$. The test for the factor structure at the market jump events is performed on a yearly basis. The reason for this is that factor loadings typically vary over longer periods of time (see, e.g., [14]) due to changes in the correlations of the cash flows of the financial assets as well as changes to discount rates.

We begin our analysis of the jump factor structure of the nine industry portfolios at the market jump times. Due to diversification, the sector portfolios are largely void of idiosyncratic risk and, hence, they also tend to have much smaller variances than individual stocks. Thus, they allow for a much sharper identification of the systematic risks in the economy and, in particular, their jump factor structure. The results from our test are reported in Table 2. Overall, the results support a low-dimensional factor structure of the jumps of the industry portfolios at the times of market jumps. Indeed, the test for rank of 1 rejects the null hypothesis at the 1% significance level only in years 2011 and 2012 when we use the mixed-scale aggregation with $k = 5$, and there is no rejection across all years at the aggregation level $k = 10$. This finding is consistent with the informal evidence from the eigenvalue structure of the jump matrix shown in Figure 1.

We continue next with the Dow 30 stocks. The results, reported in Table 3, are qualitatively similar to those for the industry portfolios. In general, we continue to find support for a low-dimensional factor structure at market jump events, though for many of the years in the sample, the p-values of the hypothesis $r = 1$ drop somewhat when compared with those for the industry portfolios. A one-factor jump model at times of market jumps is not rejected at the 1% level in six out of the nine years. This finding is again in line with the descriptive evidence seen in Figure 1. One possible explanation for the lower p-values of the test $r = 1$ in years 2007, 2012 and 2014 can be small variation in the loadings within a one-factor jump model during these years. Tests for higher ranks also suggest that a three-factor (resp. two-factor) structure is adequate at the aggregation level $k = 5$ (resp. $k = 10$) for these years.

The evidence thus far points to a one-factor model for the jumps of stocks and stock portfolios at the times of market jumps. A natural question is whether all these assets share the same factor at the market jump times. To investigate this, we merged these two cross-sections and implemented our test on the merged data. The results for the jump

Year	# of jumps	P-values (%) $k=5$			$k=10$		
		$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$
2007	11	4.26	84.34	86.98	14.36	79.94	91.02
2008	8	4.42	77.59	99.11	2.58	86.14	93.96
2009	5	37.76	67.77	94.78	9.30	29.44	40.36
2010	9	25.20	54.15	73.70	71.62	78.36	93.32
2011	6	0.62	13.72	34.12	22.06	49.02	89.90
2012	14	0.60	3.79	14.62	17.82	43.62	73.04
2013	4	16.70	77.83	85.28	35.22	88.64	66.88
2014	12	12.17	43.77	95.45	6.58	76.58	94.50
2015	15	6.57	57.46	85.69	25.46	54.94	80.98

Table 2: Test results for industry portfolios. P-values are reported for testing rank $(\boldsymbol{J}) = r$ at mixed-scale $k \in \{5, 10\}$ for nine industry portfolios in each year, 2007–2015.

factor structure are very similar to the results for the Dow 30 stocks and we do not report them to save space. This points to the fact that the stocks and industry portfolios are all driven by a single common factor during times of market jumps. This factor is treated as latent in the test but it can be easily identified from the cross-sections of assets up to a constant. To shed some light on this factor, we consider

$$\hat{f}_i^n = \sqrt{\frac{1}{d}\sum_{j=1}^{d}(\Delta_i^n X^{(j)})^2}, \quad i \in \mathcal{I}_n,$$

Year	# of jumps	P-values (%)					
		$k=5$			$k=10$		
		$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$
2007	11	0.17	0.48	7.18	0.82	3.32	5.78
2008	8	14.90	90.55	96.43	0.80	36.70	43.70
2009	5	5.49	26.79	58.04	1.26	9.66	20.22
2010	9	11.29	24.55	44.09	52.20	72.88	84.52
2011	6	5.64	11.51	9.75	63.00	83.78	91.46
2012	14	0.01	0.04	2.27	0.52	2.48	7.96
2013	4	3.14	34.93	56.21	10.58	81.44	94.36
2014	12	0.08	0.36	3.74	7.28	64.02	82.90
2015	15	3.62	5.07	11.73	8.84	22.66	52.60

Table 3: Test results for Dow 30 stocks. P-values are reported for testing rank $(\boldsymbol{J}) = r$ at mixed-scale $k \in \{5, 10\}$ for Dow 30 stocks in each year, 2007–2015.

where $X^{(j)}$ denotes the jth component of $\boldsymbol{X}$. This quantity will recover the absolute value of the jump factor up to a constant. The reason for taking the squares is because the factor loadings of the assets in the latent one factor model can be of different signs. The cross-sectional aggregation further improves the estimation accuracy. Using the above simple method, we extract the latent jump factor from the cross-sections of industry portfolios and the Dow 30 stocks. On Figure 2, we compare the recovered factor $\hat{f}_i^n$ with the absolute value of the corresponding market jump; these variables are standardized by centering and scaling using their sample means and standard deviations. The scatter plots reveal

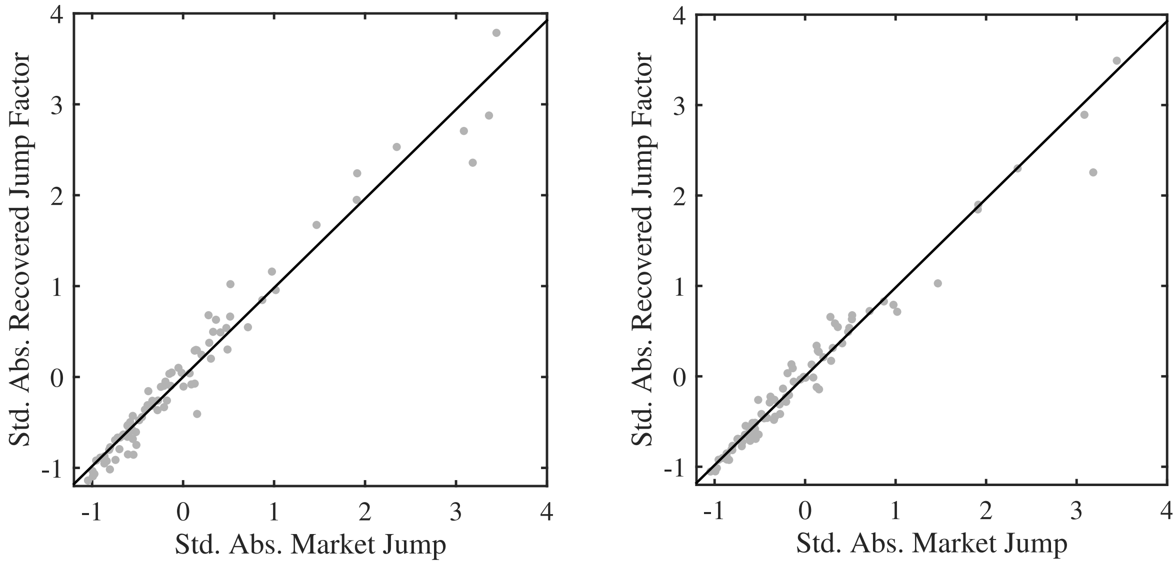

Figure 2: Recovered jump factor and market jump returns. Standardized recovered jump sizes (i.e., $\hat{f}_i^n$) are plotted versus the standardized absolute market jumps (dots), compared with 45-degree lines. Left panel corresponds to the cross-section of industry portfolios and the right panel to the Dow 30 stocks.

a remarkable similarity between the recovered factor and the market jumps. Indeed, the correlation between the series of extracted jumps from the two cross-sections and the market jumps is above 0.98 in both cases. To contrast these numbers, we performed the same latent factor extraction for all increments, instead of only those at which the market jumps. The correlation between the market return and this extracted series from the cross-section drops to 0.88 for the industry portfolios and 0.46 for the Dow 30 stocks. The significantly lower correlations outside the market jump times are due to the lack of one-factor structure of the cross-section of returns at these times.

The above evidence indicates that stock returns at market jump events can be linearly spanned by the market return. We now study the relationship between the stock market index and a set of diverse financial instruments, capturing different economic risks, again at the times of the market jumps. These additional assets are a long term bond (US 30-Year Treasury bond) and two exchange rates (Dollar-Euro and Dollar-Yen). Existing empirical work has documented a complicated and time-varying dependence between U.S. stocks and bonds. Similarly, exchange rates, which are theoretically the ratios of the pricing kernels of the two countries, are driven by investors' expectations of future central banks' policies, among other things. Given time-varying pricing of jump risks, we expect rather nontrivial connections between the jump risks in the stock market, the bond market, and the currencies.

In view of such complications, it is far more challenging for the market jump to span in a linear time-invariant manner the returns on bonds and currencies during market jump events. We provide formal evidence for this intuition using our test, with results reported in Table 4. At the aggregation level of $k = 5$ (resp. $k = 10$), our test rejects at the 1% significance level a one-factor model for the jumps of the four assets in this cross-section during the market jump times in eight (resp. five) out of the nine years in our sample. The evidence in Table 4 strongly suggests that there should be at least two factors to explain the behavior of these four assets during the market jump events. Of course, it is possible that by further conditioning on the economic source of the market jump, for example, jumps triggered by news about monetary policy, we can detect even for this cross-section a one-factor structure at this subset of market jump events. We leave this exploration for future work.

Year	# of jumps	P-values (%)					
		$k=5$			$k=10$		
		$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$
2007	11	0.00	0.07	1.84	0.00	1.34	2.08
2008	8	0.66	16.66	16.65	28.06	49.54	63.50
2009	5	0.89	0.89	4.54	2.44	0.06	75.26
2010	9	0.00	1.70	13.33	6.28	7.38	18.92
2011	6	2.35	24.59	3.92	5.28	51.08	45.26
2012	14	0.00	0.00	0.61	0.00	0.00	24.10
2013	4	0.00	68.19	96.24	0.00	42.70	59.94
2014	12	0.03	0.01	0.36	0.08	0.46	3.98
2015	15	0.08	7.33	75.16	0.26	24.68	30.64

Table 4: Test results for short-term futures. P-values are reported for testing rank $(\boldsymbol{J}) = r$ at mixed-scale $k \in \{5, 10\}$ for futures contracts on the S&P 500 index, 30-Year Treasury bond, Dollar-Euro and Dollar-Yen exchange rates in each year, 2007–2015.

5 Conclusion

We develop a formal statistical inference procedure for studying the factor structure of a general d-dimensional semimartingale at any finite set of stopping times associated with jumps of a discretely-observed process. We test for an exact rank r factor structure, versus a higher dimensional factor structure, by examining the statistical significance of the sum of the $d-r$ smallest eigenvalues of the jump covariance matrix of returns. The asymptotic theory is non-standard. The reason is that we can only determine the sampling intervals containing the jumps, not the jump times directly, so these increments are error-ridden proxies for actual jumps. The measurement errors are the diffusive components of the semimartingale across the jump intervals and the error variance in general co-jumps with the process. The observation errors shrink with the sampling interval and asymptotically follow non-pivotal heteroskedastic mixed Gaussian distributions, which leads to the non-standard asymptotic distribution of the test statistic. The statistic, however, is easy to compute via a singular value decomposition, and we present a simple resampling procedure to compute critical values.

In our empirical application, we first apply the test to high-frequency stock returns, which generally follow a three- or four-dimensional factor structure; however, the evidence from our test indicates that stock returns selected at just market jump times collapse to a one-factor structure. This application uses a panel comprised of nine well-diversified industry portfolios and another comprised of the DOW 30 stocks. For these two panels, we find very little evidence to discredit the null hypothesis of a rank-one jump covariation matrix at market jump times. We filter the panel of jump returns to recover the common factor, which is seen to be nearly collinear with the S&P 500 index return, consistent with one-factor market model for jumps. In additional work, we apply the same testing procedure to a cross-section that includes bond and currency returns, and we find strong evidence against a one-factor structure. This alternative finding highlights the different jump risk structures of bonds and currencies relative to equities, and it illustrates the power of our test against plausible alternatives in an important empirical context.

6 Appendix

In this technical appendix, we present the regularity conditions and the proofs for the theorems in the main text. We assume that $\boldsymbol{X}$ is a d-dimensional Itô semimartingale (see, e.g., [15], Section 2.1.4) of the form (1). The jump part of $\boldsymbol{X}$ can be represented as $\int_0^t \int_{\mathbb{R}} \boldsymbol{\delta}_X (s,u)\, \mu(ds,du)$ where $\boldsymbol{\delta}_X : \Omega \times \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}^d$ is a predictable function and μ is a Poisson random measure $\mathbb{R}_+ \times \mathbb{R}$ with its compensator $\nu(dt,du) = dt \otimes \lambda(du)$ for some measure λ on $\mathbb{R}$.

Assumption 1. *(a) The process $(\boldsymbol{b}_t)_{t\geq 0}$ is locally bounded; (b) $\boldsymbol{\sigma}_t \boldsymbol{\sigma}_t^\top$ is nonsingular for $t \in [0,T]$ almost surely; (c) $\nu([0,T] \times \mathbb{R}) < \infty$.*

The only nontrivial restriction in Assumption 1 is the assumption of finite-activity jumps in $\boldsymbol{X}$. This assumption is used mainly to simplify our technical exposition because our empirical focus in this paper is the big jumps. Technically speaking, this means that we can drop Assumption 1(c) and focus on jumps with size bounded away from zero. Doing so automatically verifies the finite-activity assumption, but with very little effect on the empirical investigation in the current paper.

By a standard localization procedure (see Section 4.4.1 in [15]), we can assume that the processes $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ are bounded in the proofs below. To simplify notations, we denote the conditional expectation operator $\mathbb{E}[\,\cdot\,|\mathcal{F}]$ by $\mathbb{E}_{\mathcal{F}}[\cdot]$. We also denote by $(T_m)_{m\geq 1}$ the successive jump times of the Poisson process $t \mapsto \mu((0,t]\times\mathbb{R})$. Note that these jump times are independent of the Brownian motion $\boldsymbol{W}$. Let $i(n,m)$ denote the unique integer i such that $T_m \in ((i-1)\Delta_n, i\Delta_n]$. We use K to denote a generic constant that may change from line to line.

Proof of Theorem 1. The proof is adapted to the current high-frequency setting from that of [30]. By Proposition 1 in [18], $\mathcal{I}_n$ coincides with $\{i(n,m) : \Delta Z_{T_m} \neq 0, 0 \leq T_m \leq$

$T, m \geq 1\}$ with probability approaching one. Hence, we can assume that $\hat{\boldsymbol{J}}_n$ is a $d \times p$ matrix without loss of generality. In addition, $\hat{\boldsymbol{J}}_n \xrightarrow{\mathbb{P}} \boldsymbol{J}$ and, hence, $\hat{\boldsymbol{J}}_n \hat{\boldsymbol{J}}_n^\top \xrightarrow{\mathbb{P}} \boldsymbol{J}\boldsymbol{J}^\top$. Since $(\lambda_j^2)_{1 \leq j \leq d}$ defined above are continuous functions of the elements of $\boldsymbol{J}\boldsymbol{J}^\top$, we have $\hat{\lambda}_{n,j}^2 \xrightarrow{\mathbb{P}} \lambda_j^2$, $1 \leq j \leq d$.

We start with the asymptotics of $\hat{S}_{n,r}$ under the null hypothesis. Recall (6). In restriction to $\Omega_{0,r}$, we can express the $d \times p$ jump matrix $\boldsymbol{J}$ further as $\boldsymbol{J} = \boldsymbol{U}_1 \boldsymbol{D}_r \boldsymbol{V}_1^\top$ where $\boldsymbol{D}_r$ is a $r \times r$ square diagonal matrix consisting of the first r diagonal elements of $\boldsymbol{D}$. Since $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices, we have $\boldsymbol{U}_1^\top \boldsymbol{U}_1 = \boldsymbol{I}_r$ and $\boldsymbol{U}_2^\top \boldsymbol{U}_1 = \boldsymbol{0}$, and the same holds for $\boldsymbol{V}$. In addition, we recall from Proposition 4.4.10 in [15] that

$$\Delta_n^{-1/2}(\hat{\boldsymbol{J}}_n - \boldsymbol{J}) \xrightarrow{\mathcal{L}\text{-}s} \boldsymbol{\zeta}, \tag{7}$$

where $\xrightarrow{\mathcal{L}\text{-}s}$ denotes stable convergence in law.

We now observe

$$\boldsymbol{U}_1^\top \hat{\boldsymbol{J}}_n = \boldsymbol{D}_r \boldsymbol{V}_1^\top + O_p(\Delta_n^{1/2}), \quad \Delta_n^{-1/2} \boldsymbol{U}_2^\top \hat{\boldsymbol{J}}_n = \Delta_n^{-1/2} \boldsymbol{U}_2^\top (\hat{\boldsymbol{J}}_n - \boldsymbol{J}). \tag{8}$$

Using (8) and the fact that $\hat{\lambda}_{n,j}^2 = o_p(1)$ for $j = r+1, \ldots, d$, we deduce

$$\begin{aligned}
&\left(\boldsymbol{U}_1, \Delta_n^{-1/2}\boldsymbol{U}_2\right)^\top (\hat{\boldsymbol{J}}_n \hat{\boldsymbol{J}}_n^\top - \hat{\lambda}_{n,j}^2 \boldsymbol{I}_d) \left(\boldsymbol{U}_1, \Delta_n^{-1/2}\boldsymbol{U}_2\right) \\
&\quad = \begin{pmatrix} \boldsymbol{D}_r^2 & \boldsymbol{D}_r \boldsymbol{V}_1^\top \Delta_n^{-1/2} (\hat{\boldsymbol{J}}_n - \boldsymbol{J})^\top \boldsymbol{U}_2 \\ \bullet & \Delta_n^{-1} \boldsymbol{U}_2^\top (\hat{\boldsymbol{J}}_n - \boldsymbol{J})(\hat{\boldsymbol{J}}_n - \boldsymbol{J})^\top \boldsymbol{U}_2 \end{pmatrix} - \Delta_n^{-1} \hat{\lambda}_{n,j}^2 \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \bullet & \boldsymbol{I}_{d-r} \end{pmatrix} + o_p(1),
\end{aligned}$$

where we have omitted the matrix elements in our notation that are implied by symmetry. Therefore,

$$\begin{aligned}
0 &= \left| \hat{\boldsymbol{J}}_n \hat{\boldsymbol{J}}_n^\top - \hat{\lambda}_{n,j}^2 \boldsymbol{I}_d \right| = \left| \left(\boldsymbol{U}_1, \Delta_n^{-1/2}\boldsymbol{U}_2\right)^\top (\hat{\boldsymbol{J}}_n \hat{\boldsymbol{J}}_n^\top - \hat{\lambda}_{n,j}^2 \boldsymbol{I}_d) \left(\boldsymbol{U}_1, \Delta_n^{-1/2}\boldsymbol{U}_2\right) \right| \\
&= \left| \begin{pmatrix} \boldsymbol{D}_r^2 & \boldsymbol{D}_r \boldsymbol{V}_1^\top \Delta_n^{-1/2} (\hat{\boldsymbol{J}}_n - \boldsymbol{J})^\top \boldsymbol{U}_2 \\ \bullet & \Delta_n^{-1} \boldsymbol{U}_2^\top (\hat{\boldsymbol{J}}_n - \boldsymbol{J})(\hat{\boldsymbol{J}}_n - \boldsymbol{J})^\top \boldsymbol{U}_2 \end{pmatrix} - \Delta_n^{-1} \hat{\lambda}_{n,j}^2 \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \cdot & \boldsymbol{I}_{d-r} \end{pmatrix} \right| + o_P(1) \\
&= \left| \boldsymbol{D}_r^2 \right| \left| \Delta_n^{-1} \boldsymbol{U}_2^\top (\hat{\boldsymbol{J}}_n - \boldsymbol{J}) \boldsymbol{V}_2 \boldsymbol{V}_2^\top (\hat{\boldsymbol{J}}_n - \boldsymbol{J})^\top \boldsymbol{U}_2 - \Delta_n^{-1} \hat{\lambda}_{n,j}^2 \boldsymbol{I}_{d-r} \right| + o_P(1),
\end{aligned}$$

where the last line is obtained by computing the determinant of a partitioned matrix. By construction, the $\mathcal{F}$-conditional distribution of $\boldsymbol{\zeta}$ is non-degenerate. Hence, the limiting distribution of $\Delta_n^{-1}\boldsymbol{U}_2^\top(\hat{\boldsymbol{J}}_n-\boldsymbol{J})\boldsymbol{V}_2\boldsymbol{V}_2^\top(\hat{\boldsymbol{J}}_n-\boldsymbol{J})^\top\boldsymbol{U}_2$ is also non-degenerate. By the continuous mapping theorem, the limiting distribution of $\Delta_n^{-1}\hat{\lambda}_{n,j}^2$, $j=r+1,\ldots,d$, is the same as that of the $(d-r)$ eigenvalues of $\Delta_n^{-1}\boldsymbol{U}_2^\top(\hat{\boldsymbol{J}}_n-\boldsymbol{J})\boldsymbol{V}_2\boldsymbol{V}_2^\top(\hat{\boldsymbol{J}}_n-\boldsymbol{J})^\top\boldsymbol{U}_2$. Hence, the limiting distribution of $\hat{S}_{n,r}$ is the same as that of the trace of $\Delta_n^{-1}\boldsymbol{U}_2^\top(\hat{\boldsymbol{J}}_n-\boldsymbol{J})\boldsymbol{V}_2\boldsymbol{V}_2^\top(\hat{\boldsymbol{J}}_n-\boldsymbol{J})^\top\boldsymbol{U}_2$. By (7) and the properties of stable convergence in law, $\hat{S}_{n,r}\xrightarrow{\mathcal{L}\text{-}s}\left\|\boldsymbol{U}_2^\top\boldsymbol{\zeta}\boldsymbol{V}_2\right\|^2$ in restriction to $\Omega_{0,r}$.

Under the alternative, it is easy to see that the probability limit of $\hat{S}_{n,r}$ is strictly positive. The second assertion of the theorem then readily follows. *Q.E.D.*

Proof of Theorem 2. Let $(j_{m+}^*,j_{m-}^*)_{m\geq1}$ be independent variables drawn uniformly from $\{1,\ldots,k_n\}$. Let $(\boldsymbol{\chi}_{m+}^*,\boldsymbol{\chi}_{m-}^*)_{m\geq1}$ be independent d-dimensional standard normal variables. Let $(\tilde{\kappa}_m^*)_{m\geq1}$ be independent variables from the Uniform[0,1] distribution. These variables are all independent of $\mathcal{F}$. Below, for a sequence of random variables A_n, we write $A_n\xrightarrow{\mathcal{L}|\mathcal{F}}A$ if the $\mathcal{F}$-conditional law of A_n converges in probability to that of A under any metric that is compatible with the weak convergence of probability measures.

We first show that, under the product topology,

$$\left(\Delta_n^{-1/2}\Delta_{i(n,m)+j_{m+}^*}^n\boldsymbol{X}\right)_{m\geq1}\xrightarrow{\mathcal{L}|\mathcal{F}}\left(\boldsymbol{\sigma}_{T_m}\boldsymbol{\chi}_{m+}^*\right)_{m\geq1}. \tag{9}$$

We consider a sequence of events $\Omega_n\equiv\{$there is at most one jump T_m in $[i-k_n\Delta_n,i+k_n\Delta_n]$ for all i such that $k_n\leq i\leq\lfloor T/\Delta_n\rfloor-k_n\}$. Since the jumps of $\boldsymbol{X}$ have finite activity, $\mathbb{P}(\Omega_n)\to1$. Therefore, we can restrict attention to Ω_n below. In particular, the return

$\Delta^n_{i(n,m)+j^*_{m+}}\boldsymbol{X}$ does not contain any jump. We then observe

$$
\begin{aligned}
&\Delta_n^{-1/2}\mathbb{E}_{\mathcal{F}}\left\|\Delta^n_{i(n,m)+j^*_{m+}}\boldsymbol{X}-\boldsymbol{\sigma}_{T_m}\Delta^n_{i(n,m)+j^*_{m+}}\boldsymbol{W}\right\| \\
&\quad\le \Delta_n^{-1/2}\mathbb{E}_{\mathcal{F}}\left\|\int_{\left(i(n,m)+j^*_{m+}-1\right)\Delta_n}^{\left(i(n,m)+j^*_{m+}\right)\Delta_n}\boldsymbol{b}_s ds\right\| \\
&\qquad+\Delta_n^{-1/2}\mathbb{E}_{\mathcal{F}}\left\|\int_{\left(i(n,m)+j^*_{m+}-1\right)\Delta_n}^{\left(i(n,m)+j^*_{m+}\right)\Delta_n}\left(\boldsymbol{\sigma}_s-\boldsymbol{\sigma}_{T_m}\right)d\boldsymbol{W}_s\right\| \\
&\quad\le O_p(\Delta_n^{1/2})+\Delta_n^{-1/2}\frac{1}{k_n}\sum_{j=1}^{k_n}\left\|\int_{(i(n,m)+j-1)\Delta_n}^{(i(n,m)+j)\Delta_n}\left(\boldsymbol{\sigma}_s-\boldsymbol{\sigma}_{T_m}\right)d\boldsymbol{W}_s\right\| \\
&\quad= o_p(1),
\end{aligned}
\tag{10}
$$

where the last line follows from Itô's isometry, the càdlàg property of $\boldsymbol{\sigma}$ and the assumption $k_n\Delta_n\to 0$. Next, we observe that

$$
\left(\Delta^n_{i(n,m)+j^*_{m+}}\boldsymbol{W}/\Delta_n^{1/2}\right)_{m\ge 1}\xrightarrow{\mathcal{L}|\mathcal{F}}\left(\boldsymbol{\chi}^*_{m+}\right)_{m\ge 1}. \tag{11}
$$

Indeed, for each $m\ge 1$,

$$
\begin{aligned}
&\mathbb{P}\left(\Delta^n_{i(n,m)+j^*_{m+}}\boldsymbol{W}/\Delta_n^{1/2}\le x\middle|\mathcal{F}\right) \\
&\qquad=\frac{1}{k_n}\sum_{j=1}^{k_n}\left[1\left\{\Delta^n_{i(n,m)+j}\boldsymbol{W}/\Delta_n^{1/2}\le x\right\}\right]\xrightarrow{\mathbb{P}}\Phi\left(x\right),
\end{aligned}
$$

where $\Phi\left(\cdot\right)$ is the cumulative distribution function of a d-dimensional standard Gaussian variable, and the convergence is by a law of large numbers (for which we note that $(T_m)_{m\ge 1}$ and, hence, $(i\left(n,m\right))_{m\ge 1}$, are independent of $\boldsymbol{W}$). Hence, (11) holds for each m. Since the variables $\left(j^*_{m+}\right)_{m\ge 1}$ are drawn independently conditionally on $\mathcal{F}$, (11) also hold jointly for all $m\ge 1$.

From (11), we deduce

$$
\left(\boldsymbol{\sigma}_{T_m}\Delta^n_{i(n,m)+j^*_{m+}}\boldsymbol{W}/\Delta_n^{1/2}\right)_{m\ge 1}\xrightarrow{\mathcal{L}|\mathcal{F}}\left(\boldsymbol{\sigma}_{T_m}\boldsymbol{\chi}^*_{m+}\right)_{m\ge 1}. \tag{12}
$$

From (10) and (12), (9) readily follows. Similarly, we can show (9) with j_{m+}^* replaced by $-j_{m-}^*$ and $\boldsymbol{\sigma}_{T_m}\boldsymbol{\chi}_{m+}^*$ replaced by $\boldsymbol{\sigma}_{T_m-}\boldsymbol{\chi}_{m-}^*$; in addition, these convergence results hold jointly due to the $\mathcal{F}$-conditional independence of these variables. From here, it follows that

$$\begin{aligned}\sqrt{\tilde{\kappa}_m^*}\Delta_n^{-1/2}\Delta_{i(n,m)-j_{m-}^*}^n\boldsymbol{X} + \sqrt{k-\tilde{\kappa}_m^*}\Delta_n^{-1/2}\Delta_{i(n,m)+j_{m+}^*}^n\boldsymbol{X}\\ \xrightarrow{\mathcal{L}|\mathcal{F}} \sqrt{\tilde{\kappa}_m^*}\boldsymbol{\sigma}_{T_m-}\boldsymbol{\chi}_{m-}^* + \sqrt{k-\tilde{\kappa}_m^*}\boldsymbol{\sigma}_{T_m}\boldsymbol{\chi}_{m+}^*.\end{aligned} \tag{13}$$

We denote the trimming function by

$$f\left(\boldsymbol{x};\boldsymbol{v}_n\right) = \{\min\{\max\{\boldsymbol{x}, -\boldsymbol{v}_n\}, \boldsymbol{v}_n\}.$$

Since $(\Delta_{i(n,m)\pm j_{m\pm}^*}^n\boldsymbol{X})_{m\geq 1}$ do not contain jumps, $\Delta_{i(n,m)\pm j_{m\pm}^*}^n\boldsymbol{X} = f(\Delta_{i(n,m)\pm j_{m\pm}^*}^n\boldsymbol{X};\boldsymbol{v}_n)$ with probability approaching one. Moreover, by Proposition 1 in [18], $\mathcal{I}_n$ coincides with $\{i\left(n,m\right) : \Delta Z_{T_m} \neq 0, 0 \leq T_m \leq T, m \geq 1\}$ with probability approaching one. Then, from (13) and the definition of $\boldsymbol{\zeta}_n^*$, we deduce

$$\boldsymbol{\zeta}_n^* \xrightarrow{\mathcal{L}|\mathcal{F}} \boldsymbol{\zeta}. \tag{14}$$

Since $\hat{\boldsymbol{J}}_n \xrightarrow{\mathbb{P}} \boldsymbol{J}$, $\hat{\boldsymbol{U}}_{2n}$ and $\hat{\boldsymbol{V}}_{2n}$ converge in probability to that of $\boldsymbol{U}_2$ and $\boldsymbol{V}_2$, respectively, subject to normalization and identifying constraints, to which the functions $\|\hat{\boldsymbol{U}}_{2n}\boldsymbol{\zeta}_n^*\hat{\boldsymbol{V}}_{2n}^\top\|$ and $\|\boldsymbol{U}_2\boldsymbol{\zeta}\boldsymbol{V}_2^\top\|$ are invariant. Then, from (14), we have

$$\left\|\hat{\boldsymbol{U}}_{2n}\boldsymbol{\zeta}_n^*\hat{\boldsymbol{V}}_{2n}^\top\right\|^2 \xrightarrow{\mathcal{L}|\mathcal{F}} \left\|\boldsymbol{U}_2\boldsymbol{\zeta}\boldsymbol{V}_2^\top\right\|^2.$$

From the construction of $\boldsymbol{\zeta}$, it is easy to see that the $\mathcal{F}$-conditional distribution of $\boldsymbol{\zeta}$ is not degenerate. Therefore, the $\mathcal{F}$-conditional distribution of $\left\|\boldsymbol{U}_2\boldsymbol{\zeta}\boldsymbol{V}_2^\top\right\|^2$ is continuous. By Lemma 21.2 in [32], $cv_{n,\alpha}$ converges in probability to the $1-\alpha$ quantile of $\left\|\boldsymbol{U}_2\boldsymbol{\zeta}\boldsymbol{V}_2^\top\right\|^2$. The assertions on the asymptotic level and power property then readily follows from Theorem 1. *Q.E.D.*

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