AN APPLICATION OF HARDY-LITTLEWOOD TAUBERIAN THEOREM TO HARMONIC EXPANSION OF A COMPLEX MEASURE ON THE SPHERE

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Abstract. We apply Hardy-Littlewood’s Tauberian theorem to obtain an estimate on the harmonic expansion of a complex measure on the unit sphere, using a monotonicity property for positive harmonic functions.

Let $B^n = \{ x \in \mathbb{R}^n : |x| < 1 \}$, $n \geq 2$ be the unit ball in $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$ be the unit sphere. From a monotonicity property, we obtain a precise asymptotic for the spherical harmonic expansion of a complex measure on $S^{n-1}$ by applying the Tauberian theorem of Hardy and Littlewood.

It is known [1] that a positive harmonic function $u$ in $B^n$ can be uniquely represented by the Poisson kernel $P(x, y)$ and a positive measure $\mu$ on $S^{n-1}$ as

\[
(1) \quad u(x) = P[\mu](x) = \int_{S^{n-1}} P(x, \eta) d\mu(\eta) = \int_{S^{n-1}} \frac{1 - |x|^2}{|x - \eta|^n} d\mu(\eta).
\]

In the following we state a monotonicity property for positive harmonic functions as a theorem (Theorem 1), which is the special case $\delta = 0$ of Theorem 1.1 in [5]. A corollary (Corollary 2) on asymptotic results follow. Then we apply the monotonicity and the asymptotic property to obtain an estimate on the spherical harmonic expansion of a complex measure on $S^{n-1}$ (Theorem 3) by applying Hardy-Littlewood’s Tauberian Theorem. Two corollaries follow.

**Theorem 1.** (Theorem 1.1 in [5]) Let $u$ be a positive harmonic function in $B^n$, $\zeta \in S^{n-1}$. Then the function

\[
\frac{(1-r)^{n-1}}{1+r} u(r\zeta)
\]

is decreasing and the function

\[
\frac{(1+r)^{n-1}}{1-r} u(r\zeta)
\]

is increasing for $0 \leq r < 1$.

The following is needed to prove our main result in Theorem 3.
Corollary 2. Let $u$ be a positive harmonic function in $B^n$ defined by a positive measure $\mu$ as in (1). Then

$$\lim_{r \to 1} (1 - r)^{n-1} u(r\zeta) = 2\mu(\{\zeta\})$$

and

$$\lim_{r \to 1} \frac{u(r\zeta)}{1 - r} = \int_{S^{n-1}} \frac{2}{|\zeta - \eta|^n} d\mu(\eta).$$

Proof. Applying Theorem 1 to the Poisson kernel we obtain

$$\frac{(1 - r)^{n-1}}{1 + r} P(r\zeta, \eta) = \frac{(1 - r)^n}{|r\zeta - \eta|^n} \delta(\zeta, \eta) = \begin{cases} 1, & \zeta = \eta \\ 0, & \zeta \neq \eta \end{cases} \quad \text{as } r \to 1.$$ 

By the representation (1) and Lebesgue’s dominated convergence theorem,

$$\lim_{r \to 1} (1 - r)^{n-1} u(r\zeta) = \lim_{r \to 1} \frac{1}{1 - r} \int_{S^{n-1}} P(r\zeta, \eta) d\mu(\eta)$$

Similarly, $\frac{(1 + r)^{n-1}}{1 - r} P(r\zeta, \eta) = \frac{(1 + r)^n}{|r\zeta - \eta|^n}$ increases as $r \to 1$. By Lebesgue’s monotone convergence theorem,

$$\lim_{r \to 1} \frac{u(r\zeta)}{1 - r} = \lim_{r \to 1} \frac{1}{1 - r} \int_{S^{n-1}} P(r\zeta, \eta) d\mu(\eta)$$

Let $\mathcal{H}_m(S^{n-1})$ denote the complex vector space of spherical harmonics of degree $m$. $\mathcal{H}_m(S^{n-1})$ is the restriction to $S^{n-1}$ of the complex vector space $\mathcal{H}_m(\mathbb{R}^n)$ of homogeneous harmonic polynomials of degree $m$ in $\mathbb{R}^n$. It is known [1] that

$$\dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n + m - 1}{n - 1} - \binom{n + m - 3}{n - 1},$$

and that under the inner product $\langle p, q \rangle = \int_{S^{n-1}} p(x)q(x)d\sigma(x)$, where $d\sigma$ is the normalized Lebesgue measure on $S^{n-1}$, there exists an orthogonal decomposition of the Hilbert space of square-integrable functions on $S^{n-1}$,

$$\mathcal{L}^2(S^{n-1}) = \oplus_{m=0}^{\infty} \mathcal{H}_m(S^{n-1}).$$
By the property of finite dimensional Hilbert space, \( \forall \zeta \in S^{n-1} \), there exists a unique \( Z_m(\zeta, \cdot) \in \mathcal{H}_m(S^{n-1}) \) (the zonal function of pole \( \zeta \) and order \( m \)) such that

\[
p_m(\zeta) = \int_{S^{n-1}} p_m(\eta) Z_m(\zeta, \eta) d\sigma(\eta), \quad \forall p_m \in \mathcal{H}_m(S^{n-1}).
\]

The above leads to a zonal expansion of the Poisson kernel (Theorem 5.33 in [1])

\[
P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|^n} = \sum_{m=0}^{\infty} Z_m(x, \zeta), \quad \forall x \in B^n, \ \zeta \in S^{n-1}.
\]

Consequently, any complex measure on \( S^{n-1} \) has a spherical harmonic expansion

\[
\sum_{m=0}^{\infty} p_m(\zeta), \quad p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1}), \ \zeta \in S^{n-1}.
\]

If \( f \in L^2(S^{n-1}) \) and \( d\mu = f d\sigma \), then the spherical harmonic expansion for \( \mu \) converges to \( f \) in \( L^2(S^{n-1}) \). It is known [3] that if \( 1 \leq p < 2, n > 2 \) then there is an \( \phi \in L^p(S^{n-1}) \) with spherical harmonic expansion divergent almost everywhere. There have been studies of general theory of Cesàro summability on spherical harmonic expansions of \( L^p \) functions through estimates (e.g. [2]). In this paper, we consider spherical harmonic expansion of complex measures through asymptotics, which is the exact situation applicable by the Hardy-Littlewood Tauberian theory.

In the following we provide a precise asymptotics for the spherical harmonic expansion of complex measures on \( S^{n-1} \).

**Theorem 3.** Let \( \mu \) be a complex Borel measure on the unit sphere \( S^{n-1} \). Let \( \sum_{m=0}^{\infty} p_m(\zeta) \) be the spherical harmonic expansion of \( \mu \). Then

\[
\sum_{m=0}^{N} p_m(\zeta) \sim \frac{2}{(n-1)!} \mu(\{\zeta\}) N^{n-1} \quad \text{as} \quad N \to \infty.
\]

The proof of Theorem 3 is an application of the well-known Hardy-Littlewood Tauberian Theorem [4] stated below.

**Hardy-Littlewood Tauberian Theorem.** Assume that \( \sum_{m=0}^{\infty} a_m x^m \) converges on \( |x| < 1 \). Suppose that for some number \( \alpha \geq 0 \),

\[
\sum_{m=0}^{\infty} a_m x^m \sim \frac{A}{(1-x)^{\alpha}} \quad \text{as} \quad x \nearrow 1
\]

while

\[
am_m \geq -C m^{\alpha}, \quad m \geq 1,
\]

then

\[
\sum_{m=0}^{N} a_m \sim \frac{A}{\Gamma(\alpha+1)} N^\alpha.
\]
Another known result crucial in our proof is stated below as Lemma 4, which is a modified version of Corollary 5.34 in [1].

**Lemma 4.** Let $\mu$ be a complex measure on $S^{n-1}$ and $u(x) = P[\mu](x)$ as in (1). Then there exist $p_m \in \mathcal{H}_m(\mathbb{R}^n)$, $m = 0, 1, 2, \ldots$ such that
\[
u(x) = \sum_{m=0}^{\infty} p_m(x), \quad x \in B^n
\]
and the series converges absolutely and uniformly on compact subsets of $B^n$. Furthermore, there is a positive constant $C$ such that
\[
|p_m(x)| \leq C|\mu(S^{n-1})|n-2|x|^m, \quad m = 0, 1, 2, \ldots.
\]
If $x = |x|\zeta$, then $p_m(\zeta)$ is given by
\[
p_m(\zeta) = \int_{S^{n-1}} Z_m(x, \zeta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1}).
\]

**Proof.** Our proof of Lemma 4 is a modified version of the proof of Corollary 5.34 in [1] in terms of measures. By Theorem 5.33 of [1], the Poisson kernel expansion by zonal harmonics (2) converges absolutely and uniformly on $K \times S^{n-1}$ for every compact set $K \subset B^n$. So for any $x \in B^n$,
\[
u(x) = \int_{S^{n-1}} P(x, \zeta) d\mu(\zeta) = \sum_{m=0}^{\infty} \int_{S^{n-1}} P(x, \zeta) Z_m(x, \zeta) d\mu(\zeta) = \sum_{m=0}^{\infty} p_m(x)
\]
where
\[
p_m(x) = \int_{S^{n-1}} Z_m(x, \zeta) d\mu(\zeta), \quad x \in B^n.
\]
Since $p_m(x) \in \mathcal{H}_m(\mathbb{R}^n)$, so for $x = |x|\eta$, we have $Z_m(x, \zeta) = |x|^m Z_m(\eta, \zeta)$. Furthermore, it is known that [1]
\[
|Z_m(\eta, \zeta)| \leq \dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}
\]
By Pascal’s triangle,
\[
\dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2} = \frac{1}{(n-2)!} \left( \frac{n+2m-2}{m} \right) \frac{(n+m-3)!}{(m-1)!}
\]
Applying Stirling’s formula,
\[
\frac{\dim \mathcal{H}_m(\mathbb{R}^n)}{m^{n-2}} \to \frac{2}{(n-2)!} \quad \text{as} \quad m \to \infty.
\]
Therefore there exists $C = C(n) > 0$ such that $|Z_m(x, \zeta)| \leq Cm^{n-2}$ and
\[
|p_m(x)| \leq \int_{S^{n-1}} |Z_m(x, \zeta)| d\mu(\zeta) \leq C|\mu(S^{n-1})|m^{n-2}|x|^m.
\]
This completes the proof of Lemma 4.
Below is the proof of our main result.

**Proof of Theorem 3.** From the above results, for \( x = r\zeta, \zeta \in S^{n-1} \), we can write

\[
 u(x) = P[\mu](x) = \sum_{m=0}^{\infty} p_m(x) = \sum_{m=0}^{\infty} p_m(\zeta)r^m,
\]

and the last series converges for \(|r| < 1\) by Lemma 4. The complex Borel measure \( \mu \) can be decomposed as

\[
 \mu = \text{Re}(\mu) + i \text{IM}(\mu) = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)
\]

where \( \mu_j, j = 1, 2, 3, 4 \) are positive Borel measures. Applying Lemma 4 and Corollary 2 to the \( \mu_j \)'s and combining the resulting expansions, we have

\[
 \sum_{m=0}^{\infty} p_m(\zeta)r^m \sim \frac{2\mu(\{\zeta\})}{(1-r)^{n-1}} \quad \text{as} \quad r \uparrow 1
\]

Taking real and imaginary parts we have

\[
 \sum_{m=0}^{\infty} \text{Re}\{p_m(\zeta)\}r^m \sim \frac{2\text{Re}\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \quad \text{as} \quad r \uparrow 1
\]

and

\[
 \sum_{m=0}^{\infty} \text{Im}\{p_m(\zeta)\}r^m \sim \frac{2\text{Im}\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \quad \text{as} \quad r \uparrow 1.
\]

By Lemma 4, there exists a positive constant \( C \) so that

\[
 |p_m(\zeta)| \leq Cm^{n-2}
\]

It follows that

\[
 m \text{ Re}\{p_m(\zeta)\} \geq -Cm^{n-1}, \quad m \text{ Im}\{p_m(\zeta)\} \geq -Cm^{n-1}.
\]

Applying Hardy-Littlewood Tauberian Theorem with \( \alpha = n - 1 \) we obtain (3). This completes the proof of Theorem 3. \( \square \)

**Corollary 5.** Let \( \mu \) be a complex Borel measure on \( S^{n-1} \). If \( \mu(\{\zeta\}) > 0 \) for some \( \zeta \in S^{n-1} \) then the spherical expansion series of \( \mu \) is divergent:

\[
 \sum_{m=0}^{\infty} p_m(\zeta) = +\infty.
\]

If \( \mu(\{\zeta\}) = 0 \) then

\[
 \sum_{m=0}^{N} p_m(\zeta) = o(N^{n-1}).
\]

When the dimension of the space \( n = 2 \), the spherical expansion corresponds to Fourier series, and Theorem 3 has the following form which is a well known classical result.
Corollary 6. Let \( \mu \) be a complex Borel measure on \( S^1 \). Let

\[
\sum_{m=-\infty}^{\infty} a_m e^{im\theta}, \quad a_m = \int_{-\pi}^{\pi} e^{-im\theta} d\mu(e^{i\theta})
\]

be the Fourier series of \( \mu \). Then

\[
\sum_{m=-N}^{N} a_m e^{im\theta} \sim 2\mu(\{e^{i\theta}\})N \quad \text{as} \quad N \to \infty.
\]

Proof. In \( \mathbb{R}^2 \) the zonal functions are given by

\[
Z_m(e^{i\theta}, e^{i\phi}) = e^{im(\theta-\phi)} + e^{-im(\theta-\phi)}
\]

for \( m > 0 \), and \( Z_0(e^{i\theta}, e^{i\phi}) = 1 \). So Corollary 6 follows from Theorem 3. \( \square \)

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References


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