

**AN APPLICATION OF HARDY-LITTLEWOOD TAUBERIAN  
THEOREM TO HARMONIC EXPANSION OF A COMPLEX  
MEASURE ON THE SPHERE**

YIFEI PAN AND MEI WANG

ABSTRACT. We apply Hardy-Littlewood's Tauberian theorem to obtain an estimate on the harmonic expansion of a complex measure on the unit sphere, using a monotonicity property for positive harmonic functions.

Let  $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ ,  $n \geq 2$  be the unit ball in  $\mathbb{R}^n$  and  $S^{n-1} = \partial B^n$  be the unit sphere. From a monotonicity property, we obtain a precise asymptotic for the spherical harmonic expansion of a complex measure on  $S^{n-1}$  by applying the Tauberian theorem of Hardy and Littlewood.

It is known [1] that a positive harmonic function  $u$  in  $B^n$  can be uniquely represented by the Poisson kernel  $P(x, y)$  and a positive measure  $\mu$  on  $S^{n-1}$  as

$$(1) \quad u(x) = P[\mu](x) = \int_{S^{n-1}} P(x, \eta) d\mu(\eta) = \int_{S^{n-1}} \frac{1 - |x|^2}{|x - \eta|^n} d\mu(\eta).$$

In the following we state a monotonicity property for positive harmonic functions as a theorem (Theorem 1), which is the special case  $\delta = 0$  of Theorem 1.1 in [4]. A corollary (Corollary 2) on asymptotic results follow. Then we apply the monotonicity and the asymptotic property to obtain an estimate on the spherical harmonic expansion of a complex measure on  $S^{n-1}$  (Theorem 3) by applying Hardy-Littlewood's Tauberian Theorem. Two corollaries follow.

**Theorem 1.** *(Theorem 1.1 in [4]) Let  $u$  be a positive harmonic function in  $B^n$ ,  $\zeta \in S^{n-1}$ . Then the function*

$$\frac{(1 - r)^{n-1}}{1 + r} u(r\zeta)$$

*is decreasing and the function*

$$\frac{(1 + r)^{n-1}}{1 - r} u(r\zeta)$$

*is increasing for  $0 \leq r < 1$ .*

The following is needed to prove our main result in Theorem 3.

**Corollary 2.** *Let  $u$  be a positive harmonic function in  $B^n$  defined by a positive measure  $\mu$  as in (1). Then*

$$\lim_{r \rightarrow 1} (1-r)^{n-1} u(r\zeta) = 2\mu(\{\zeta\})$$

and

$$\lim_{r \rightarrow 1} \frac{u(r\zeta)}{1-r} = \int_{S^{n-1}} \frac{2}{|\zeta - \eta|^n} d\mu(\eta).$$

*Proof.* Applying Theorem 1 to the Poisson kernel we obtain

$$\frac{(1-r)^{n-1}}{1+r} P(r\zeta, \eta) = \frac{(1-r)^n}{|r\zeta - \eta|^n} \searrow \delta(\zeta, \eta) = \begin{cases} 1, & \zeta = \eta \\ 0, & \zeta \neq \eta \end{cases} \quad \text{as } r \rightarrow 1.$$

By the representation (1) and Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{r \rightarrow 1} (1-r)^{n-1} u(r\zeta) &= \lim_{r \rightarrow 1} (1-r)^{n-1} \int_{S^{n-1}} P(r\zeta, \eta) d\mu(\eta) \\ &= \lim_{r \rightarrow 1} (1+r) \int_{S^{n-1}} \lim_{r \rightarrow 1} \left\{ \frac{(1-r)^{n-1}}{1+r} P(r\zeta, \eta) \right\} d\mu(\eta) = 2\mu(\{\zeta\}). \end{aligned}$$

Similarly,  $\frac{(1+r)^{n-1}}{1-r} P(r\zeta, \eta) = \frac{(1+r)^n}{|r\zeta - \eta|^n}$  increases as  $r \rightarrow 1$ . By Lebesgue's monotone convergence theorem,

$$\begin{aligned} \lim_{r \rightarrow 1} \frac{u(r\zeta)}{1-r} &= \lim_{r \rightarrow 1} \frac{1}{1-r} \int_{S^{n-1}} P(r\zeta, \eta) d\mu(\eta) \\ &= \lim_{r \rightarrow 1} \frac{1}{(1+r)^{n-1}} \int_{S^{n-1}} \lim_{r \rightarrow 1} \left\{ \frac{(1+r)^{n-1}}{1-r} P(r\zeta, \eta) \right\} d\mu(\eta) \\ &= \frac{1}{2^{n-1}} \int_{S^n} \frac{2^n}{|\zeta - \eta|^n} d\mu(\eta) = \int_{S^n} \frac{2}{|\zeta - \eta|^n} d\mu(\eta). \end{aligned}$$

□

Let  $\mathcal{H}_m(S^{n-1})$  denote the complex vector space of spherical harmonics of degree  $m$ .  $\mathcal{H}_m(S^{n-1})$  is the restriction to  $S^{n-1}$  of the complex vector space  $\mathcal{H}_m(\mathbb{R}^n)$  of homogeneous harmonic polynomials of degree  $m$  in  $\mathbb{R}^n$ . It is known [1] that

$$\dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1},$$

and that under the inner product  $\langle p, q \rangle = \int_{S^{n-1}} p(x)q(x) d\sigma(x)$ , where  $d\sigma$  is the normalized Lebesgue measure on  $S^{n-1}$ , there exists an orthogonal decomposition of the Hilbert space of square-integrable functions on  $S^{n-1}$ ,

$$\mathcal{L}^2(S^{n-1}) = \bigoplus_0^\infty \mathcal{H}_m(S^{n-1}).$$

By the property of finite dimensional Hilbert space,  $\forall \zeta \in S^{n-1}$ , there exists a unique  $Z_m(\zeta, \cdot) \in \mathcal{H}_m(S^{n-1})$  (the *zonal function* of pole  $\zeta$  and order  $m$ ) such that

$$p_m(\zeta) = \int_{S^{n-1}} p_m(\eta) Z_m(\zeta, \eta) d\sigma(\eta), \quad \forall p_m \in \mathcal{H}_m(S^{n-1}).$$

The above leads to a zonal expansion of the Poisson kernel (Theorem 5.33 in [1])

$$(2) \quad P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|^n} = \sum_{m=0}^{\infty} Z_m(x, \zeta), \quad \forall x \in B^n, \zeta \in S^{n-1}.$$

Consequently, any complex measure on  $S^{n-1}$  has a spherical harmonic expansion

$$\sum_{m=0}^{\infty} p_m(\zeta), \quad p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1}), \zeta \in S^{n-1}.$$

If  $f \in L^2(S^{n-1})$  and  $d\mu = f d\sigma$ , then the spherical harmonic expansion for  $\mu$  converges to  $f$  in  $\mathcal{L}^2(S^{n-1})$ . It is known [2] that if  $1 \leq p < 2$  then there is an  $\phi \in L^p(S^{n-1})$  with spherical harmonic expansion divergent almost everywhere. In the following we provide a precise asymptotics for the spherical harmonic expansion of complex measures on  $S^{n-1}$ .

**Theorem 3.** *Let  $\mu$  be a complex Borel measure on the unit sphere  $S^{n-1}$ . Let  $\sum_{m=0}^{\infty} p_m(\zeta)$  be the spherical harmonic expansion of  $\mu$ . Then*

$$(3) \quad \sum_{m=0}^N p_m(\zeta) \sim \frac{2}{(n-1)!} \mu(\{\zeta\}) N^{n-1} \quad \text{as } N \rightarrow \infty.$$

The proof of Theorem 3 is an application of the well-known Hardy-Littlewood Tauberian Theorem [3] stated below.

**Hardy-Littlewood Tauberian Theorem.** *Assume that  $\sum_{m=0}^{\infty} a_m x^m$  converges on  $|x| < 1$ . Suppose that for some number  $\alpha \geq 0$ ,*

$$\sum_{m=0}^{\infty} a_m x^m \sim \frac{A}{(1-x)^\alpha} \quad \text{as } x \nearrow 1$$

while

$$m a_m \geq -C m^\alpha, \quad m \geq 1,$$

then

$$\sum_{m=0}^N a_m \sim \frac{A}{\Gamma(\alpha+1)} N^\alpha.$$

Another known result crucial in our proof is stated below as Lemma 4, which is a modified version of Corollary 5.34 in [1].

**Lemma 4.** *Let  $\mu$  be a complex measure on  $S^{n-1}$  and  $u(x) = P[\mu](x)$  as in (1). Then there exist  $p_m \in \mathcal{H}_m(\mathbb{R}^n)$ ,  $m = 0, 1, 2, \dots$  such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x), \quad x \in B^n$$

*and the series converges absolutely and uniformly on compact subsets of  $B^n$ . Furthermore, there is a positive constant  $C$  such that*

$$|p_m(x)| \leq C|\mu(S^{n-1})|m^{n-2}|x|^m, \quad m = 0, 1, 2, \dots$$

*If  $x = |x|\zeta$ , then  $p_m(\zeta)$  is given by*

$$p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1}).$$

*Proof.* Our proof of Lemma 4 is a modified version of the proof of Corollary 5.34 in [1] in terms of measures. By Theorem 5.33 of [1], the Poisson kernel expansion by zonal harmonics (2) converges absolutely and uniformly on  $K \times S^{n-1}$  for every compact set  $K \subset B^n$ . So for any  $x \in B^n$ ,

$$u(x) = \int_{S^{n-1}} P(x, \zeta) d\mu(\zeta) = \sum_{m=0}^{\infty} \int_{S^{n-1}} P(x, \zeta) Z_m(x, \zeta) d\mu(\zeta) = \sum_{m=0}^{\infty} p_m(x)$$

where

$$p_m(x) = \int_{S^{n-1}} Z_m(x, \zeta) d\mu(\zeta), \quad x \in B^n.$$

Since  $p_m(x) \in \mathcal{H}_m(\mathbb{R}^n)$ , so for  $x = |x|\eta$ , we have  $Z_m(x, \zeta) = |x|^m Z_m(\eta, \zeta)$ . Furthermore, it is known that [1]

$$|Z_m(\eta, \zeta)| \leq \dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$$

By Pascal's triangle,

$$\dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2} = \frac{1}{(n-2)!} \binom{n+2m-2}{m} \frac{(n+m-3)!}{(m-1)!}.$$

Applying Stirling's formula,

$$\frac{\dim \mathcal{H}_m(\mathbb{R}^n)}{m^{n-2}} \rightarrow \frac{2}{(n-2)!} \quad \text{as } m \rightarrow \infty.$$

Therefore there exists  $C = C(n) > 0$  such that  $|Z_m(x, \zeta)| \leq Cm^{n-2}$  and

$$|p_m(x)| \leq \left| \int_{S^{n-1}} |Z_m(x, \zeta)| d\mu(\zeta) \right| \leq C|\mu(S^{n-1})|m^{n-2}|x|^m.$$

This completes the proof of Lemma 4. □

Below is the proof of our main result.

*Proof of Theorem 3.* From the above results, for  $x = r\zeta$ ,  $\zeta \in S^{n-1}$ , we can write

$$u(x) = P[\mu](x) = \sum_{m=0}^{\infty} p_m(x) = \sum_{m=0}^{\infty} p_m(\zeta)r^m,$$

and the last series converges for  $|r| < 1$  by Lemma 4. The complex Borel measure  $\mu$  can be decomposed as

$$\mu = \text{Re}(\mu) + i \text{IM}(\mu) = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$$

where  $\mu_j$ ,  $j = 1, 2, 3, 4$  are positive Borel measures. Applying Lemma 4 and Corollary 2 to the  $\mu_j$ 's and combining the resulting expansions, we have

$$\sum_{m=0}^{\infty} p_m(\zeta)r^m \sim \frac{2\mu(\{\zeta\})}{(1-r)^{n-1}} \quad \text{as } r \nearrow 1$$

Taking real and imaginary parts we have

$$\sum_{m=0}^{\infty} \text{Re}\{p_m(\zeta)\}r^m \sim \frac{2 \text{Re}\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \quad \text{as } r \nearrow 1$$

and

$$\sum_{m=0}^{\infty} \text{Im}\{p_m(\zeta)\}r^m \sim \frac{2 \text{Im}\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \quad \text{as } r \nearrow 1.$$

By Lemma 4, there exists a positive constant  $C$  so that

$$|p_m(\zeta)| \leq Cm^{n-2}$$

It follows that

$$m \text{Re}\{p_m(\zeta)\} \geq -Cm^{n-1}, \quad m \text{Im}\{p_m(\zeta)\} \geq -Cm^{n-1}.$$

Applying Hardy-Littlewood Tauberian Theorem with  $\alpha = n - 1$  we obtain (3). This completes the proof of Theorem 3.  $\square$

**Corollary 5.** *Let  $\mu$  be a complex Borel measure on  $S^{n-1}$ . If  $\mu(\{\zeta\}) > 0$  for some  $\zeta \in S^{n-1}$  then the spherical expansion series of  $\mu$  is divergent:*

$$\sum_{m=0}^{\infty} p_m(\zeta) = +\infty.$$

*If  $\mu(\{\zeta\}) = 0$  then*

$$\sum_{m=0}^N p_m(\zeta) = o(N^{n-1}).$$

When the dimension of the space  $n = 2$ , the spherical expansion corresponds to Fourier series, and Theorem 3 has the following form.

**Corollary 6.** *Let  $\mu$  be a complex Borel measure on  $S^1$ . Let*

$$\sum_{m=-\infty}^{\infty} a_m e^{im\theta}, \quad a_m = \int_{-\pi}^{\pi} e^{-im\theta} d\mu(e^{i\theta})$$

*be the Fourier series of  $\mu$ . Then*

$$\sum_{m=-N}^N a_m e^{im\theta} \sim 2\mu(\{e^{i\theta}\})N \quad \text{as } N \rightarrow \infty.$$

*Proof.* In  $\mathbb{R}^2$  the zonal functions are given by

$$Z_m(e^{i\theta}, e^{i\phi}) = e^{im(\theta-\phi)} + e^{-im(\theta-\phi)}$$

for  $m > 0$ , and  $Z_0(e^{i\theta}, e^{i\phi}) = 1$ . So Corollary 6 follows from Theorem 3.  $\square$

#### REFERENCES

- [1] S. AXLER, P. BOURDON and W. RAMEY (2001) *Harmonic Function Theory*, 2nd ed. Springer, New York.
- [2] C. KENIG, R. STANTON and P. TOMAS (1982) *Divergence of eigenfunction expansions*, J. Funct. Anal. **46**, no. 1, 28-44.
- [3] J. KOREVAAR (2004) *Tauberian Theory: a century of developments*, Springer.
- [4] Y. PAN and M. WANG (2007) Monotonicity of Harnack inequality for positive invariant harmonic functions, *J. Appl. Math. Stoc. Anal. 2007*: doi:10.1155/2007/39171.

YIFEI PAN, DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY - PURDUE UNIVERSITY FORT WAYNE, FORT WAYNE, IN 46805-1499

SCHOOL OF MATHEMATICS AND INFORMATICS, JIANGXI NORMAL UNIVERSITY, NANCHANG, CHINA

*E-mail address:* pan@ipfw.edu

MEI WANG, DEPARTMENT OF STATISTICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637

*E-mail address:* meiwang@galton.uchicago.edu