A NOTE ON LINEAR FUNCTIONAL NORMS

YIFEI PAN AND MEI WANG

Abstract. For a vector $u$ in a normed linear space, Hahn-Banach Theorem provides the existence of a linear functional $f$, $f(u) = \|u\|$ such that $\|f\| = 1$. In this paper we prove the existence of functionals with $|f(u_j)| = \|u_j\|$ for linearly independent vectors and characterize the norm-one property $\|f\| = 1$ in terms of the triangle inequality.

Key words: Hahn-Banach Theorem; triangle inequality

Let $\mathcal{X}$ be a normed vector space over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathcal{X}^*$ be the space of $\mathbb{K}$-valued, continuous linear functionals on $\mathcal{X}$. For any $u \in \mathcal{X} \setminus \{0\}$, Hahn-Banach Theorem [1] gives the existence of $f \in \mathcal{X}^*, f(u) = \|u\|$ with the property $\|f\| = 1$. In this paper we prove the existence of $f \in \mathcal{X}^*, |f(u_j)| = \|u_j\|$ for linearly independent vectors $u_j$, $j = 1, \cdots, k$ and investigate the implications of the norm-one property of such functionals. This paper is motivated by the study of Schwarz lemma for higher order derivatives of holomorphic mappings from unit ball to itself in $\mathbb{C}^n$. The following are the results.

Theorem 1. Let $(\mathcal{X}, \| \cdot \|)$ be a normed vector space over $\mathbb{K}$. Let $u_1, \cdots, u_k \in \mathcal{X} \setminus \{0\}$ be linearly independent. Then $\exists f \in \mathcal{X}^*$ such that

$$|f(u_j)| = \|u_j\|, \quad j = 1, \cdots, k.$$ 

Remarks. Here the condition of linear independence is necessary (see Example 1 below). The functional in Theorem 1 has $\|f\| \geq 1$ by definition. The following shows that in most situations, $\|f\| > 1$.

2000 Mathematics Subject Classification. 46B20.
Proposition 1. Let \((\mathcal{X}, \| \cdot \|)\) be a normed vector space over \(\mathbb{K}\), \(u_1, \cdots, u_k \in \mathcal{X} \setminus \{0\}\). If \(\exists f \in \mathcal{X}^*\) such that \(\|f\| = 1\) and

\[
f(u_j) = \|u_j\|, \quad j = 1, \cdots, k,
\]

then the triangle equality holds, i.e.,

\[
\left\| \sum_{j=1}^{k} u_j \right\| = \sum_{j=1}^{k} \|u_j\|.
\]

Furthermore,

\[
\left\| \frac{u_{j_1}}{\|u_{j_1}\|} + \cdots + \frac{u_{j_m}}{\|u_{j_m}\|} \right\| = m, \quad \forall \{j_1, \cdots, j_m\} \subset \{1, \cdots, k\}.
\]

Remarks. The following figure is an example of the triangle equality case under maximum norm.

![Diagram](image)

**Figure 1.** Triangle equality case for \(k = 2\) under maximum norm.

Our proofs are based on applications of the Hahn-Banach Theorem [1]. Here we quote a corollary of the Hahn-Banach Theorem as a lemma.
Lemma 1. Let $L$ be a linear subspace of a normed vector space $X$ over $K$. Then for each $u_o \in X$ with $\text{dist}(u_o, L) = \inf_{u \in L} \| u_o - u \| > 0$, $\exists f \in X^*$ such that

$$f(u) = 0, \forall u \in L, \quad f(u_o) = \text{dist}(u_o, L), \quad \| f \| = 1.$$ 

Proof of Theorem 1. Let $L_1 = \text{span}\{ u_2, \cdots, u_k \}, \quad v_1 = \frac{1}{\| u_1 \|} u_1.$

$L_1$ is a finite dimensional subspace thus $L_1$ is closed. Since $v_1$ is independent of $u_2, \cdots, u_k$, $\text{dist}(u_1, L_1) > 0$ so

$$d_1 = \text{dist}(v_1, L_1) = \frac{1}{\| u_1 \|} \text{dist}(u_1, L_1) > 0.$$ 

By Lemma 1, $\exists h \in X^*$ such that

$$\| h \| = 1, \quad h(v_1) = d_1, \quad h(u_j) = 0, \forall j = 2, \cdots, k \quad (\text{since } u_j \in L_1).$$

Let

$$f_1 = \frac{1}{d_1} h = \frac{\| u_1 \|}{\text{dist}(u_1, L_1)} h = \frac{\| u_1 \|}{\text{dist}(u_1, \text{span}\{ u_2, \cdots, u_k \})} h.$$ 

Then $f_1 \in X^*$ and

$$\| f_1 \| = \frac{\| u_1 \|}{\text{dist}(u_1, L_1)}, \quad f_1(u_j) = 0, \forall j = 2, \cdots, k, \quad f_1(u_1) = \frac{1}{d_1} h(\| u_1 \| v_1) = \| u_1 \|.$$ 

Define

$$L_m = \text{span}\{ u_j, \; j \in \{ 1, \cdots, k \} \setminus \{ m \} \}, \quad m = 2, \cdots, k.$$ 

We can construct $f_m \in X^*$ in a similar way such that $\forall m = 2, \cdots, k,$

$$\| f_m \| = \frac{\| u_m \|}{\text{dist}(u_m, L_m)}, \quad f_m(u_j) = 0, \forall j \in \{ 1, \cdots, k \} \setminus \{ m \}, \quad f_m(u_m) = \| u_m \|.$$ 

Now let

$$f = f_1 + \cdots + f_k.$$
Then $f \in X^*$ and
\[ f(u_j) = \|u_j\| \quad \text{for} \quad j = 1, \cdots, k. \]
This completes the proof of Theorem 1. \qed

**Corollary 1.** Let $(X, \| \cdot \|)$ be a normed vector space over $\mathbb{K}$. Let $u_1, \cdots, u_k \in X \setminus \{0\}$ be linearly independent. Then $\exists f \in X^*$ and $\theta_1, \cdots, \theta_k \in [0, 2\pi)$ such that
\[ f(u_j^*) = \|u_j^*\| \quad \text{for} \quad u_j^* = e^{i\theta_j}u_j, \quad j = 1, \cdots, k. \]

**Proof.** By Theorem 1, there exists a $f \in X$ such that
\[ |f(u_j)| = \|u_j\|, \quad j = 1, \cdots, k. \]
Thus there are $\theta_1, \cdots, \theta_k \in [0, 2\pi)$ such that for $j = 1, \cdots, k$,
\[ f(u_j) = e^{-i\theta_j}\|u_j\|. \]
Consequently,
\[ f(u_j^*) = f(e^{i\theta_j}u_j) = e^{i\theta_j}f(u_j) = e^{i\theta_j}e^{-i\theta_j}\|u_j\| = \|u_j\| = \|e^{-i\theta_j}u_j\| = \|u_j^*\|. \]
\qed

The following example shows that Theorem 1 and Corollary 1 may not hold if the vectors are linearly dependent.

**Example 1.** Let $X = \mathbb{R}^2$ with the Euclidean norm. Let $u_1 = (1, 0), u_2 = (0, 1)$ and $u_3 = (1, 1)$. If there were a functional $f \in X^*$ such that $|f(u_j)| = \|u_j\|$ for $j = 1, 2, 3$, then $f(u_i) = \pm 1$ for $i = 1, 2$. By linearity $|f(u_3)| = |f(u_1 + u_2)| = |f(u_1) + f(u_2)| = 0$ or $2$, which contradicts $|f(u_3)| = \|u_3\| = \sqrt{2}$. Therefore $|f(u_j)| = \|u_j\|$ may not exist for linearly dependent $u_j$’s.
Proof of Proposition 1. From (1), \( f(u_j) = \|u_j\| \) for \( j = 1, \ldots, k \). If \( \|f\| = 1 \), by the linearity of \( f \) and the definition of functional norm,

\[
\frac{\|u_1\| + \cdots + \|u_k\|}{\|u_1 + \cdots + u_k\|} = \frac{|f(u_1 + \cdots + u_k)|}{\|u_1 + \cdots + u_k\|} \leq \sup_{u \in X \setminus \{0\}} \frac{|f(u)|}{\|u\|} = 1
\]

\[
\implies \|u_1\| + \cdots + \|u_k\| \leq \|u_1 + \cdots + u_k\|.
\]

The triangle inequality implies that the equality holds. Consequently, by applying the above result to subsets of the \( u_j \)'s we obtain

\[
\|u_{j_1} + \cdots + u_{j_m}\| = \|u_{j_1}\| + \cdots + \|u_{j_m}\|, \quad \forall \ \{j_1, \ldots, j_m\} \subset \{1, \ldots, k\}.
\]

Consider \( w_j = \frac{u_j}{\|u_j\|} \) for \( j = 1, \ldots, k \). Then \( f(w_j) = \|w_j\| = 1 \). Applying the above result to the \( w_j \)'s we have

\[
\left\| \sum_{j=1}^{k} \frac{u_j}{\|u_j\|} \right\| = \left\| \sum_{j=1}^{k} w_j \right\| = \sum_{j=1}^{k} \|w_j\| = k,
\]

and

\[
\left\| \frac{u_{j_1}}{\|u_{j_1}\|} + \cdots + \frac{u_{j_m}}{\|u_{j_m}\|} \right\| = \|w_{j_1} + \cdots + w_{j_m}\| = \|w_{j_1}\| + \cdots + \|w_{j_m}\| = m
\]

for any \( \{j_1, \ldots, j_m\} \subset \{1, \ldots, k\} \). This completes the proof of Proposition 1. \( \Box \)

Remark. Notice that our result is consistent with [2], where the sharpness of the triangle inequality was discussed thoroughly and extensively.

Corollary 2. Let \((X, \|\cdot\|)\) be a normed vector space over \( \mathbb{K} \) and \( u_1, \ldots, u_k \in X \setminus \{0\} \). If \( \exists f \in X^* \) such that

\[
f(u_j) = \|u_j\|, \quad j = 1, \ldots, k, \quad \left\| \sum_{j=1}^{k} u_j \right\| < \sum_{j=1}^{k} \|u_j\|,
\]

(3)
then

\[ \|f\| > 1. \]

**Proof.** By the definition of functional norm and (3),

\[ \|f\| = \sup_{u \in X \setminus \{0\}} \frac{|f(u)|}{\|u\|} \geq \frac{|f(u_1 + \cdots + u_k)|}{\|u_1 + \cdots + u_k\|} = \frac{\|u_1\| + \cdots + \|u_k\|}{\|u_1 + \cdots + u_k\|} > 1. \]

\[ \square \]

The example below illustrates that, when a functional \( f \) with \( |f(u_j)| = \|u_j\| \) for \( k \) linearly independent vectors exists (Theorem 1) and the triangle equality \( \| \sum u_j \| = \sum \| u_j \| \) holds (ref. (2) in Proposition 1), \( f \) is not necessarily of norm one. As indicated in Proposition 1, \( \|f\| = 1 \) is a very restricted condition. In fact the corresponding \( \|f\| \) in \( X \) can be arbitrarily large.

**Example 2.** Let

\[ X = C_0 = \left\{ u = (a_1, a_2, \cdots) = (a_n) : a_n \in \mathbb{K}, \lim_{n \to \infty} a_n = 0 \right\}, \quad \| (a_n) \| = \max_n |a_n|. \]

\((X, \| \cdot \|)\) is a complete normed vector space (Banach space) with the dual space

\[ X^* = C_0^* = \ell^1 = \left\{ f = (\alpha_1, \alpha_2, \cdots) = (\alpha_n) : \alpha_n \in \mathbb{K}, \sum_{n=1}^{\infty} |\alpha_n| < \infty \right\} \]

where

\[ f(u) = f_{\{\alpha_n\}} (u_{(a_n)}) = \sum_{n=1}^{\infty} \alpha_n a_n, \quad \|f\| = \sum_{n=1}^{\infty} |\alpha_n|. \]

For \( k > 0 \) and \( a_1, \cdots, a_k \) such that \( 0 < a_1 < a_2 < \cdots < a_k \), let

\[ u_1 = (a_1, 0, 0, \cdots), \quad u_2 = (a_2, a_1, 0, \cdots), \quad \cdots, \quad u_k = (a_k, a_{k-1}, \cdots, a_1, 0, \cdots). \]
Then \( u_1, \ldots, u_k \) are linearly independent vectors in \( C_0 \). For \( r \in (0, 1) \), consider

\[
f = \left\{ 1, 0, \ldots, 0, r, r^2, r^3, \ldots \right\} \in C_0^* = \ell^1.
\]

\( f \) has the property (as in Theorem 1)

\[
f(u_j) = a_j = \|u_j\|, \quad \forall j = 1, \ldots, k.
\]

Furthermore,

\[
\|u_j\| = a_j, \quad \forall j = 1, \ldots, k, \quad \|u_1 + \cdots + u_k\| = a_1 + \cdots + a_k \implies \left\| \sum_{j=1}^{k} u_j \right\| = \sum_{j=1}^{k} \|u_j\|.
\]

However

\[
\|f\| = \sum_{n=1}^{\infty} |\alpha_n| = \frac{1}{1 - r} > 1.
\]

Furthermore, for any \( u = (a_1, a_2, \ldots) \in C_o \) with \( \|u\| = 1 \), we have \( a_k \leq 1 \) for all \( k \) and \( a_j < 1 \) for some \( j \). Let \( v = (1, 1, \ldots) \). Then

\[
|f(u)| = \sum_{n=1}^{\infty} \alpha_n a_n < \sum_{n=1}^{\infty} \alpha_n = |f(v)| = \frac{1}{1 - r} = \|f\|.
\]

Thus

\[
|f(u)| < \|f\|, \quad \forall u \text{ with } \|u\| = 1,
\]

which implies that the functional \( f \) is not norm-attaining, that is,

\[
\frac{|f(u)|}{\|u\|} < \|f\|, \quad \forall u \in X \setminus \{0\}.
\]

**Remark.** An immediate corollary of the above results is that, if \( X \) is strictly convex and \( u_1, \ldots, u_k \in X \setminus \{0\} \) are linearly independent, then a functional \( f \) with \( |f(u_j)| = \|u_j\|, \quad j = 1, \ldots, k \), \( \|f\| = 1 \) implies

\[
\frac{|f(u_1)|}{\|u_1\|} = \frac{|f(u_2)|}{\|u_2\|} = \cdots = \frac{|f(u_k)|}{\|u_k\|}.
\]
REFERENCES


YIFEI PAN, DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY - PURDUE UNIVERSITY FORT WAYNE, FORT WAYNE, IN 46805-1499

School of Math and Informatics, Jiangxi Normal University, Nanchang, China
E-mail address: pan@ipfw.edu

MEI WANG, DEPARTMENT OF STATISTICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637
E-mail address: meiwang@galton.uchicago.edu