# WHEN IS A FUNCTION NOT FLAT?

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Abstract. In this paper we prove a unique continuation property for vector valued functions of one variable satisfying certain differential inequality.

*Key words: unique continuation, Carleman's method.*

## 1. Introduction

The function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is well known for its property

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \underline{\text{but}} \qquad f \not\equiv 0.$$

Such a function is called *flat* at the origin. On the other hand, if $f$ is a real analytic function with Taylor expansion on an open interval containing 0, then $f^{(k)}(0) = 0$, $\forall k \geq 0$ implies $f \equiv 0$. The unique continuation problem in PDE is to find conditions such that the solutions of PDE enjoy the same property. There are a large amount of literature in this area originated from the ideas by Carleman [1], called Carleman's method. In this paper we consider the simplest case of one variable. The following is the main theorem. The results can be generalized to complex or vector valued function case (Theorem 5). The proofs are in the subsequent sections.

**Theorem 1.** *Let $f(x) \in C^\infty([a,b])$, $0 \in [a,b]$, and*

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}}, \quad x \in [a,b] \tag{1}$$

*for some constant $C$ and $n \geq 2$. Then*

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \text{implies} \qquad f \equiv 0 \quad \textit{on} \quad [a,b].$$

**Remarks.** The conditions in Theorem 1 are optimal in the following sense: the infinite order vanishing $f^{(k)}(0) = 0$, $\forall k \geq 0$ can not be relaxed (e.g. $f(x) = x^N$), and the singularity order in (1) is sharp, as illustrated in the example given after Corollary 2.

From Theorem 1 we obtain the following corollary.

**Corollary 2.** *Let $f(x) \in C^\infty([a,b])$, $0 \in [a,b]$, and (1) hold for some constant $C$ with $n \geq 2$. Then*

$$f \not\equiv 0 \qquad \text{implies} \qquad \textit{the zero set} \quad \{f^{-1}(0)\} \subset [a,b] \quad \textit{is finite.}$$

**An example.**

We use an example in Garofalo and Lin [2] to show that the order of vanishing in (1) is sharp in the following sense. There exists a function $f(x) \in C^\infty([-a,a])$, $a > 0$,

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k+\varepsilon}} \quad for \quad x \in [-a,a] \qquad and \qquad f^{(k)}(0) = 0, \ \forall k \geq 0$$

for some constant $C$ and $\varepsilon > 0$, but $f \not\equiv 0$ on $[-a,a]$.

For $m > 1$, $\varepsilon > 0$, the following equation is considered in [2]:

$$x^2 u''(x) + mxu'(x) - cx^{-\varepsilon} u(x) = 0, \qquad x \in (0,1),$$

or equivalently,

$$u''(x) + \frac{m}{x} u'(x) - \frac{c}{x^{2+\varepsilon}} u(x) = 0, \qquad x \in (0,1), \tag{2}$$

— a Bessel differential equation [4]. The general Bessel differential equation takes the form

$$z^2 u''(z) + (1-2\alpha) z u'(z) + \left\{ \beta^2 \gamma^2 z^{2\gamma} + (\alpha^2 - \nu^2 \gamma^2) \right\} u(z) = 0, \qquad z \in \mathbb{C}. \tag{3}$$

It is well known that for (non-integer) $\nu \notin \mathbb{Z}$, the solution for (3) is

$$u(z) = z^\alpha \left[ \, C_1 J_\nu(\beta z^\gamma) + C_2 J_{-\nu}(\beta z^\gamma) \, \right],$$

where $C_1$, $C_2$ are arbitrary complex numbers, and $J_\nu$ is the Bessel function of order $\nu$, i.e., a solution of equation (3) with $\alpha = 0$, $\gamma = 1$. Notice that equation (2) with $c > 0$ is equation (3) with

$$\alpha = -\frac{m-1}{2}, \quad \beta = i\frac{2\sqrt{c}}{\varepsilon}, \quad \gamma = -\frac{\varepsilon}{2}, \quad \nu = \frac{m-1}{\varepsilon}.$$

By choosing $m > 1$, $\varepsilon \in (0,1)$ such that

$$C_1 = - \, C_2 = -\frac{\pi e^{-i\nu\pi}}{2 \, \sin(\nu\pi)}, \qquad \nu = \frac{m-1}{\varepsilon} \notin \mathbb{Z},$$

the solution of equation (2) can be written as

$$u(x) = |x|^{-(m-1)/2} K_{(m-1)/\varepsilon} \left( \frac{2\sqrt{c}}{\varepsilon} |x|^{-\varepsilon/2} \right), \qquad x \in (0,1) \tag{4}$$

where

$$K_\nu(z) = \frac{\pi}{2} \, \frac{e^{-i\nu\pi} \left( J_{-\nu}(iz) - J_\nu(iz) \right)}{\sin(\nu\pi)}, \qquad \arg z \in (-\pi, \pi/2)$$

is the modified Bessel function of the third kind [4], with the asymptotic property

$$K_\nu(x) \approx \frac{\pi}{2} x^{-1/2} e^{-x} \qquad as \quad x \to +\infty.$$

Therefore in (4), the function

$$u(x) \approx \frac{\pi}{2}\left(\frac{2\sqrt{c}}{\varepsilon}\right)^{-1/2} x^{-\frac{m-1}{2}+\frac{\varepsilon}{4}} \exp\left\{-\frac{2\sqrt{c}}{\varepsilon}x^{-\varepsilon/2}\right\} \qquad as \quad x \to 0$$

is a nontrivial solution of (2) vanishing at $x = 0$ of infinite order. Hence

$$f(x) = u(|x|), \qquad x \in [-a, a], \quad a \in (0, 1)$$

is well defined and $f \in \mathcal{C}^\infty([-a, a])$. Taking derivatives of equation (2) $n-2$ times,

$$\frac{d^{n-2}}{dx^{n-2}}\left\{u''(x) + \frac{m}{x}u'(x) - \frac{c}{x^{2+\varepsilon}}u(x)\right\} = 0, \qquad x \in (0, 1),$$

we obtain

$$u^{(n)}(x) + a_{n-1}(x)u^{(n-1)}(x) + \cdots + a_0(x)u(x) = 0, \qquad x \in (0, 1).$$

For given $m,\ c$ and $\varepsilon$, the coefficients

$$|a_j(x)| \le C_o\left(\frac{1}{|x|^{n-j+\varepsilon}}\right), \quad j = 0, 1, \cdots, n-1, \quad x \in (0, 1)$$

for some constant $C_o > 0$. By the property of $u(x)$ near $x = 0$, we have

$$f^{(n)}(x) + a_{n-1}(x)f^{(n-1)}(x) + \cdots + a_0(x)f(x) = 0, \qquad x \in [-a, a]$$

with

$$f^{(k)}(0) = 0,\ \forall k \ge 0, \qquad |a_j(x)| \le C\left(\frac{1}{|x|^{n-j+\varepsilon}}\right), \quad j = 0, 1, \cdots, n-1.$$

for some constant $C > 0$. Thus

$$f^{(k)}(0) = 0,\ \forall k \ge 0, \qquad |f^{(n)}(x)| \le C\sum_{k=0}^{n-1}\frac{|f^{(k)}(x)|}{|x|^{n-k+\varepsilon}}, \quad x \in [-a, a],$$

and $f \not\equiv 0$ from the non-triviality of $u$.

Theorem 1 leads to applications in ODE as stated in the following two propositions. Based on the above example, the order of the singularity of the coefficients in the assumption of the propositions is sharp.

**Proposition 3.** *Let $f(x) \in \mathcal{C}^\infty$ be a solution of*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0, \qquad x \in [-a, a],\ a > 0$$

*with*

$$|a_k(x)| = O\left(\frac{1}{|x|^{n-k}}\right) \qquad as \quad x \to 0, \qquad k = 0, 1, \cdots, n-1.$$

*Then*

$$f^{(k)}(0) = 0, \quad \forall k \ge 0 \qquad \Longrightarrow \qquad f \equiv 0 \quad on \quad [-a, a].$$

**Proposition 4.** *Let $f(x),\ g(x) \in \mathcal{C}^\infty$ be solutions of*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = b(x), \qquad x \in [-a, a],\ a > 0$$

*with*

$$|a_k(x)| = O\left(\frac{1}{|x|^{n-k}}\right) \qquad as \quad x \to 0, \qquad k = 0, 1, \cdots, n-1.$$

*Then*

$$f^{(k)}(0) = g^{(k)}(0), \quad \forall k \geq 0 \qquad \Longrightarrow \qquad f \equiv g \quad on \quad [-a, a].$$

The result in Theorem 1 can be generalized to complex or vector valued functions as stated in the following theorem.

**Theorem 5.** *Let $0 \in [a, b]$, $f(x) = u(x) + iv(x)$, where $u(x),\ v(x) \in C^\infty([a, b])$, or $f(x) = (u_1(x), \cdots, u_d(x))$ for some $d \geq 1$, where $u_1(x),\ u_2(x), \cdots, u_d(x) \in C^\infty([a, b])$. Assume*

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}}, \quad x \in [a, b] \tag{5}$$

*for some constant $C$ and $n \geq 2$. Then*

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \text{implies} \qquad f \equiv 0 \quad on \quad [a, b].$$

**Remarks.** Theorem 5 has interesting geometric meanings. A $d$-dimensional curve $((u_1(t), \cdots, u_d(t)), d \geq 2$, satisfying (5) may have at most finite order singularity at $t = 0$, which means there must be at least one component $u_i$ with $u_i^{(k)}(0) \neq 0$ for some $k \geq 0$.

## 2. Proof of Theorem 1 and its corollary

Several lemmas are needed for the proof of Theorem 1. The basic idea of the following lemma was considered in [3].

**Lemma 6.** *Let $v(x) \in C^\infty([0, b])$. Assume $v^{(k)}(0) = 0$, $\forall k \geq 0$. Then for $\alpha \geq 1$,*

$$\int_0^b \frac{[v(x)]^2}{x^{\alpha+2}} dx \leq \frac{4}{(\alpha+1)^2} \int_0^b \frac{[v'(x)]^2}{x^\alpha} dx \tag{6}$$

*Proof.*

$$\frac{d}{dx}\left(x^{-(\alpha+1)} v^2(x)\right) = -(\alpha+1)x^{-(\alpha+2)} v^2(x) + x^{-(\alpha+1)} 2v(x)v'(x)$$

Since $v^{(k)}(0) = 0$ for $k \geq [\alpha/2] + 1$, we have $[v(x)]^2 \sim O(x^{2(k+1)}) = o(x^{\alpha+1})$, thus

$$\int_0^b \frac{d}{dx}\left(x^{-(\alpha+1)} v^2(x)\right) dx = \frac{[v(b)]^2}{b^{\alpha+1}} - \lim_{x \to 0+} \frac{[v(x)]^2}{x^{\alpha+1}} = \frac{[v(b)]^2}{b^{\alpha+1}} \geq 0.$$

Therefore,

$$
\begin{aligned}
(\alpha+1)\int_0^b \frac{v^2(x)}{x^{\alpha+2}}dx &\le \int_0^b \frac{2v(x)v'(x)}{x^{\alpha+1}}dx \\
&= \int_0^b 2\left\{\left(\frac{\alpha+1}{2}\right)^{1/2}\frac{v(x)}{x^{(\alpha+2)/2}}\right\}\left\{\left(\frac{2}{\alpha+1}\right)^{1/2}\frac{v'(x)}{x^{\alpha/2}}\right\}dx \\
&\le \int_0^b \frac{\alpha+1}{2}\frac{v^2(x)}{x^{\alpha+2}}dx + \int_0^b \frac{2}{\alpha+1}\frac{(v'(x))^2}{x^{\alpha}}dx
\end{aligned}
$$

by applying $2ab \le a^2+b^2$ to the last inequality. Consequently,

$$
\frac{\alpha+1}{2}\int_0^b \frac{v^2(x)}{x^{\alpha+2}}dx \le \frac{2}{\alpha+1}\int_0^b \frac{(v'(x))^2}{x^{\alpha}}dx,
$$

which is equivalent to (6). □

**Lemma 7.** *Let* $u(x) \in C^\infty([0,b])$. *Assume* $u^{(k)}(0) = 0,\ \forall k \ge 0$. *Then for* $\beta \ge 1,\ n \ge 1,$

$$
\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx \le \frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^{\beta}}dx, \qquad for \quad k = 0,\cdots,n-1.
$$

*Proof.* Applying Lemma 6 to $v(x) = u^{(n-1-j)}(x),\ \alpha = \beta+2j,\ \beta \ge 1,\ j = 0,1,\cdots,n-1$, we obtain

$$
\int_0^b \frac{\left[u^{(n-1-j)}(x)\right]^2}{x^{\beta+2j+2}}dx \le \frac{4}{(\beta+2j+1)^2}\int_0^b \frac{[u^{(n-j)}(x)]^2}{x^{\beta+2j}}dx, \quad j = 0,1,\cdots,n-1,
$$

or equivalently,

$$
\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx \le \frac{4}{(\beta+2(n-k-1)+1)^2}\int_0^b \frac{[u^{(k+1)}(x)]^2}{x^{\beta+2(n-k-1)}}dx, \quad k = 0,1,\cdots,n-1.
$$

Applying the above inequality repeatedly, we have

$$
\begin{aligned}
\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx &\le \left\{\prod_{m=k}^{n-2}\frac{4}{(\beta+2(n-1-m)+1)^2}\right\}\frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^{\beta}}dx \\
&\le \frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^{\beta}}dx,
\end{aligned}
$$

because $\beta \ge 1,\ \beta+2(n-m-1)+1 \ge 2$ for $m = 0,\ldots,n-1$. □

**Lemma 8.** *Let* $u(x) \in C^\infty([0,b])$. *Assume* $u^{(k)}(0) = 0,\ \forall k \ge 0$. *Then for* $\beta \ge 1,\ n \ge 1,$

$$
\int_0^b \frac{1}{x^\beta}\sum_{k=0}^{n-1}\left(\frac{u^{(k)}(x)}{x^{n-k}}\right)^2 dx \quad \le \quad \frac{4n}{(\beta+1)^2}\int_0^b \frac{1}{x^\beta}[u^{(n)}(x)]^2 dx.
$$

*Proof.* Apply Lemma 7 to $k = 0, \cdots, n-1$ and sum up both sides. $\square$

**Lemma 9.** *Let* $f(x) \in C^\infty([a, b]),\ 0 \in [a, b]$. *Assume* $f^{(k)}(0) = 0,\ \forall k \geq 0$. *Then for* $\beta \geq 1,\ n \geq 1$,

$$\int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{f^{(k)}(x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \frac{4n}{(\beta+1)^2} \int_a^b \frac{1}{|x|^\beta} [f^{(n)}(x)]^2 dx.$$

*Proof.* The function $u(x) = f(-x)$ satisfies the assumptions for Lemma 8 on $[0, -a]$, so

$$\int_0^{-a} \frac{1}{x^\beta} \sum_{k=0}^{n-1} \left( \frac{f^{(k)}(-x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \frac{4n}{(\beta+1)^2} \int_0^{-a} \frac{1}{x^\beta} [f^{(n)}(-x)]^2 dx.$$

Substitute the variable $x$ by $-x$. Since the terms in the sum are of even powers and both sides have the same $x^\beta$ term, the above inequality can be written as

$$\int_a^0 a \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{f^{(k)}(x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \frac{4n}{(\beta+1)^2} \int_a^0 \frac{1}{|x|^\beta} [f^{(n)}(x)]^2 dx.$$

From Lemma 8 the desired inequality is already true for $f(x)$ on $[0, b]$. Combining the results on $[a, 0]$ and $[a, b]$, Lemma 9 follows. $\square$

The following is the proof of Theorem 1.

*Proof.* $f(x)$ satisfies the assumptions in Lemma 9 on $[a, b]$, so for any $\beta \geq 1$,

$$\frac{(\beta+1)^2}{4n} \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{f^{(k)}(x)}{x^{n-k}} \right)^2 dx \leq \int_a^b \frac{1}{|x|^\beta} [f^{(n)}(x)]^2 dx. \tag{7}$$

From (1),

$$|f^{(n)}(x)|^2 \leq C_1 \sum_{k=0}^{n-1} \left( \frac{|f^{(k)}(x)|}{|x|^{n-k}} \right)^2$$

for some $C_1 > 0$, thus

$$\int_a^b \frac{1}{|x|^\beta} |f^{(n)}(x)|^2 dx \leq C_1 \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{|f^{(k)}(x)|}{|x|^{n-k}} \right)^2 dx. \tag{8}$$

Combining (7) and (8) we have

$$\int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{f^{(k)}(x)}{x^{n-k}} \right)^2 dx \leq \frac{4nC_1}{(\beta+1)^2} \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{|f^{(k)}(x)|}{|x|^{n-k}} \right)^2 dx.$$

If $f \not\equiv 0$ on $[a,b]$, we would have $|f^{(k)}(x)| > 0$ on some sub-interval of $[a,b]$ thus the integrals $> 0$, which would imply

$$1 \leq \frac{4nC_1}{(\beta+1)^2}, \qquad \forall \beta \geq 1 \qquad \Longrightarrow \quad \textit{Contradiction.}$$

Therefore $f(x) \equiv 0$ on $[a,b]$. This completes the proof of Theorem 1. □

The proof of Corollary 2 follows immediately.

*Proof.* Under the assumption of Corollary 2, $f \not\equiv 0$ implies $f^{(k)}(0) = \beta \neq 0$ for some $k \geq 0$ based on the result of Theorem 1. If $\beta > 0$, then $f \in \mathcal{C}^\infty$ yields $f^{(k)}(x) > \beta/2 > 0$, $x \in [-\delta_0, \delta_0] \subset [a,b]$ for some $\delta_0 > 0$. Consequently the $k$-fold integral

$$f(x) = \int_0^x \cdots \int_0^x f^{(k)}(y)dy \cdots dy > x^k\beta/2 > 0, \qquad 0 < |x| < \delta_0.$$

The case $\beta < 0$ implies $f(x) < 0$ on an open interval containing 0.

For any $x' \in [a,b]$ such that $f(x') = 0$, the condition $f \not\equiv 0$ implies $f(x) \neq 0$, $0 < |x - x'| < \delta_{x'}$, $x \in [a,b]$ for some $\delta_{x'} > 0$. By the compactness of $[a,b]$, the zero set $\{f^{-1}(0)\}$ is at most finite in $[a,b]$. This completes the proof of Corollary 2. □

## 3. Proof of Theorem 5

*Proof.* First consider the complex valued case $f(x) = u(x) + iv(x)$, $u(x)$, $v(x) \in C^\infty([a,b])$. Then

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \text{implies} \qquad u^{(k)}(0) = 0,\ v^{(k)}(0) = 0 \quad \forall k \geq 0.$$

Therefore both $u(x)$ and $v(x)$ satisfy Lemma 9 on $[a,b]$, so for any $\beta \geq 1$,

$$\frac{(\beta+1)^2}{4n} \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{u^{(k)}(x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \int_a^b \frac{1}{|x|^\beta} [u^{(n)}(x)]^2 dx,$$

$$\frac{(\beta+1)^2}{4n} \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{v^{(k)}(x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \int_a^b \frac{1}{|x|^\beta} [v^{(n)}(x)]^2 dx.$$

Since

$$\left| f^{(k)}(x) \right|^2 = \left| u^{(k)}(x) \right|^2 + \left| v^{(k)}(x) \right|^2, \qquad \forall k \geq 0,$$

we have inequality (7) for $|f(x)|$:

$$\frac{(\beta+1)^2}{4n} \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{\left|f^{(k)}(x)\right|}{x^{n-k}} \right)^2 dx \quad \leq \quad \int_a^b \frac{1}{|x|^\beta} [f^{(n)}(x)]^2 dx.$$

The rest of the proof is the same as in that for Theorem 1. From the assumption in Theorem 5 we have

$$|f^{(n)}(x)|^2 \leq C_1 \sum_{k=0}^{n-1} \left( \frac{|f^{(k)}(x)|}{|x|^{n-k}} \right)^2$$

for some $C_1 > 0$, thus

$$\int_a^b \frac{1}{|x|^\beta} |f^{(n)}(x)|^2 dx \leq C_1 \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{|f^{(k)}(x)|}{|x|^{n-k}} \right)^2 dx.$$

Combining the above we have

$$\int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{|f^{(k)}(x)|}{x^{n-k}} \right)^2 dx \leq \frac{4nC_1}{(\beta+1)^2} \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left( \frac{|f^{(k)}(x)|}{|x|^{n-k}} \right)^2 dx.$$

If $f \not\equiv 0$ on $[a,b]$, we would have $|f^{(k)}(x)| > 0$ on some sub-interval of $[a,b]$ thus the integrals $> 0$, which would imply

$$1 \leq \frac{4nC_1}{(\beta+1)^2}, \qquad \forall \beta \geq 1 \qquad \Longrightarrow \quad \textit{Contradiction.}$$

Therefore $f(x) \equiv 0$ on $[a,b]$. The proof for vector valued case is immediate, because

$$\left| f^{(k)}(x) \right|^2 = \left| u_1^{(k)}(x) \right|^2 + \cdots + \left| u_d^{(k)}(x) \right|^2, \qquad \forall k \geq 0$$

holds. This completes the proof of Theorem 5. □

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