Posterior Distribution for Negative Binomial Parameter p Using a Group Invariant Prior *

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Abstract

We obtain a noninformative prior measure for the p parameter of the negative binomial distribution by use of a group theoretic method. Heretofore, group theoretic inference methods have not been applicable in the case of discrete distributions. A linear representation of a group leads to quantities whose squared moduli constitute the probability distribution. The group invariant measure yields prior measure dp/p^2.

Key words: Statistical inference, group invariant measure, noninformative prior, Bayesian posterior distribution, coherent states.

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1 Introduction

The object of this paper is to construct a posterior distribution for the parameter p of the negative binomial distribution by the use of a non-informative prior obtained from group theoretic methods. Efron (1998) has indicated a relationship between the fiducial method of inference and the Bayesian method as follows. ".... By 'objective' Bayes I mean a Bayesian theory in which the subjective element is removed form the choice of prior distributions, in practical terms a universal recipe for applying Bayes theory in the absence of prior information. A widely accepted objective Bayes theory, which fiducial inference was intended to be, would be of immense theoretical and practical importance."

Here we demonstrate a group invariant method of obtaining such a prior. We illustrate the method by an example which is associated with a particular matrix group; namely $SU(1,1)$ (defined in Section 2.1). The choice of this group is determined by the fact that the negative binomial family is obtained by the action of this group when it is represented by certain linear operators acting in a certain Hilbert space. Quantities which may be characterized as complex valued "square roots" of the negative binomial family are obtained by expanding a family of vectors of the Hilbert space with respect to a discrete basis. The squared moduli of those square root quantities are then the probabilities constituting the negative binomial family. Similar results for the binomial and Poisson families were described in Heller and Wang (2006).

Group theoretic methods for inferential and other purposes abound in the statistical literature, for example, Fraser (1961), Eaton (1989), Helland (2004), Kass and Wasserman (1996), and many others. In all of these accounts, the group acts on the sample space of the statistic of interest as well as on the parameter space of the postulated statistical model. That requirement is not met by discrete families with continuous parameter spaces, many of which are useful in statistical inference. For example, the fiducial method of inference as described in Fraser (1961) would not be applicable to discrete distributions; a remark also attributable to R. A. Fisher. However, Efron's view of it as quoted above, which interprets fiducial inference in terms of Bayesian posteriors, opens the door to a possible group theoretic approach which is different from that of Fraser. In this paper we describe such an approach.

In Section 2.1 we briefly describe the matrix group $SU(1,1)$. Section 2.2 describes a representation of the group by certain linear operators in a Hilbert space and identifies an orthonormal basis for that space. Section 2.3 depicts a family of vectors in the Hilbert space which act as generating functions for the quantities whose squared moduli constitute the negative binomial family. In Section 2.4, the relation of the negative binomial parameter space to the space of group parameters is established. In Section 2.5, a measure on this space is described which is invariant to the action of the group and this measure is used to obtain a posterior distribution for the negative binomial parameter p.

2 The method and results

2.1 The group related to the negative binomial distribution

The negative binomial distribution derives from the matrix group $G = SU(1,1)$. Matrices g of $SU(1,1)$ may be written in the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1, \qquad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix}.$$

The group may be viewed as a transformation group on the complex unit disc $\mathbb{D} = \{w \in \mathbb{C}, |w| < 1\}$. An element g of G acts on $\mathbb{D}$ as a linear fractional transformation defined by

$$w \to gw = \frac{\alpha w + \beta}{\bar{\beta} w + \bar{\alpha}}. \tag{1}$$

Note that $|gw| < 1$ whenever $|w| < 1$ since $|\alpha|^2 - |\beta|^2 = 1$, and these transformations satisfy the conditions for a transformation group (for any $g_1, g_2 \in G, w \in \mathbb{D}$, $(g_1 g_2)w = g_1(g_2 w)$). The group acts transitively on $\mathbb{D}$: there is a $g \in G$ such that $w_2 = g w_1$ for any $w_1, w_2 \in \mathbb{D}$.

2.2 A concrete representation of the group

We consider a linear space (a representation space of the group) wherein we can construct a family of vectors which act as generating functions leading to the negative binomial distribution.

For each integer or semi-integer $k \geq 1$, the linear space considered is a complex Hilbert space $\mathcal{H}_k$ of functions $f(z), z \in \mathbb{D}$ which satisfy the following conditions:

1. The functions are analytic on $\mathbb{D}$.

2. The functions are square-integrable with respect to the measure

$$d\mu_k(z) = \frac{2k-1}{\pi}(1-|z|^2)^{2k-2} d^2 z, \qquad z = x + iy, \; d^2 z = dxdy.$$

The inner product in $\mathcal{H}_k$ is given by $(f_1, f_2) = \int_{\mathbb{D}} \overline{f_1(z)} f_2(z) d\mu_k(z)$. Notice that it is linear in the second argument and complex conjugate linear in the first argument. An orthonormal basis for $\mathcal{H}_k$ is $\{\Phi_m^k(z)\}, \; m = 0, 1, 2, \cdots$ where

$$\Phi_m^k(z) = c_m z^m, \quad c_m = \left(\frac{\Gamma(m+2k)}{m! \; \Gamma(2k)} \right)^{1/2}. \tag{2}$$

On this space we now describe linear representation operators of the group G (nonsingular continuous linear transformations $T(g)$ satisfying $T(g_1 g_2) = T(g_1)T(g_2)$ and $T(e) = I$, the identity operator) that will be used to construct the aforementioned generating functions. To each $g \in G$, associate the linear operator

$$(U_k(g)f)(z) = (-\bar{\beta}z + \alpha)^{-2k} f(g^{-1}z) = (-\bar{\beta}z + \alpha)^{-2k} f\left(\frac{\bar{\alpha}z - \beta}{-\bar{\beta}z + \alpha} \right), \quad f \in \mathcal{H}_k, \; g \in G. \tag{3}$$

This representation is a "multiplier representation". The factor $(-\bar{\beta}z+\alpha) = A(g^{-1}, z)$ has the properties of an automorphic factor (i.e. $A(g_1g_2, z) = A(g_1, g_2z)A(g_2, z)$ and $A(e, z) = 1$) in order that the representation be a homomorphism. It is a standard result that $\{\Phi^k_m(z)\}$ is a complete orthonormal system in $\mathcal{H}_k$ and for each k, the representation given above is unitary (i.e. $(U_k(g)f_1, U_k(g)f_2) = (f_1, f_2)$ for all $f_1, f_2 \in \mathcal{H}_k$) and irreducible (no non-trivial subspace $\mathcal{H}_o \subset \mathcal{H}_k$ such that for all $f_o \in \mathcal{H}_o$, $U_k(g)f_o \in \mathcal{H}_o$ for all $g \in G$); see, for example, Sugiura (1990) and Ali, Antoine and Gazeau (2000).

2.3 A generating function for quantities whose squared moduli constitute the negative binomial distribution

For basis function $\Phi^k_0(z) \equiv 1$, from (3) we have

$$U_k(g)\Phi^k_0(z) = (-\bar{\beta}z+\alpha)^{-2k} = \alpha^{-2k}(1-(\bar{\beta}/\alpha)z)^{-2k}. \tag{4}$$

Reparameterize

$$(\alpha,\beta) \to (\zeta, t), \quad by \quad \alpha = |\alpha|e^{-it/2},\ t \in [0, 4\pi), \quad \zeta = \beta/\bar{\alpha}. \tag{5}$$

Note that $|\zeta| < 1$ as well as $|z| < 1$. Using (4) and (5), and $|\alpha|^{-2} = 1 - |\zeta|^2$, put

$$f^k_{\zeta,t}(z) = (U_k(\zeta,t)\Phi^k_0)(z) = e^{ikt}(1-|\zeta|^2)^k(1-\bar{\zeta}z)^{-2k}. \tag{6}$$

In the expanded form, this is the family of generating functions we seek. Expand the factor $(1-\bar{\zeta}z)^{-2k}$ in the power series of the form $(1-w)^{-a} = \sum_{m=0}^{\infty} \frac{\Gamma(m+a)}{\Gamma(a)m!} w^m$ which is convergent for complex w, $|w| < 1$. From (2), the family of generating functions $f^k_{\zeta,t}$ has the form

$$f^k_{\zeta,t}(z) \quad = \quad \sum_{m=0}^{\infty} \left\{ \left(\frac{\Gamma(m+2k)}{m!\ \Gamma(2k)} \right)^{1/2} e^{ikt}(1-|\zeta|^2)^k \bar{\zeta}^m \right\} \Phi^k_m(z) = \sum_{m=0}^{\infty} v^k_m(\zeta,t)\Phi^k_m(z). \tag{7}$$

The squared moduli $\{|v^k_m(\zeta,t)|^2\}$ of the coefficients of the basis functions $\{\Phi^k_m(z)\}$ constitute the negative binomial distribution. To see this, consider a common representation of the negative binomial distribution as given by a random variable Y, the number of independent trials it takes to obtain N occurrences of an event which occurs with probability p.

$$P(Y = N+m) = \frac{\Gamma(N+m)}{m!\ \Gamma(N)} p^N (1-p)^m, \quad m = 0, 1, 2, \cdots.$$

Put $2k = N$ and $|\zeta|^2 = 1-p$. Then $|v_m(\zeta,t)|^2 = P(Y = N+m)$, the negative binomial distribution. Notice that $k = 1, \frac{3}{2}, 2, \cdots$, thus the possible values of N are all integers > 1.

2.4 The parameter space

We have the family of generating functions $f^k_{\zeta,t}(z) = (U_k(\zeta,t)\Phi^k_0)(z)$ yielding the coefficients $v^k_m(\zeta,t)$, functions of the parameters ζ and t. For the purpose of constructing a probability distribution, the parameter t is irrelevant since it appears only as a factor of modulus one in the expression for $v_m^k(\zeta, t)$ and therefore $|v_m^k(\zeta, t)|^2$ is the same as $|v_m^k(\zeta)|^2$. We only need a family of generating functions indexed by $\zeta \in \mathbb{D}$ for a fixed t, say, $t = 0$. More formally, we consider representation operators U_k which, from the beginning, are indexed, not by the group elements g as in (3), but by elements $\tilde{g}$ of the coset space G/H, where the subgroup H is comprised of diagonal matrices of the form $h = diag\{e^{-it/2}, e^{it/2}\}$. A coset matrix $\tilde{g}$ has the form $\begin{pmatrix} a & \beta \\ \bar{\beta} & a \end{pmatrix}$, where a is a real number, $a > 0$ and $a^2 - |\beta|^2 = 1$. We have the decomposition of elements $g \in SU(1,1)$, $g = \tilde{g}h$ for $\tilde{g} \in G/H$ and $h \in H$, which, by virtue of the representation homomorphic property implies $U_k(g) = U_k(\tilde{g})(U_k(h))$. From (4), we have $(U_k(h)\Phi_0^k)(z) = e^{ikt}$. Thus, taking $t = 0$, we consider generating functions $f_\zeta^k(z) = (U_k(\tilde{g})\Phi_0^k)(z)$ for $\tilde{g} \in G/H$.

To reparameterize, put $\zeta = \beta/a$, obtaining, from (6), (7) and (2),

$$f_\zeta^k(z) = (U_k(\tilde{g})\Phi_0^k)(z) = (1 - |\zeta|^2)^k(1 - \bar{\zeta}z)^{-2k}, \quad v_m^k(\zeta) = c_m(1 - |\zeta|^2)^k\bar{\zeta}^m. \tag{8}$$

It can be shown that the elements of the coset space G/H are indexed precisely by the parameter ζ, $|\zeta| < 1$. Thus the family of generating functions $\{f_\zeta^k(z)\}$ is indexed by ζ as is the family of coefficient quantities $v_m^k(\zeta)$ for each $m = 0, 1, 2, \cdots$. It can also be shown that the coset space G/H is homeomorphic to the complex unit disc $\mathbb{D}$. (Perelomov (1986)).

Thus the space $\mathbb{D}$ serves three purposes. It is the space upon which the group $SU(1,1)$ acts as a transformation group according to (1). It is the argument space for the generating functions $f_\zeta^k(z)$. In addition, it is the parameter space for those generating functions and for the coefficient quantities $v_m^k(\zeta)$ whose squared moduli constitute the negative binomial distribution. This triple function of the space $\mathbb{D}$ is the key for the inferential method described in this paper. We will make use of the fact that $SU(1,1)$ acts on the parameter space $\mathbb{D}$ of those $v_m^k(\zeta)$ quantities to derive an invariant prior measure.

2.5 An inferred distribution on the parameter space

Previously we had, for a given value of $|\zeta|^2$, a probability distribution in the form of the squared moduli of the quantities $\{v_m^k(\zeta),\ m = 0, 1, 2, \cdots\}$. This probability family was generated by applying a unitary representation operator of the group $SU(1,1)$, to the basis vector Φ_0. Now we consider the inverted situation where we have an observed negative binomial value, say m, and will use it to infer a probability distribution on the parameter space $\mathbb{D}$.

We focus upon the generating function $f_\zeta^k(z)$, a vector in $\mathcal{H}_k$. Suppose $f_\zeta^k(z)$ is acted upon by an arbitrary U_k operator, say $U_k(g_0)$, $g_0 \in G$. We will see that the result is a generating function of the same form but with parameter value, say ζ_1, where ζ_1 is related to ζ by the linear fractional transformation (1) corresponding to group element g_0.

Proposition 1. $U_k(g_0)f_\zeta^k(z) = f_{\zeta_1}^k(z)$ where $\zeta_1 = \dfrac{\alpha_0\zeta + \beta_0}{\bar{\beta}_0\zeta + \bar{\alpha}_0} = g_0\zeta$.

Proof. From (8), $U_k(g_0)f_\zeta^k(z) = (U_k(g_0)U_k(\tilde{g})\Phi_0^k)(z) = (U_k(g_0\tilde{g})\Phi_0^k(z)$. The matrix $\tilde{g} = \begin{pmatrix} a & \beta \\ \bar{\beta} & a \end{pmatrix} \in G/H$ is obtained, as in (5), from parameter ζ by the map $\mathbb{D} \to G/H$, $a =$

$(1-|\zeta|^2)^{-1/2}$, $\beta = a\zeta$. The final equality is due to the homomorphic property of the group representations. The product $g_0\tilde{g} = \begin{pmatrix} \alpha_0 & \beta_0 \\ \bar{\beta}_0 & \bar{\alpha}_0 \end{pmatrix}\begin{pmatrix} a & \beta \\ \bar{\beta} & a \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \bar{\beta}_1 & \bar{\alpha}_1 \end{pmatrix}$. From (5) and $\zeta = \dfrac{\beta}{a}$ we have $\zeta_1 = \dfrac{\beta_1}{\bar{\alpha}_1} = \dfrac{\alpha_0\beta + \beta_0 a}{\bar{\beta}_0\beta + \bar{\alpha}_0 a} = \dfrac{\alpha_0\zeta + \beta_0}{\bar{\beta}_0\zeta + \bar{\alpha}_0} = g_0\zeta$. □

Remark. In the construction given in Section 2.4, $\mathbb{D}$ acts as the parameter space for generating functions $f_\zeta^k(z)$ and thus for the coefficients $\{v_m^k(\zeta), m = 0, 1.2, ...\}$. For inference on the parameter space $\mathbb{D}$, we now consider $f_\zeta^k(z)$ for fixed z, as a function of ζ noting the result of Proposition 1. Consequently, for any two elements ζ_1 and ζ_2 in $\mathbb{D}$, $v_m^k(\zeta_1)$ is carried to $v_m^k(\zeta_2)$ by the action of $SU(1,1)$. The space $\mathbb{D}$ is the parameter space for the quantities $v_m^k(\zeta)$ where, for fixed m, $|v_m^k(\zeta)|^2$ is the likelihood function for the negative binomial distribution. Therefore, for integrating likelihood functions over $\mathbb{D}$, we seek a measure on $\mathbb{D}$ which is invariant to the action of $SU(1,1)$.

Proposition 2. The measure $d\nu(\zeta) = \dfrac{2k-1}{\pi}(1-|\zeta|^2)^{-2}d^2\zeta$ is invariant on $\mathbb{D}$ with respect to the action of the group $SU(1,1)$.

Proof. (From Sugiura (1990).) Put $\zeta = x + iy$ and $\zeta' = g\zeta = u + iv$. The Jacobian of the transformation is $J(x,y) = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial y} = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2$, using the Cauchy-Riemann conditions. Since $\zeta' = \dfrac{\alpha\zeta + \beta}{\bar{\beta}\zeta + \bar{\alpha}}$ where $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, we have $\dfrac{d\zeta'}{d\zeta} = (\bar{\beta}\zeta + \bar{\alpha})^{-2}$. But we can write $\dfrac{d\zeta'}{d\zeta} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$, which implies that $J(x,y) = \left|\dfrac{d\zeta'}{d\zeta}\right|^2 = \left|\bar{\beta}\zeta + \bar{\alpha}\right|^{-4}$. Put $d^2\zeta = dxdy$, then $d^2\zeta' = |\bar{\beta}\zeta + \bar{\alpha}|^{-4}d^2\zeta$, and

$$d\nu(g\zeta) = C(1-|g\zeta|^2)^{-2}\ d^2\zeta' = C(1-|g\zeta|^2)^{-2}|\bar{\beta}\zeta + \bar{\alpha}|^{-4}\ d^2\zeta, \qquad C = (2k-1)/\pi.$$

But $(1-|g\zeta|^2) = (1-|\zeta|^2)\ |\bar{\beta}\zeta + \bar{\alpha}|^{-2}$, using $|\alpha|^2 - |\beta|^2 = 1$. So we have

$$d\nu(g\zeta) = C(1-|\zeta|^2)^{-2}\ |\bar{\beta}\zeta + \bar{\alpha}|^4\ |\bar{\beta}\zeta + \bar{\alpha}|^{-4}\ d^2\zeta = d\nu(\zeta).$$

□

Proposition 3. The functions $v_m^k(\zeta)$ are square integrable with respect to the invariant measure $d\nu$, and their squared moduli integrate to unity. That is,

$$\frac{2k-1}{\pi}\int_{\mathbb{D}} |v_m^k(\zeta)|^2 d\nu(\zeta) = \frac{2k-1}{\pi}\int_{\mathbb{D}} c_m^2(1-|\zeta|^2)^{2k-2}(|\zeta|^2)^m d^2\zeta = 1.$$

Writing $|\zeta|^2 = 1-p$ and $N = 2k$, we have $(N-1)\int_0^1 c_m^2 p^N(1-p)^m p^{-2}dp = 1$.

Proof. Write $\zeta = re^{i\theta}$ for $0 \le r < 1, !0 < \theta \le 2\pi$. Then $d^2\zeta = rdrd\theta$. From (8), we have

$$\frac{2k-1}{\pi}\int_{\mathbb{D}} |v_m^k(\zeta)|^2 d\nu(\zeta) = \left((2k-1)\int_0^1 c_m^2(1-r^2)^{2k-2}(r^2)^m 2rdr\right)\left(\frac{1}{2\pi}\int_0^{2\pi} d\theta\right).$$

Write the change of variable $p = 1 - r^2,\ dp = -2rdr$. Since the second integral is unity, putting $N = 2k$, we have

$$(N-1)\int_0^1 p^{N-2}(1-p)^m dp = \frac{(N-1)\Gamma(N-1)\Gamma(m+1)}{\Gamma(N+m)} = \frac{\Gamma(N)m!}{\Gamma(N+m)} = \frac{1}{c_m^2}.$$

□

Remark. Integrated as a Bayesian posterior distribution for p with likelihood function $c_m^2 p^N (1-p)^m$, we have

$$\mathcal{P}_r(p \in \Delta) = (N-1)\int_\Delta c_m^2 p^N (1-p)^m \frac{1}{p^2} dp.$$

We see that the $SU(1,1)$ invariant measure $d\nu$ on complex parameter space $\mathbb{D}$ led to the Bayesian prior measure dp/p^2 for real parameter p.

3 Conclusion

We have obtained a noninformative prior by establishing a connection between the matrix group $SU(1,1)$ and the negative binomial distribution. The parameter space of the complex index $\zeta \in \mathbb{D}$ was seen to be in one-to-one correspondence with a space upon which the group acts transitively. The negative binomial parameter $p = 1 - |\zeta|^2$. By using an invariant measure on the parameter space $\mathbb{D}$, we have been able to construct a posterior distribution for the parameter p in a group theoretic context. This led to a Bayesian prior measure for p which in this case was dp/p^2. This result is different from some other known priors such as the ones described in Box and Tiao (1973) and Bernardo and Smith (1994). Most noticeably it is different from but comparable with Jeffreys prior $dp/p\sqrt{1-p}$. Perhaps this indicates an interesting method alternative to other approaches.

Previously we have found similar constructions for two other discrete distributions. We found a relationship between the Poisson distribution and the Weyl-Heisenberg group which resulted in a uniform prior measure for the Poisson mean parameter λ, and for the binomial distribution, the matrix group $SU(2)$ (related to the rotation group), resulting in a uniform prior measure for the binomial parameter p. (Heller and Wang, 2006.) We have also found a similar relationship between the normal family, indexed by the mean parameter μ and the Weyl-Heisenberg group which resulted in a uniform prior measure for μ. (Heller and Wang, 2004). The group related to the negative binomial distribution presented in this paper is comparatively more complicated, and the harmonic analysis construction differs from the previous approach.

In the present construction, we have used a representation of $SU(1,1)$ in the form of a set of linear operators in a Hilbert space. That resulted in the construction of generating functions for a set of complex valued "square root" quantities in the form of inner products in the Hilbert space, whose square moduli constitute the negative binomial distribution. We have not yet addressed the question of uniqueness in the choice of group. By choosing the group $SU(1,1)$ we have indeed *found* a group which leads to the negative binomial distribution. Whether there exist other groups which would serve the same purpose remains an open question at the present time.

This method for the construction of probabilities is found in the quantum mechanics literature. Vectors of the form $U(g)\Phi_0$, which we referred to as generating functions (whose index spans the whole parameter space of interest), are known as *coherent states* in quantum mechanics. (See Perelomov (1986).) A use of coherent states for the purpose of constructing probability distributions for applications in physics is given in Ali, Antoine, and Gazeau (2000), more examples and discussions are in Novaes and Gazeau (2003) and Gazeau (2005). The idea of using a kind of "square root" probability for expansion purposes and then taking the square afterwards was proposed in Good and Gaskins (1971), where the connection to quantum mechanics was noted. In Barndorff-Nielsen, Gill, and Jupp (2003), this linear space context for the construction of probability distributions is described and used for the purpose of statistical inference for quantum measurements (also see Malley and Hornstein (1993)). The gap between the linear, group theoretic methods of quantum probability and the general form of statistical inference is being bridged (see Helland (2006)).

As mentioned above and in the introduction, group theoretic methods for statistical inference have been in use for many years. However, none of those methods are applicable to discrete distributions with continuous parameter spaces. Although there is no known systematic approach to choose a group for an arbitrary family of probability distribution, we have found it possible to obtain a group theoretic prior measure for the negative binomial distribution as well as the other discrete distributions mentioned above.

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References

[1] Ali, S., Antoine, J. and Gazeau, J. (2000). *Coherent States, Wavelets and Their Generalizations* (Springer-Verlag, Berlin).

[2] Barndorff-Nielsen,O., Gill, R. E. and Jupp, P. D. (2003). On quantum statistical inference. *J. R. Statist. Soc.* **B 65** 775–816.

[3] Bernardo, J. M. and Smith, A. F. M. (1994). *Bayesian Theory* (Wiley, New York).

[4] Box, G. E. P. and Tiao, G. C. (1973). *Bayesian Inference in Statistical Analysis* (Reading, MA, Addison-Wesley).

[5] Eaton, M. L. (1989). *Group Invariance Applications In Statistics.* Inst. Math. Stat. (Hayward, California).

[6] Efron, B. (1998). R. A. Fisher in the 21st century. *Statistical Science* **13** 95–122.

[7] Fraser, D. A. S. (1961). On fiducial inference. *Ann. Math. Statist.* **32** 661–676.

[8] Gazeau, J. (2005). Quantization is just a certain regard to... . *Concepts of Physics* **2** 1–36.

[9] Good, I. J. and Gaskins R. A. (1971). Nonparametric roughness penalties for probability densities. *Biometrika* **58**(2) 255–277.

[10] Helland, I. (2004). Statistical inference under symmetry. *Internat. Statist. Rev.* **72**(3), 409–422.

[11] Helland, I. (2006). Extended statistical modeling under symmetry: The link towards quantum mechanics. *Ann. Statistics*, **34**(1) 42–77.

[12] Heller, B. and Wang, M. (2006). Group invariant inferred distributions via noncommutative probability. *Lecture Notes and Monograph Series*, Inst. Math. Stat. **50** 1–19.

[13] Heller, B. and Wang, M. (2004). Posterior distributions on certain parameter spaces obtained by using group theoretic methods adopted from quantum physics. *Technical Report* **546**. Department of Statistics, the University of Chicago.

[14] Malley, J. and Hornstein, J. (1993). Quantum statistical inference. *Statistical Science* **8**(4) 433–457.

[15] Novaes, M. and Gazeau, J. P. (2003). Multidimensional generalized coherent states. J. Phys. A: Math. Gen. **36** 199–212.

[16] Kass, R. and Wasserman, L. (1996). The selection of prior distributions by formal rules. *Journal of the American Statistical Association*, **91**(435) 1343–1370.

[17] Perelomov, A. (1986). *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin).

[18] Sugiura, M. (1990). *Unitary Representations and Harmonic Analysis - An Introduction* (North-Holland, New York. 2nd ed.).