The role of tensor rank in the complexity analysis of bilinear forms

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1 Preliminaries
   • Tensors
   • Tensor rank
   • Bilinear forms

2 Properties of tensor rank
   • Border rank
   • Lower bounds

3 Open problems
   • The direct sum conjecture
   • Some tensors of unknown rank
Let $\mathbb{F}$ be a number field, say, $\mathbb{R}, \mathbb{C}$; tensors of the kind

$$\mathcal{A} = (a_{i_1,i_2,...,i_h}) \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_h},$$

that is, h-way arrays, are encountered in many problems of very different nature [Comon 2001], [Comon, Golub, Lim, Mourrain, 2006], [De Silva, Lim 2006]

- Blind source separation
- High order factor analysis
- Independent component analysis
- Candecomp/Parafac model
- Complexity analysis
- Psycometric, Chemometric, Economy,...
It is surprising that little interplay occurred among these different research areas
Some properties have been rediscovered in different contexts
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In this talk I wish to provide an overview of the main results concerning tensors obtained in the research field of computational complexity (starting from 1969) with the aim of

- creating a synergic exchange of information between these research areas
- presenting problems which might be solved with the more recent tools
- presenting old results that might be adapted and extended to the new needs
Tensor rank

Definition (Hitchcock 1927)

A tensor \( \mathcal{T} = (t_{i_1}, \ldots, i_h) \) has rank 1 if there exist vectors \( u^{(k)} = (u_i^{(k)}) \in \mathbb{F}^{n_k} \), \( k = 1 : h \) such that \( t_{i_1}, \ldots, i_h = u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_h}^{(h)} \),

\[ \mathcal{T} = u^{(1)} \circ u^{(2)} \circ \cdots \circ u^{(h)} \]

Definition (Hitchcock 1927)

The tensor rank \( \text{rk}(\mathcal{A}) \) of \( \mathcal{A} = (a_{i_1}, \ldots, i_h) \) is the minimum number \( r \) of rank-1 tensors \( \mathcal{T}_i \in \mathbb{F}^{n_1 \times \cdots \times n_h} \) such that

\[ \mathcal{A} = \mathcal{T}_1 + \mathcal{T}_2 + \cdots + \mathcal{T}_r \quad \text{canonical decomposition} \]
Some remarks

For $\mathcal{A} \in \mathbb{F}^{m\times n\times p}$ a canonical decomposition

$$\mathcal{A} = u^{(1)} \circ v^{(1)} \circ w^{(1)} + u^{(2)} \circ v^{(2)} \circ w^{(2)} + \cdots + u^{(r)} \circ v^{(r)} \circ w^{(r)}$$

is defined by three matrices

$$U = (u_{i,j}) \in \mathbb{F}^{m \times r}, \quad V = (v_{i,j}) \in \mathbb{F}^{n \times r}, \quad W = (w_{i,j}) \in \mathbb{F}^{p \times r},$$

whose columns are the vectors $u^{(i)}$, $v^{(i)}$, $w^{(i)}$, $i = 1 : r$, respectively.
Some remarks

A tensor $\mathcal{A} = (a_{i,j,k}) \in \mathbb{F}^{m \times n \times p}$, can be represented by means of the set of the 3-slabs $A_k = (a_{i,j,k})_{i,j} \in \mathbb{F}^{m \times n}$ of $\mathcal{A}$ or by a single matrix of variables

$$A = \sum_{k=1}^{p} s_k A_k$$

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \leftrightarrow \quad \begin{bmatrix} s_1 & s_2 \\ s_2 & -s_1 \end{bmatrix}$$
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$$A = \sum_{k=1}^{p} s_k A_k$$

$$\mathcal{A} = \left[ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \leftrightarrow \begin{bmatrix} s_1 & s_2 \\ s_2 & -s_1 \end{bmatrix}$$

This suggests a different point of view for tensor rank:

$\text{rk}(\mathcal{A})$ is the minimum set of rank one matrices which span the linear space generated by the 3-slabs $A_1, \ldots, A_p$ [Gastinel 71, Fiduccia 72].
**Problem:** Given matrices $A_k = (a_{i,j,k})_{i=1:m, j=1:n, k=1:p}$ compute the set of bilinear forms

$$f_k(x, y) = x^T A_k y, \quad k = 1:p$$

with the **minimum number** of nonscalar multiplications with no use of commutativity (noncommutative bilinear complexity)

A nonscalar multiplication is a multiplication of the kind

$$s = \left(\sum_{i=1}^{m} \alpha_i x_i\right)\left(\sum_{j=1}^{n} \beta_j y_j\right), \quad \alpha_i, \beta_j \in \mathbb{F}$$

**Remark:** The set of bilinear forms is uniquely determined by the tensor $A = (a_{i,j,k})$. 

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The role of tensor rank
Bilinear forms

A canonical decomposition of the tensor $\mathcal{A}$ associated with the set of bilinear forms provides an algorithm of complexity $r$. In fact

$$\mathcal{A} = \sum_{\ell=1}^{r} u^{(\ell)} \circ v^{(\ell)} \circ w^{(\ell)} \quad \rightarrow \quad A_k = \sum_{\ell=1}^{r} w_{k,\ell} u^{(\ell)} \circ v^{(\ell)}$$

$$\rightarrow \quad x^T A_k y = \sum_{\ell=1}^{r} w_{k,\ell} (x^T u^{(\ell)}) (v^{(\ell)^T} y)$$

Theorem (Strassen 1975)

The noncommutative bilinear complexity of a set of bilinear forms $f_k(x, y) = x^T A_k y$, $k = 1 : p$, $A_k = (a_{i,j,k})$, is given by the tensor rank of the associated tensor $\mathcal{A} = (a_{i,j,k})$. 

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The role of tensor rank
An example

Multiplication of complex numbers

\[(x_1 + ix_2)(y_1 + iy_2) = (x_1y_1 - x_2y_2) + i(x_1y_2 + x_2y_1) = f_1 + if_2\]

Apparently, 4 multiplications are needed
An example

Multiplication of complex numbers

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Tensor:\n\[\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\]
An example

Multiplication of complex numbers

\[(x_1 + ix_2)(y_1 + iy_2) = (x_1y_1 - x_2y_2) + i(x_1y_2 + x_2y_1) = f_1 + if_2\]

Apparently, 4 multiplications are needed

Tensor: \[A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\]

The tensor rank is at most 3:

\[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\]

\[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\]

Algorithm:

\[s_1 = (x_1 + x_2)(y_1 + y_2),\]
\[s_2 = x_1y_1,\]
\[s_3 = x_2y_2\]
\[f_1 = s_2 - s_3,\]
\[f_2 = s_1 - s_2 - s_3\]
Main problem: For a given tensor, compute its rank and a canonical decomposition.

Two ways of attacking the problem
- looking for lower bounds to the tensor rank
- looking for upper bounds to the tensor rank

Things get more complicated: unlike the case of $m \times n$ matrices
- The rank depends on the ground field $\mathbb{F}$
- High rank tensors can be approximated by low rank tensors

Rational algorithms for tensor rank, like Gaussian elimination, cannot exist.
Topological properties of tensors: Border Rank

For $m \times n$ matrices, any sequence $\{A_k\}$ of $m \times n$ matrices of rank $r$ with limit $A = \lim_k A_k$ is such that $\text{rank}(A) \leq r$. 

Example (Bini, Capovani, Lotti, Romani 1980)

The tensor $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & \epsilon & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ has rank 3. The tensor $A \epsilon = \begin{bmatrix} 1 & 0 & \epsilon & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ has rank 2 for any $\epsilon \neq 0$. 

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The role of tensor rank
For $m \times n$ matrices, any sequence $\{A_k\}$ of $m \times n$ matrices of rank $r$ with limit $A = \lim_k A_k$ is such that $\text{rank}(A) \leq r$.

For higher order tensors this property does not hold anymore.

**Example (Bini, Capovani, Lotti, Romani 1980)**

The tensor

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has rank 3. The tensor

$$A_\epsilon = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} ; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has rank 2 for any $\epsilon \neq 0$. 
Canonical decomposition

\[
\begin{bmatrix}
1 & 0 \\
\epsilon & 1
\end{bmatrix}
= \begin{bmatrix}
1 & \epsilon^{-1} \\
\epsilon & 1
\end{bmatrix}
- \epsilon^{-1} \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]
**Definition (Bini, Capovani, Lotti, Romani 1980)**

The border rank of $A \in \mathbb{F}^{m \times n \times p}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$) is

$$\text{brk}(A) = \min\{ r : \forall \epsilon > 0 \exists E \in \mathbb{F}^{m \times n \times p} : ||E|| < \epsilon, \text{rk}(A+E) = r \}$$

where $|| \cdot ||$ is any norm

Some properties:

- $\text{brk}(A) \leq \text{rk}(A)$
- $\text{brk}(A)$ is the minimum number of nonscalar multiplications sufficient to approximate the set of bilinear forms associated with $A$ with arbitrarily small nonzero error
More on rank and border rank

Let $\mathcal{A}_\epsilon = \mathcal{A} + \mathcal{E}_\epsilon$ be such that

- $\text{rk}(\mathcal{A}_\epsilon) = \text{brk}(\mathcal{A})$
- the entries of $\mathcal{E}_\epsilon$ are polynomials of degree $d$

Then

$$\text{rk}(\mathcal{A}) \leq (d + 1)\text{brk}(\mathcal{A})$$
Let $A_\epsilon = A + E_\epsilon$ be such that

- $\text{rk}(A_\epsilon) = \text{brk}(A)$
- the entries of $E_\epsilon$ are polynomials of degree $d$

Then

$$\text{rk}(A) \leq (d + 1)\text{brk}(A)$$

Proof:

- Write $d + 1$ copies of the canonical decomposition of length $\text{rk}(A_\epsilon)$ obtained with $(d + 1)$ pairwise different values of $\epsilon$
More on rank and border rank

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**Proof:**

- Write $d + 1$ copies of the canonical decomposition of length $\text{rk}(A_\epsilon)$ obtained with $(d + 1)$ pairwise different values of $\epsilon$
- Take linear combinations of these decompositions with coefficients $\gamma_j$, $j = 1 : d + 1$ in order to kill the terms in $\epsilon^i$, $i = 1 : d$ and to have $\sum \gamma_j = 1$
More on rank and border rank

Let $A_\epsilon = A + E_\epsilon$ be such that
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Proof:
- Write $d + 1$ copies of the canonical decomposition of length $\text{rk}(A_\epsilon)$ obtained with $(d + 1)$ pairwise different values of $\epsilon$
- Take linear combinations of these decompositions with coefficients $\gamma_j$, $j = 1 : d + 1$ in order to kill the terms in $\epsilon^i$, $i = 1 : d$ and to have $\sum \gamma_j = 1$
- Obtain a decomposition of length $(d + 1) \text{brk}(A)$ with no error
Simple criteria for providing lower bounds on tensor rank and border rank can be given.

**Trivial bounds**

\[ \text{brk}(\mathcal{A}) \geq \dim(\text{span}(A_1, \ldots, A_p)) \]

Similar inequalities are valid w.r.t. the other coordinates.

Assume for simplicity \( p = \dim(\text{span}(A_1, \ldots, A_p)) \).
Let $U, V, W$ be the matrices defining a canonical factorization of $A$ of length $\text{rk}(A)$

$$A = \sum_{\ell=1}^{\text{rk}(A)} u^{(\ell)} \circ v^{(\ell)} \circ w^{(\ell)}$$
Lower bounds and linear algebra

Let $U, V, W$ be the matrices defining a canonical factorization of $A$ of length $\text{rk}(A)$

$$A = \sum_{\ell=1}^{\text{rk}(A)} u^{(\ell)} \circ v^{(\ell)} \circ w^{(\ell)}$$

Assume w.l.o.g. that the first $p$ columns of $W$ are linearly independent, that is

$$W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}, \quad W_1 \in \mathbb{F}^{p \times p}, \quad \det W_1 \neq 0$$
Lower bounds and linear algebra

Let $U, V, W$ be the matrices defining a canonical factorization of $A$ of length $\text{rk}(A)$

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$$W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}, \quad W_1 \in \mathbb{F}^{p \times p}, \quad \det W_1 \neq 0$$

Let $A'_k = \sum_{j=1}^{p} w_{k,j}^{(-1)} A_j$, define

$$A' = [A'_1, A'_2, \ldots, A'_p] = A \cdot_3 W_1^{-1},$$

i.e., choose a different basis to represent the space spanned by the slabs $A_1, \ldots, A_p$
Evidently, $\mathcal{A}'$ has the same rank of $\mathcal{A}$ and a canonical decomposition is given by $U' = U, V' = V, W' = W_1^{-1}W$, where

$$W' = \begin{bmatrix} I & W_1^{-1}W_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \end{bmatrix}$$

**Remark**

Any subtensor $\hat{\mathcal{A}}$ formed by $k$ 3-slabs $[A'_{\sigma_1}, \ldots, A'_{\sigma_k}]$ is such that

$$\text{rk}(\hat{\mathcal{A}}) \leq \text{rk}(\mathcal{A}) - (p - k)$$
More generally,

**Theorem**

\[ \text{rk}(A) \geq \max_{T} \min_{S} (p - k + \text{rk}(A \bullet_3 TS)) \]

\(T: k \times p \) submatrix of \( I_p\), \hspace{1em} \(S: p \times p \) nonsingular
More generally,

**Theorem**

\[ rk(A) \geq \max_T \min_S (p - k + rk(A \cdot_3 TS)) \]

\( T: k \times p \) submatrix of \( I_p \), \( S: p \times p \) nonsingular

**Corollary**

If for any basis of \( \text{span}(A_1, \ldots, A_p) \) there exists a matrix of rank \( q \) then

\[ rk(A) \geq p + q - 1 \]
Higher order generalization:

**Corollary**

Let $\mathcal{A} = [\mathcal{A}_1, \ldots, \mathcal{A}_{n_h}] \in \mathbb{F}^{n_1 \times \cdots \times n_h}$, $\mathcal{A}_k \in \mathbb{F}^{n_1 \times \cdots \times n_{h-1}}$ be such that $n_h = \dim(\text{span}(\mathcal{A}_1, \ldots, \mathcal{A}_{n_h}))$. If for any basis of $\text{span}(\mathcal{A}_1, \ldots, \mathcal{A}_{n_h})$ there exists a tensor in the basis of rank $q$ then

$$\text{rk}(\mathcal{A}) \geq n_h + q - 1$$
Higher order generalization:

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Let $\mathcal{A} = [A_1, \ldots, A_{n_h}] \in \mathbb{F}^{n_1 \times \cdots \times n_h}$, $A_k \in \mathbb{F}^{n_1 \times \cdots \times n_{h-1}}$ be such that $n_h = \dim(\text{span}(A_1, \ldots, A_{n_h}))$. If for any basis of $\text{span}(A_1, \ldots, A_{n_h})$ there exists a tensor in the basis of rank $q$ then

$$\text{rk}(\mathcal{A}) \geq n_h + q - 1$$

Example: For

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

any basis must contain a nonsingular matrix, therefore

$$\text{rk}(\mathcal{A}) \geq 2 + 2 - 1 = 3$$
Lower bounds to border rank

Unfortunately the same technique cannot be applied to border rank $W_1$ may be singular in the limit as $\epsilon \to 0$ so that $E \bullet_3 W_1^{-1}$ may not be infinitesimal anymore.

**Remedy:** compute the QR factorization of $W$ and multiply $W$ by $Q^T$ which has Euclidean norm 1 for any $\epsilon$. One has

$$W' = Q^T W = \begin{bmatrix}
* & * & * & * & * & * & * \\
0 & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * \\
\end{bmatrix}$$

**Remark**

There exists an $m \times n \times k$ subtensor $\hat{A}$ such that

$$\text{brk}(\hat{A}) \leq \text{brk}(A) - (p - k)$$
**Theorem**

\[ \text{brk}(A) \geq \min_T \min_S \left( p - k + \text{brk}(A \bullet_3 TS) \right) \]

\( T: k \times p \) submatrix of \( l_p \), \( S: p \times p \) nonsingular
Theorem

\[ \text{brk}(\mathcal{A}) \geq \min_T \min_S (p - k + \text{brk}(\mathcal{A} \bullet_T T S)) \]

\( T: k \times p \) submatrix of \( I_p \), \( S: p \times p \) nonsingular

Corollary

If any basis of the linear space spanned by \( A_1, \ldots, A_p \) is made up by matrices of rank at most \( q \) then \( \text{brk}(\mathcal{A}) \geq p + q - 1 \)
Theorem

\[
\text{brk}(\mathcal{A}) \geq \min_T \min_S (p - k + \text{brk}(\mathcal{A} \bullet_3 TS))
\]

\(T: k \times p\) submatrix of \(I_p\), \(S: p \times p\) nonsingular

Corollary

If any basis of the linear space spanned by \(A_1, \ldots, A_p\) is made up by matrices of rank at most \(q\) then \(\text{brk}(\mathcal{A}) \geq p + q - 1\)

Corollary (Generalization to higher dimension)

Let \(\mathcal{A} = [A_1, \ldots, A_{n_h}]\). If any basis of the linear space spanned by \(A_1, \ldots, A_{n_h}\) is made up by tensors of rank at most \(q\) then \(\text{brk}(\mathcal{A}) \geq n_h + q - 1\)
Example

The tensor $\mathcal{A} \in \mathbb{R}^{n^2 \times n^2 \times n^2}$ whose 3-slabs span the linear space

$$I \otimes S = \begin{bmatrix} S \\ \vdots \\ S \end{bmatrix}$$

is associated with $n \times n$ matrix multiplication.

$S = \begin{bmatrix} s_{1,1} & \ldots & s_{1,n} \\
\vdots & \ddots & \vdots \\
s_{n,1} & \ldots & s_{n,n} \end{bmatrix}$
The tensor $\mathcal{A} \in \mathbb{F}^{n^2 \times n^2 \times n^2}$ whose 3-slabs span the linear space

$$I \otimes S = \begin{bmatrix} S \\ \vdots \\ S \end{bmatrix}, \quad S = \begin{bmatrix} s_{1,1} & \cdots & s_{1,n} \\ \vdots & \ddots & \vdots \\ s_{n,1} & \cdots & s_{n,n} \end{bmatrix}$$

is associated with $n \times n$ matrix multiplication.

All the matrices in any basis of the linear space have rank at least $n$. Therefore

$$\text{brk}(\mathcal{A}) \geq n^2 + n - 1$$

The bound can be improved to

$$\text{brk}(\mathcal{A}) \geq n^2 + 2n - 2$$
Example

The tensor $\mathcal{A} \in \mathbb{R}^{4 \times 4 \times 3}$ whose 3-slabs span the linear space

\[
\begin{bmatrix}
s_1 & s_2 & 0 & 0 \\
s_3 & s_4 & 0 & 0 \\
0 & 0 & s_1 & s_2
\end{bmatrix}
\]

is associated with the multiplication of a $2 \times 2$ triangular matrix and a full $2 \times 2$ matrix.
Example

The tensor $\mathcal{A} \in \mathbb{R}^{4 \times 4 \times 3}$ whose 3-slabs span the linear space

$$
\begin{bmatrix}
  s_1 & s_2 & 0 & 0 \\
s_3 & s_4 & 0 & 0 \\
0 & 0 & s_1 & s_2
\end{bmatrix}
$$

is associated with the multiplication of a $2 \times 2$ triangular matrix and a full $2 \times 2$ matrix.

For any basis of the linear space there exist two matrices which form a tensor of rank at least 4.

$$\text{rk}(\mathcal{A}) \geq 4 - 2 + 4 = 6$$

Since there exists a canonical approximate decomposition of length 5 of $\mathcal{A}$ then $\text{brk}(\mathcal{A}) \leq 5 < 6 = \text{rk}(\mathcal{A})$
The direct sum conjecture

Let $A \in \mathbb{F}^{m \times n \times p}$, $B \in \mathbb{F}^{m' \times n' \times p'}$. Consider the direct sum of $A$ and $B$

$$C = A \oplus B \in \mathbb{F}^{(m+m') \times (n+n') \times (p+p')}$$

Direct sum conjecture, Strassen 1973

$$\text{rk}(C) = \text{rk}(A) + \text{rk}(B)$$

The complexity of two disjoint sets of bilinear forms is the sum of the complexities of each set.
The direct sum conjecture

Let $\mathcal{A} \in \mathbb{F}^{m \times n \times p}$, $\mathcal{B} \in \mathbb{F}^{m' \times n' \times p'}$. Consider the direct sum of $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{C} = \mathcal{A} \oplus \mathcal{B} \in \mathbb{F}^{(m+m') \times (n+n') \times (p+p')}$$

Direct sum conjecture, Strassen 1973

$$\text{rk}(\mathcal{C}) = \text{rk}(\mathcal{A}) + \text{rk}(\mathcal{B})$$

The complexity of two disjoint sets of bilinear forms is the sum of the complexities of each set.

Fact

There are cases where $\text{brk}(\mathcal{C}) < \text{brk}(\mathcal{A}) + \text{brk}(\mathcal{B})$
Example (A. Schoenhage 81)

\[ \text{brk}(\mathcal{A}) = 10 < 9 + 4 \]

\[ \mathcal{A} = \begin{bmatrix}
    s_{11} & s_{12} & s_{13} \\
    s_{21} & s_{22} & s_{23} \\
    s_{21} & s_{22} & s_{23} \\
\end{bmatrix} \begin{bmatrix}
    t \\
    t \\
    t \\
\end{bmatrix} \]
Dario A. Bini
The role of tensor rank
Matrix multiplication and open problems

- **2 × 2 matrix product**:
  \[ \mathcal{A} = l_2 \otimes \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \]

  \[ \text{rk}(\mathcal{A}) = \text{brk}(\mathcal{A}) = 7 \text{ [Strassen 69, Landsberg 05]} \]

- **3 × 3 matrix product** \( \mathcal{A} = l_3 \otimes \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \)

  \[ 19 \leq \text{rk}(\mathcal{A}) \leq 23 \text{ [Laderman 76], [Bläser 03]} \]

  \[ 13 \leq \text{brk}(\mathcal{A}) \leq 22 \text{ [Schönhage 81]} \]
Matrix multiplication and open problems

- $4 \times 4$ matrix product $\mathcal{A} = I_4 \otimes \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix}$

$34 \leq \text{rk}(\mathcal{A}) \leq 49$

$22 \leq \text{brk}(\mathcal{A}) \leq 49$