The role of tensor rank in the complexity analysis of bilinear forms

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Tensors

Let $\mathbb{F}$ be a number field, say, $\mathbb{R}, \mathbb{C}$; tensors of the kind

$$\mathcal{A} = (a_{i_1,i_2,\ldots,i_h}) \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_h},$$

that is, h-way arrays, are encountered in many problems of very different nature [Comon 2001], [Comon, Golub, Lim, Mourrain, 2006], [De Silva, Lim 2006]

- Blind source separation
- High order factor analysis
- Independent component analysis
- Candecomp/Parafac model
- Complexity analysis
- Psycometric, Chemometric, Economy,...

Tensors

- It is surprising that little interplay occurred among these different research areas
- Some properties have been rediscovered in different contexts
- Apparently, results obtained in one field have not migrated to the other fields

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In this talk I wish to provide an overview of the main results concerning tensors obtained in the research field of computational complexity (starting from 1969) with the aim of

- creating a synergic exchange of information between these research areas
- presenting problems which might be solved with the more recent tools
- presenting old results that might be adapted and extended to the new needs

Tensor rank

Definition (Hitchcock 1927)

A tensor $\mathcal{T} = (t_{i_1,\dots,i_h})$ has rank 1 if there exist vectors $\mathbf{u}^{(k)} = (u_i^{(k)}) \in \mathbb{F}^{n_k}$, $k = 1 : h$ such that $t_{i_1,\dots,i_h} = u_{i_1}^{(1)} u_{i_2}^{(2)} \cdots u_{i_h}^{(h)}$,

$$\mathcal{T} = \mathbf{u}^{(1)} \circ \mathbf{u}^{(2)} \circ \cdots \circ \mathbf{u}^{(h)}$$

Definition (Hitchcock 1927)

The tensor rank $\mathrm{rk}(\mathcal{A})$ of $\mathcal{A} = (a_{i_1,\dots,i_h})$ is the minimum number r of rank-1 tensors $\mathcal{T}_i \in \mathbb{F}^{n_1 \times \cdots \times n_h}$ such that

$$\mathcal{A} = \mathcal{T}_1 + \mathcal{T}_2 + \cdots + \mathcal{T}_r \quad \text{canonical decomposition}$$

Some remarks

For $\mathcal{A} \in \mathbb{F}^{m\times n\times p}$ a canonical decomposition

$$\mathcal{A} = \mathbf{u}^{(1)} \circ \mathbf{v}^{(1)} \circ \mathbf{w}^{(1)} + \mathbf{u}^{(2)} \circ \mathbf{v}^{(2)} \circ \mathbf{w}^{(2)} + \cdots + \mathbf{u}^{(r)} \circ \mathbf{v}^{(r)} \circ \mathbf{w}^{(r)}$$

is defined by three matrices

$$U = (u_{i,j}) \in \mathbb{F}^{m\times r}, \quad V = (v_{i,j}) \in \mathbb{F}^{n\times r}, \quad W = (w_{i,j}) \in \mathbb{F}^{p\times r},$$

whose columns are the vectors $\mathbf{u}^{(i)}$, $\mathbf{v}^{(i)}$, $\mathbf{w}^{(i)}$, $i = 1 : r$, respectively.

Some remarks

A tensor $\mathcal{A} = (a_{i,j,k}) \in \mathbb{F}^{m \times n \times p}$, can be represented by means of the set of the 3-slabs $A_k = (a_{i,j,k})_{i,j} \in \mathbb{F}^{m \times n}$ of $\mathcal{A}$ or by a single matrix of variables

$$A = \sum_{k=1}^{p} s_k A_k$$

$$\mathcal{A} = \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \quad \leftrightarrow \quad \begin{bmatrix} s_1 & s_2 \\ s_2 & -s_1 \end{bmatrix}$$

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This suggests a different point of view for tensor rank:

$\mathrm{rk}(\mathcal{A})$ is the minimum set of rank one matrices which span the linear space generated by the 3-slabs $A_1, \ldots, A_p$ [Gastinel 71, Fiduccia 72].

Bilinear forms

Problem: Given matrices $A_k = (a_{i,j,k})_{i=1:m,j=1:n}$, $k = 1:p$ compute the set of bilinear forms

$$f_k(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A_k \mathbf{y}, \quad k = 1:p$$

with the minimum number of nonscalar multiplications with no use of commutativity (noncommutative bilinear complexity)

A nonscalar multiplication is a multiplication of the kind

$$s = (\sum_{i=1}^{m} \alpha_i x_i)(\sum_{j=1}^{n} \beta_j y_j), \quad \alpha_i, \beta_j \in \mathbb{F}$$

Remark: The set of bilinear forms is uniquely determined by the tensor $\mathcal{A} = (a_{i,j,k})$.

Bilinear forms

A canonical decomposition of the tensor $\mathcal{A}$ associated with the set of bilinear forms provides an algorithm of complexity r. In fact

$$\mathcal{A} = \sum_{\ell=1}^{r} \mathbf{u}^{(\ell)} \circ \mathbf{v}^{(\ell)} \circ \mathbf{w}^{(\ell)} \quad \rightarrow \quad A_k = \sum_{\ell=1}^{r} w_{k,\ell} \mathbf{u}^{(\ell)} \circ \mathbf{v}^{(\ell)}$$

$$\rightarrow \quad \mathbf{x}^T A_k \mathbf{y} = \sum_{\ell=1}^{r} w_{k,\ell} (\mathbf{x}^T \mathbf{u}^{(\ell)})(\mathbf{v}^{(\ell)T} \mathbf{y})$$

Theorem (Strassen 1975)

The noncommutative bilinear complexity of a set of bilinear forms $f_k(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A_k \mathbf{y}$, $k = 1 : p$, $A_k = (a_{i,j,k})$, is given by the tensor rank of the associated tensor $A = (a_{i,j,k})$.

An example

Multiplication of complex numbers

$$(x_1 + ix_2)(y_1 + iy_2) = (x_1y_1 - x_2y_2) + i(x_1y_2 + x_2y_1) = f_1 + if_2$$

Apparently, 4 multiplications are needed

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Tensor: $\mathcal{A} = \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]$

The tensor rank is at most 3:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Algorithm:

$$\begin{aligned} s_1 &= (x_1 + x_2)(y_1 + y_2), \\ s_2 &= x_1y_1, \\ s_3 &= x_2y_2 \end{aligned} \qquad \begin{aligned} f_1 &= s_2 - s_3, \\ f_2 &= s_1 - s_2 - s_3 \end{aligned}$$

Tensor rank

Main problem: For a given tensor, compute its rank and a canonical decomposition.

Two ways of attacking the problem

- looking for lower bounds to the tensor rank
- looking for upper bounds to the tensor rank

Things get more complicate: unlike the case of $m \times n$ matrices

- The rank depends on the ground field $\mathbb{F}$
- High rank tensors can be approximated by low rank tensors

Rational algorithms for tensor rank, like Gaussian elimination, cannot exist

Topological properties of tensors: Border Rank

For $m \times n$ matrices, any sequence $\{A_k\}$ of $m \times n$ matrices of rank r with limit $A = \lim_k A_k$ is such that $\text{rank}(A) \leq r$.

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For higher order tensor this property does not hold anymore

Example (Bini, Capovani, Lotti, Romani 1980)

The tensor

$$\mathcal{A} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right]$$

has rank 3. The tensor

$$\mathcal{A}_\epsilon = \left[\begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right]$$

has rank 2 for any $\epsilon \neq 0$

Canonical decomposition

$$\begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} = \begin{bmatrix} 1 & \epsilon^{-1} \\ \epsilon & 1 \end{bmatrix} - \epsilon^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Border rank

Definition (Bini, Capovani, Lotti, Romani 1980)

The border rank of $\mathcal{A} \in \mathbb{F}^{m\times n\times p}$ $(\mathbb{F} = \mathbb{R}, \mathbb{C})$ is

$$\text{brk}(\mathcal{A}) = \min\{r : \forall \epsilon > 0\, \exists \mathcal{E} \in \mathbb{F}^{m\times n\times p} : ||\mathcal{E}|| < \epsilon,\ \text{rk}(\mathcal{A}+\mathcal{E}) = r\}$$

where $||\cdot||$ is any norm

Some properties:

- $\text{brk}(\mathcal{A}) \leq \text{rk}(\mathcal{A})$
- $\text{brk}(\mathcal{A})$ is the minimum number of nonscalar multiplications sufficient to approximate the set of bilinear forms associated with $\mathcal{A}$ with arbitrarily small nonzero error

More on rank and border rank

Let $\mathcal{A}_\epsilon = \mathcal{A} + \mathcal{E}_\epsilon$ be such that

- $\mathrm{rk}(\mathcal{A}_\epsilon) = \mathrm{brk}(\mathcal{A})$
- the entries of $\mathcal{E}_\epsilon$ are polynomials of degree d

Then

$$\mathrm{rk}(\mathcal{A}) \leq (d+1)\mathrm{brk}(\mathcal{A})$$

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- Write $d+1$ copies of the canonical decomposition of length $\mathrm{rk}(\mathcal{A}_\epsilon)$ obtained with $(d+1)$ pairwise different values of ϵ

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- Take linear combinations of these decompositions with coefficients γ_j, $j = 1 : d+1$ in order to kill the terms in ϵ^i, $i = 1 : d$ and to have $\sum \gamma_j = 1$
- Obtain a decomposition of length $(d+1)\mathrm{brk}(\mathcal{A})$ with no error

Lower bounds and linear algebra

Simple criteria for providing lower bounds on tensor rank and border rank can be given
Trivial bounds

$$\mathrm{brk}(\mathcal{A}) \geq \dim(\mathrm{span}(A_1, \ldots, A_p))$$

Similar inequalities are valid w.r.t. the other coordinates

Assume for simplicity $p = \dim(\mathrm{span}(A_1, \ldots, A_p))$

Lower bounds and linear algebra

Let U, V, W be the matrices defining a canonical factorization of $\mathcal{A}$ of length $\mathrm{rk}(\mathcal{A})$

$$\mathcal{A} = \sum_{\ell=1}^{\mathrm{rk}(\mathcal{A})} \mathbf{u}^{(\ell)} \circ \mathbf{v}^{(\ell)} \circ \mathbf{w}^{(\ell)}$$

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Assume w.l.o.g. that the first p columns of W are linearly independent, that is

$$W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}, \quad W_1 \in \mathbb{F}^{p \times p}, \quad \det W_1 \neq 0$$

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Let $A'_k = \sum_{j=1}^{p} w_{k,j}^{(-1)} A_j$, define
$\mathcal{A}' = [A'_1, A'_2, \ldots, A'_p] = \mathcal{A} \bullet_3 W_1^{-1}$,

i.e., choose a different basis to represent the space spanned by the slabs $A_1, \ldots, A_p$

Evidently, $\mathcal{A}'$ has the same rank of $\mathcal{A}$ and a canonical decomposition is given by $U' = U, V' = V, W' = W_1^{-1}W$, where

$$W' = \begin{bmatrix} I & W_1^{-1}W_2 \end{bmatrix} = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * \\ \hline 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \end{array}\right]$$

Remark

Any subtensor $\widehat{\mathcal{A}}$ formed by k 3-slabs $[A'_{\sigma_1}, \ldots, A'_{\sigma_k}]$ is such that

$$\mathrm{rk}(\widehat{\mathcal{A}}) \leq \mathrm{rk}(\mathcal{A}) - (p - k)$$

More generally,

Theorem

$$rk(\mathcal{A}) \geq \max_{T} \min_{S} (p - k + rk(\mathcal{A} \bullet_3 TS))$$

T: $k \times p$ *submatrix of* I_p, $\quad$ S: $p \times p$ *nonsingular*

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$$rk(\mathcal{A}) \geq \max_T \min_S (p - k + rk(\mathcal{A} \bullet_3 TS))$$

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Corollary

If for any basis of $span(A_1, \ldots, A_p)$ *there exists a matrix of rank* q *then*

$$rk(\mathcal{A}) \geq p + q - 1$$

Higher order generalization:

Corollary

Let $\mathcal{A} = [\mathcal{A}_1, \ldots, \mathcal{A}_{n_h}] \in \mathbb{F}^{n_1 \times \cdots \times n_h}$, $\mathcal{A}_k \in \mathbb{F}^{n_1 \times \cdots \times n_{h-1}}$ *be such that* $n_h = dim(span(\mathcal{A}_1, \ldots, \mathcal{A}_{n_h}))$. *If for any basis of* $span(\mathcal{A}_1, \ldots, \mathcal{A}_{n_h})$ *there exists a tensor in the basis of rank* q *then*

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$$rk(\mathcal{A}) \geq n_h + q - 1$$

Example: For

$$\mathcal{A} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

any basis must contain a nonsingular matrix, therefore

$$\mathrm{rk}(\mathcal{A}) \geq 2 + 2 - 1 = 3$$

Lower bounds to border rank

Unfortunately the same technique cannot be applied to border rank

W_1 may be singular in the limit as $\epsilon \to 0$ so that $\mathcal{E} \bullet_3 W_1^{-1}$ may not be infinitesimal anymore

Remedy: compute the QR factorization of W and multiply W by Q^T which has Euclidean norm 1 for any ϵ. One has

$$W' = Q^T W = \left[\begin{array}{cccc|ccc} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ \hline 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \end{array}\right]$$

Remark

There exists an $m \times n \times k$ subtensor $\widehat{\mathcal{A}}$ such that

$$\mathrm{brk}(\widehat{\mathcal{A}}) \leq \mathrm{brk}(\mathcal{A}) - (p - k)$$

Theorem

$$brk(\mathcal{A}) \geq \min_T \min_S (p - k + brk(\mathcal{A} \bullet_3 TS))$$

T: $k \times p$ submatrix of I_p, S: $p \times p$ nonsingular

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Corollary

If any basis of the linear space spanned by $A_1, \ldots, A_p$ *is made up by matrices of rank at most* q *then* $brk(\mathcal{A}) \geq p + q - 1$

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Corollary (Generalization to higher dimension)

Let $\mathcal{A} = [\mathcal{A}_1, \ldots, \mathcal{A}_{n_h}]$. *If any basis of the linear space spanned by* $\mathcal{A}_1, \ldots, \mathcal{A}_{n_h}$ *is made up by tensors of rank at most* q *then* $brk(\mathcal{A}) \geq n_h + q - 1$

Example

The tensor $\mathcal{A} \in \mathbb{F}^{n^2 \times n^2 \times n^2}$ whose 3-slabs span the linear space

$$I \otimes S = \begin{bmatrix} S & & \\ & \ddots & \\ & & S \end{bmatrix}, \quad S = \begin{bmatrix} s_{1,1} & \cdots & s_{1,n} \\ \vdots & \ddots & \vdots \\ s_{n,1} & \cdots & s_{n,n} \end{bmatrix}$$

is associated with $n \times n$ matrix multiplication.

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is associated with $n \times n$ matrix multiplication.

All the matrices in any basis of the linear space have rank at least n. Therefore

$$\mathrm{brk}(\mathcal{A}) \geq n^2 + n - 1$$

The bound can be improved to

$$\mathrm{brk}(\mathcal{A}) \geq n^2 + 2n - 2$$

Example

The tensor $\mathcal{A} \in \mathbb{F}^{4\times 4\times 3}$ whose 3-slabs span the linear space

$$\left[\begin{array}{cc|cc} s_1 & s_2 & 0 & 0 \\ s_3 & s_4 & 0 & 0 \\ \hline 0 & 0 & s_1 & s_2 \end{array}\right]$$

is associated with the multiplication of a 2×2 triangular matrix and a full 2×2 matrix

Example

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For any basis of the linear space there exist two matrices which form a tensor of rank at least 4.

$$\mathrm{rk}(\mathcal{A}) \geq 4 - 2 + 4 = 6$$

Since there exists a canonical approximate decomposition of length 5 of $\mathcal{A}$ then $\mathrm{brk}(\mathcal{A}) \leq 5 < 6 = \mathrm{rk}(\mathcal{A})$

The direct sum conjecture

Let $\mathcal{A} \in \mathbb{F}^{m \times n \times p}$, $\mathcal{B} \in \mathbb{F}^{m' \times n' \times p'}$. Consider the direct sum of $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{C} = \mathcal{A} \oplus \mathcal{B} \in \mathbb{F}^{(m+m') \times (n+n') \times (p+p')}$$

Direct sum conjecture, Strassen 1973

$\mathrm{rk}(\mathcal{C}) = \mathrm{rk}(\mathcal{A}) + \mathrm{rk}(\mathcal{B})$

The complexity of two disjoint sets of bilinear forms is the sum of the complexities of each set.

The direct sum conjecture

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Direct sum conjecture, Strassen 1973

$\mathrm{rk}(\mathcal{C}) = \mathrm{rk}(\mathcal{A}) + \mathrm{rk}(\mathcal{B})$

The complexity of two disjoint sets of bilinear forms is the sum of the complexities of each set.

Fact

There are cases where $\mathrm{brk}(\mathcal{C}) < \mathrm{brk}(\mathcal{A}) + \mathrm{brk}(\mathcal{B})$

Example (A. Schoenhage 81)

$\mathrm{brk}(\mathcal{A}) = 10 < 9 + 4$

$$\mathcal{A} = \left[\begin{array}{ccc|cccc} s_{11} & s_{12} & s_{13} & & & & \\ s_{21} & s_{22} & s_{23} & & & & \\ s_{21} & s_{22} & s_{23} & & & & \\ \hline & & & t & & & \\ & & & & t & & \\ & & & & & t & \\ & & & & & & t \end{array}\right]$$

$$\left[\begin{array}{ccc|cccc} . & . & . & & & & \\ . & 1 & 1 & \epsilon & \epsilon & & \\ . & 1 & 1 & & & \epsilon & \epsilon \\ \hline & \epsilon & & \epsilon^2 & & & \\ & & \epsilon & & \epsilon^2 & & \\ & \epsilon & & & & \epsilon^2 & \\ & & \epsilon & & & & \epsilon^2 \end{array}\right]$$

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 1 & & & & \\ 1 & . & . & -\epsilon & -\epsilon & & \\ 1 & . & . & & & -\epsilon & -\epsilon \\ \hline & -\epsilon & & & & & \\ & & -\epsilon & & & & \\ & -\epsilon & & & & & \\ & & -\epsilon & & & & \end{array}\right]$$

Matrix multiplication and open problems

- 2×2 matrix product:

$$\mathcal{A} = I_2 \otimes \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & & \\ s_{21} & s_{22} & & \\ & & s_{11} & s_{12} \\ & & s_{21} & s_{22} \end{bmatrix}$$

$\mathrm{rk}(\mathcal{A}) = \mathrm{brk}(\mathcal{A}) = 7$ [Strassen 69, Landsberg 05]

- 3×3 matrix product $\mathcal{A} = I_3 \otimes \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$

$19 \leq \mathrm{rk}(\mathcal{A}) \leq 23$ [Laderman 76], [Bläser 03]

$13 \leq \mathrm{brk}(\mathcal{A}) \leq 22$ [Schönhage 81]

Matrix multiplication and open problems

- 4×4 matrix product $\mathcal{A} = I_4 \otimes \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix}$

$34 \leq \mathrm{rk}(\mathcal{A}) \leq 49$

$22 \leq \mathrm{brk}(\mathcal{A}) \leq 49$