Prediction Market and Parimutuel Mechanism

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Outline

- **World-Cup Betting Example**
- **Market for Contingent Claims**
- **Parimutuel Mechanism**
- **Sequential Parimutuel Mechanism**
- **Betting on Permutation**

World Cup Betting Market

- **Market for World Cup Winner (2006)**
  - **Assume 5 teams have a chance to win the World Cup: *Argentina, Brazil, Italy, Germany and France***
  - **We'd like to have a standard payout of $1 per share if a participant has a claim where his selected team won**
- **Sample Input Orders**

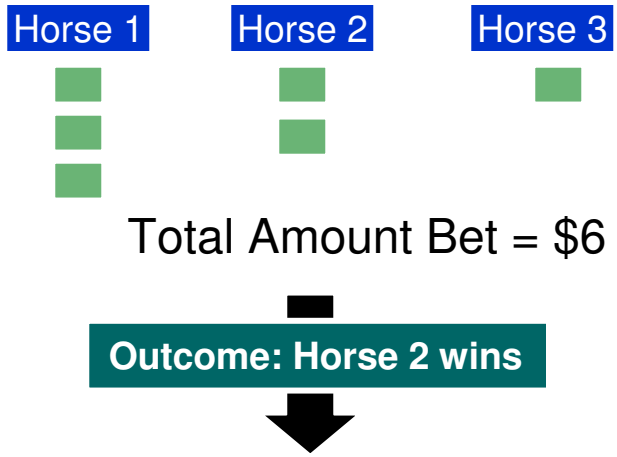
Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

Principles of the Market Maker

- Monotonicity
  - Given any two orders $(\mathbf{a}_1,\pi_1)$ and $(\mathbf{a}_2,\pi_2)$, if $\mathbf{a}_2 \leq \mathbf{a}_1$ and $\pi_2 \geq \pi_1$, then order 1 is awarded implies that order 2 must be awarded.
  - Given any three orders $(\mathbf{a}_1,\pi_1)$, $(\mathbf{a}_2,\pi_2)$, and $(\mathbf{a}_3,\pi_2)$, if $\mathbf{a}_3 \leq \mathbf{a}_1+\mathbf{a}_2$, and $\pi_3 \geq \pi_1+\pi_2$, then orders 1 and 2 are awarded implies that order 3 must be awarded.
  - …
- Truthfulness
  - A charging rule that each order reports π truthfully.
- Parimutuel-ness
  - The market is self funded, and, if possible, even making some profit.

Parimutuel Principle

- **Definition**
  - **Etymology: French *pari mutuel*, literally, mutual stake**
    A system of betting on races whereby the winners divide the total amount bet, after deducting management expenses, in proportion to the sums they have wagered individually.
- **<u>Example: Parimutuel Horseracing Betting</u>**



Winners earn $2 per bet plus stake back: Winners have stake returned then divide the winnings among themselves

The Model and Mechanism

- **A Contingent Claim or Prediction Market**
  - **S possible states of the world (one will be realized)**
  - **N participants who, *k*, submit orders to a market organizer containing the following information:**
    - $a_{i,k}$ **- State bid (either 1 or 0)**
    - q_k **– Limit share quantity**
    - π_k **– Limit price per share**
  - **One market organizer who will determine the following:**
    - x_k **– Order fill**
    - p_i **– State price**
  - **Call or online auction mechanism is used.**

Research Evolution

Call Auction Mechanisms

2002 – Bossaerts, et al.
- Issues with double auctions that can lead to thinly traded markets
- Call auction mechanism helps

2003 – Fortnow et al.
- Solution technique for the call auction mechanism

2005 – Lange and Economides
- Non-convex call auction formulation with unique state prices

2005 – Peters, So and Ye
- Convex programming of call auction with unique state prices

2008 – Agrawal, Wang and Ye
- Parimutuel Betting on Permutations
- (WINE 2008)

Automated Market Makers

2003 – Hanson
- Combinatorial information market design

2004, 2006 – Chen, Pennock et al.
- Dynamic Pari-mutuel market

2007 – Peters, So and Ye
- Dynamic market-maker implementation of call auction mechanism (WINE2007)

2009 – Agrawal et al
- Unified model for PM (EC2009)

LP Market Mechanism

Boosaerts et al. [2001], Lange and Economides [2001], Fortnow et al. [2003], Yang and Ng [2003], Peters et al. [2005], etc

Colleted revenue

Worst-case cost

Cost if state i is realized

$$\max \quad \sum_k \pi_k x_k - z$$

$$\text{s.t.} \quad \sum_k a_{ik} x_k \le z \qquad \forall\, i \in S$$

$$0 \le x_k \le q_k \qquad \forall\, k \in N$$

An LP pricing mechanism for the call auction market the optimal dual solution gives prices of each state

Its dual is to minimize the “information loss” for each order

World Cup Betting Results

Orders Filled

Order	Price Limit	Quantity Limit	Fill	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00

More Issues

- **Online pricing**
  - Make order-fill decisions instantly when an order arrives
  - The state prices could be updated in real time
- **How much is the worst case loss incurred from offline to online?**
- **Could the risk of loss be controlled**
- **Maintain truthfulness**
- **Update prices super fast**

Dynamic Pari-mutuel Mechanisms

- **Logarithmic Market Scoring Rule (Hanson, 2003)**
  - **Based on logarithmic scoring function and its associated price function**
- **Dynamic Pari-mutuel Market Maker (Chen and Pennock, 2004 & 2006)**
  - **Based on a cost function for purchasing shares in a state and a price function for that state**
- **Sequential Convex Programming Mechanism (Peters et al. 2007)**
  - **Sequential application of the CPM**

Sequential Convex Programming Mechanism

- **As soon as bid $(\mathbf{a}, \pi, q)$ arrives, market maker solves**

Immediate revenue

Future value

$$\max_{\{x,\mathbf{s}\}} \pi x - z + u(\mathbf{s})$$

$$\text{s.t.} \quad \mathbf{a}x + \mathbf{s} = ze - \mathbf{q},$$

$$0 \le x \le q$$

Available shares for allocation

where the ith entry of $\mathbf{q}$ is the shares already sold to earlier traders on state i, x is the order fill variable for the new order, and $u(\mathbf{s})$ is any concave and increasing value function of remaining good quantity vector, $\mathbf{s}$.

An Equivalence Theorem
(Agrawal et al EC2009)

- SCPM is a unified frame work for all online prediction market mechanisms and non-regret learning algorithms.
- A new SCPM mechanism
  - Efficient computation for price update, linear in the number of states and loglog(1/ε)
  - Truthfulness
  - Strict Properness
  - Bounded worst-case loss
  - Controllable risk measure of market-maker

Typical Value Functions

- **LMSR:**

$$u(\mathbf{s}) = -b \ln\left(\sum_i e^{-s_i/b}\right)$$

- **QMSR*:**

$$u(\mathbf{s}) = \frac{\mathbf{e}^T\mathbf{s}}{N} - \frac{1}{4b}\mathbf{s}^T\left(1 - \frac{1}{N}\mathbf{e}\mathbf{e}^T\right)\mathbf{s}$$

- **Log-SCPM:**

$$u(\mathbf{s}) = (b/N)\sum_i \ln(s_i)$$

More Issues

(Agrawal et al. WINE2008)

- **Bet on permutations?**
  - **First, second, …; or any combination**
- **Reward rule?**

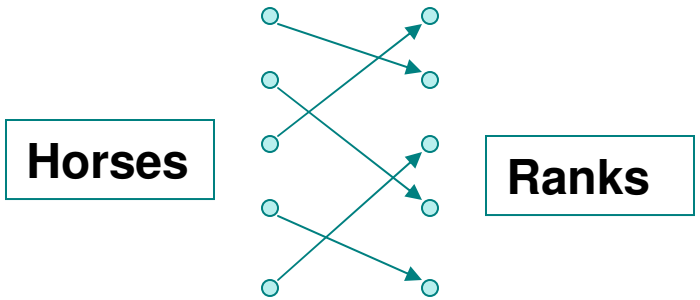
Parimutuel Betting on Permutations

Challenges

- n! outcomes

- **Betting languages/mechanism**
- **How to price them effectively**

Permutation Betting Mechanism



Permutation realization

Outcome

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Bid

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

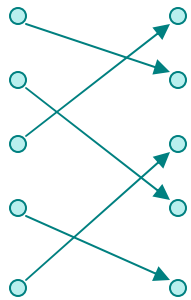
Fixed reward Betting

Reward = $1

Theorem 1: Harder than maximum satisfiability problem!

Permutation Betting Mechanism

Horses



Ranks

Permutation realization

Outcome

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Bid

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Proportional Betting Market

Reward = $3

Marginal Prices

Q

Ranks

Horses

$$\begin{bmatrix} 0.05 & 0.2 & 0.35 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.1 & 0.4 & 0.2 \\ 0.25 & 0.35 & 0.2 & 0.1 & 0.1 \\ 0.4 & 0.05 & 0.25 & 0.1 & 0.2 \end{bmatrix}$$

=1

=1

Marginal Distributions

Theorem One can compute in polynomial-time, an $n \times n$ marginal price matrix Q which is sufficient to price the bets in the Proportional Betting Mechanism. Further, the price matrix is unique, parimutuel, and satisfies the desired price-consistency constraints.

Pricing the Permutations

Q **Ranks**

Horses

$$\begin{bmatrix} 0.05 & 0.2 & 0.35 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.1 & 0.4 & 0.2 \\ 0.25 & 0.35 & 0.2 & 0.1 & 0.1 \\ 0.4 & 0.05 & 0.25 & 0.1 & 0.2 \end{bmatrix}$$

=1

Marginal Distributions

=1

$$= p_1 \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} + p_2 \begin{bmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} + p_2 \begin{bmatrix} & & & 1 & \\ & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & & 1 \end{bmatrix} + \ldots\ldots + p_{n!} \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}$$

Joint Distribution *p* over permutations

Maximum Entropy Criteria

$$\begin{array}{ll} \min & \sum_\sigma p_\sigma \log p_\sigma \\ \text{s.t.} & \sum_\sigma p_\sigma M_\sigma = Q \\ & p_\sigma \geq 0 \end{array} \Rightarrow \boxed{p_\sigma = \frac{1}{e} e^{Y \bullet M_\sigma}}$$

- **Closest distribution to uniform prior**
- **Maximum likelihood estimator**
- **Completely specified by n^2 parameters *Y***
- **Concentration theorem applies**

Hardness and Approximation Results

- **Theorem** It's #P-hard to compute the parameter Y
  - Reduction from the problem of computing permanent of a non-negative matrix

- **Theorem** Using a separating oracle given with the ellipsoid method, a distribution generator $\{Y\}$ over permutations can be constructed in time $poly(n, 1/\varepsilon, 1/q_{min})$ to arbitrary accuracy ε.