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Decision Aiding

Decision-making with the AHP: Why is the principal eigenvector necessary

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Abstract

In this paper it is shown that the principal eigenvector is a necessary representation of the priorities derived from a positive reciprocal pairwise comparison judgment matrix $A = (a_{ij})$ when A is a small perturbation of a consistent matrix. When providing numerical judgments, an individual attempts to estimate sequentially an underlying ratio scale and its equivalent consistent matrix of ratios. Near consistent matrices are essential because when dealing with intangibles, human judgment is of necessity inconsistent, and if with new information one is able to improve inconsistency to near consistency, then that could improve the validity of the priorities of a decision. In addition, judgment is much more sensitive and responsive to large rather than to small perturbations, and hence once near consistency is attained, it becomes uncertain which coefficients should be perturbed by small amounts to transform a near consistent matrix to a consistent one. If such perturbations were forced, they could be arbitrary and thus distort the validity of the derived priority vector in representing the underlying decision.

Keywords: Consistent; Near consistent; Order; Priority; Principal eigenvector

1. Introduction

In the field of decision-making, the concept of priority is quintessential and how priorities are derived influences the choices one makes. Priorities should be unique and not one of many possibilities, they must also capture the dominance of the order expressed in the judgments of the pairwise comparison matrix. The idea of a priority vector has much less validity for an arbitrary positive reciprocal matrix $A = (a_{ij})$ than for a consistent and a near consistent matrix. A positive n by n matrix is reciprocal if $a_{ji} = 1/a_{ij}$. It is consistent if $a_{ij}a_{jk} = a_{ik},\ i,j,k = 1,\dots,n$. From $a_{ij} = a_{ik}/a_{jk}$ we have $a_{ji} = a_{jk}/a_{ik} = a_{ij}^{-1}$ and a consistent matrix is reciprocal. A matrix is said to be near consistent if it is a small perturbation of a consistent matrix. The custom is to look for a vector $w = (w_1,\dots,w_n)$ such that the matrix $W = (w_i/w_j)$ is "close" to $A = (a_{ij})$ by minimizing a metric. Metric closeness to the numerical values of the a_{ij} by itself says little about the numerical precision with which one element dominates another directly as in the

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matrix itself and indirectly through other elements as represented by the powers of the matrix. In this paper we show that with dominance order, the *principal eigenvector*, known to be unique to within a positive multiplicative constant (thus defining a ratio scale), and made unique through normalization, is the *only plausible candidate for representing priorities derived from a positive reciprocal near consistent pairwise comparison matrix*.

The Analytic Hierarchy Process (AHP) allows for inconsistency because in making judgments people are more likely to be cardinally inconsistent than cardinally consistent because they cannot estimate precisely measurement values even from a known scale and worse when they deal with intangibles (a is preferred to b twice and b to c three times, but a is preferred to c only five times) and ordinally intransitive (a is preferred to b and b to c but c is preferred to a). One reason for filling out an entire matrix is to improve the validity of the judgments in the real world. When we deal with tangibles, a pairwise comparison judgment matrix may be perfectly consistent but irrelevant and far off the mark of the true values. For several reasons a modicum of inconsistency may be considered as a good thing and forced consistency without knowledge of the precise values as an undesirable compulsion. If one insists on consistency, people would be required to be like robots unable to change their minds with new evidence and unable to look within for judgments that represent their thoughts, feelings and preferences.

The AHP also uses a principle of hierarchic composition to derive composite priorities of alternatives with respect to multiple criteria from their priorities with respect to each criterion. It consists of multiplying each priority of an alternative by the priority of its corresponding criterion and adding over all the criteria to obtain the overall priority of that alternative. This is perhaps the simplest way for composing priorities. The additive approach is also crucial in doing composition using the limiting powers of a priority rather than a judgment matrix when dependence and feedback are considered in decision-making. Different methods for deriving priorities within the same hierarchy can lead to different final values for the alternatives [7].

2. What is a priority vector?

Now we ask the question, what is priority or more generally what meaning should we attach to a priority vector of a set of alternatives? We can think of two meanings. The first is a numerical ranking of the alternatives that indicates an order of preference among them. The other is that the ordering should also reflect intensity or cardinal preference as indicated by the ratios of the numerical values and is thus unique to within a positive multiplicative constant (a similarity transformation). It is the latter that interests us here as it relates to the principle of hierarchic composition under a single criterion. Given the priorities of the alternatives and given the matrix of preferences for each alternative over every other alternative, what meaning do we attach to the vector obtained by weighting the preferences by the corresponding priorities of the alternatives and adding? It is another priority vector for the alternatives. We can use it again to derive another priority vector ad infinitum. Even then what is the limit priority and what is the real priority vector to be associated with the alternatives? It all comes down to this: What condition must a priority vector satisfy to remain invariant under the hierarchic composition principle? A priority vector must reproduce itself on a ratio scale because it is ratios that preserve the strength of preferences. Thus a necessary condition that the priority vector should satisfy is not only that it should belong to a ratio scale, which means that it should remain invariant under multiplication by a positive constant c, but also that it should be invariant under hierarchic composition for its own judgment matrix so that one does not keep getting new priority vectors from that matrix. In sum, a priority vector x must satisfy the relation $Ax = cx$, $c > 0$. We will show that as a result of the need for invariance to produce a unique priority vector, x must be the principal right eigenvector of A and c is its corresponding principal eigenvalue. Our problem for positive reciprocal matrices and their priorities is a special case of the following:

Theorem. *For a given positive matrix A, the only positive vector x and only positive constant c that satisfy $Ax = cx$, is a vector x that is a positive* *multiple of the Perron vector (principal eigenvector) of A, and the only such c is the Perron value (principal eigenvalue) of A.*

Proof. We know that the (right) Perron vector and Perron value satisfy our requirements. We also know that the algebraic multiplicity of the Perron value is one, and that there is a positive left eigenvector of A (call it z) corresponding to the Perron value. Suppose there is a positive vector y and a (necessarily positive) scalar d such that $Ay = dy$. If d and c are not equal, then by biorthogonality [2] y is orthogonal to z, which is impossible since both vectors are positive. If d and $c$are equal, then y and x are dependent since c has algebraic multiplicity one, and y is a positive multiple of x.

It is also true that if one starts with any priority vector and transforms it through multiplication by A any number of times, in the limit, the product converges to the Perron vector of A. Significantly and interestingly, for our purpose to derive priorities for a special type of positive matrices, the foregoing theorem can also be shown to hold for a class of positive reciprocal matrices that are consistent and near consistent without the use of the theorem of Perron. We know that the principal eigenvector is the priority vector of a consistent matrix. For such a matrix $a_{ij} = w_i/w_j$, and it follows from $Aw = nw$ that the vector $w = (w_1, \ldots, w_n)$ that is also the principal eigenvector of A is its priority vector with corresponding principal eigenvalue $c = n$. We can show by small and continuous perturbation [3,8] of a consistent matrix A that the resulting near consistent matrix (see the next section), has its priority vector as its principal eigenvector obtained as a perturbation of the corresponding principal eigenvector of A. Thus if we assume that a judgment matrix is obtained as a small perturbation of an underlying consistent matrix constructed from a ratio scale $w = (w_1, \ldots, w_n)$, its priority vector coincides with its principal eigenvector obtained as a small perturbation of w. For the perturbation proof, which is fairly long and elaborate, see [4].

That would end our quest if we could also say what to do about a positive inconsistent matrix with large inconsistency. We need to improve its consistency to speak of its priority vector. Using the Perron vector and Perron root of such a matrix, we show that it can be transformed in steps to a near consistent matrix whose priority vector we now know is its principal eigenvector. □

3. Some observations on positive reciprocal matrices and their perturbation

We have for an n by n consistent matrix $A: A^k = n^{k-1}A,\ A = (w_i/w_j)$. A near consistent matrix is a small reciprocal (multiplicative) perturbation of a consistent matrix. It is given by the Hadamard product: $A = W \circ E$, where $W = (w_i/w_j)$ and $E \equiv (\varepsilon_{ij}), \varepsilon_{ji} = \varepsilon_{ij}^{-1}$. Small means ε_{ij} is close to one. Unlike an additive perturbation of the form $a_{ij} + \gamma_{ij}$, a reciprocal perturbation $a_{ij}\varepsilon_{ij}$, $\varepsilon_{ji} = \varepsilon_{ij}^{-1}$ is multiplicative. It can be transformed to an additive perturbation of a consistent matrix by writing:

$$\frac{w_i}{w_j} + \gamma_{ij} = \frac{w_i}{w_j}\varepsilon_{ij}, \quad \varepsilon_{ij} = 1 + \frac{w_j}{w_i}\gamma_{ij},$$

$$\varepsilon_{ji} = \varepsilon_{ij}^{-1} = \frac{w_j}{w_i} + \gamma_{ji} = \frac{1}{1 + \frac{w_j}{w_i}\gamma_{ij}}.$$

Note that with a reciprocal perturbation we ensure that $\lambda_{\max} \geqslant n$ which helps determine the validity of w as a priority vector of a near consistent matrix. We have

$$\sum_{j=1}^{n} \varepsilon_{ij} = \sum_j a_{ij} w_j / w_i = [Aw]_i / w_i = \lambda_{\max} w_i / w_i = \lambda_{\max}.$$

The computation

$$n\lambda_{\max} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\varepsilon_{ij}\right) = \sum_{i=1}^{n}\varepsilon_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(\varepsilon_{ij} + \varepsilon_{ji}) = n + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(\varepsilon_{ij} + \varepsilon_{ij}^{-1}) \geqslant n + (n^2 - n)/2 = n^2$$

reveals that $\lambda_{\max} \geqslant n$. Moreover, since $x + 1/x \geqslant 2$ for all $x > 0$, with equality if and only if $x = 1$, we see that $\lambda_{\max} = n$ if and only if all $\varepsilon_{ij} = 1$, which is equivalent to having all $a_{ij} = w_i/w_j$. The foregoing arguments show that a positive reciprocal matrix

A has $\lambda_{\max} \geqslant n$, with equality if and only if A is consistent.

4. The general case: How to transform a positive reciprocal matrix to a near consistent matrix

To improve the validity of the priority vector, we must transform a given reciprocal judgment matrix to a near consistent matrix. In practice, the judgments available to make the comparisons may not be sufficient to bring the matrix to near consistency. In that case we abandon making a decision based on the information we have, and must seek additional knowledge to modify the judgments.

The final question then is how to construct ε the perturbations in a general reciprocal matrix. A judgment matrix already has some built in consistency; it is not an arbitrary reciprocal matrix. Among others, inconsistency in a matrix may be due to an error such as putting a_{ji} instead of a_{ij} in the i, j position which if appropriately detected and changed the matrix may become near consistent or at least improve the consistency of A. Because the principal eigenvector is necessary for representing dominance (and priorities when near consistency is obtained), we must use an algorithm based on the eigenvector, whose existence is assured by Perron's theory for positive matrices, to improve the consistency of a reciprocal matrix until it is near consistent. Can we do that?

For a given positive reciprocal matrix $A = [a_{ij}]$ and a given pair of distinct indices $k > l$, define $A(t) = [a_{ij}(t)]$ by $a_{kl}(t) = a_{kl} + t, a_{lk}(t) = (a_{lk} + t)^{-1}$, and $a_{ij}(t) = a_{ij}$ for all $i > k,\ j > l$, so $A(0) = A$. Let $\lambda_{\max}(t)$ denote the Perron eigenvalue of $A(t)$ for all $t$in a neighborhood of $t = 0$ that is small enough to ensure that all entries of the reciprocal matrix $A(t)$ are positive there. Finally, let $v = [v_i]$ be the unique positive eigenvector of the positive matrix A^{T} that is normalized so that $v^{\mathrm{T}}w = 1$. Then a classical perturbation formula [2, Theorem 6.3.12] tells us that

$$\left.\frac{\mathrm{d}\lambda_{\max}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{v^{\mathrm{T}}A'(0)w}{v^{\mathrm{T}}w} = v^{\mathrm{T}}A'(0)w = v_k w_l - \frac{1}{a_{kl}^2} v_l w_k.$$

We conclude that

$$\frac{\partial \lambda_{\max}}{\partial a_{ij}} = v_i w_j - a_{ji}^2 v_j w_i \quad \text{for all } i, j = 1, \ldots, n.$$

Because we are operating within the set of positive reciprocal matrices,

$$\frac{\partial \lambda_{\max}}{\partial a_{ji}} = -\frac{\partial \lambda_{\max}}{\partial a_{ij}} \quad \text{for all } i \text{ and } j.$$

Thus, to identify an entry of A whose adjustment within the class of reciprocal matrices would result in the largest rate of change in $\lambda_{\max}$ we should examine the $n(n-1)/2$ values $\{v_i w_j - a_{ji}^2 v_j w_i\}$, $i > j$, and select (any) one of largest absolute value. This is the method proposed for positive reciprocal matrices by Harker [1].

To illustrate the methods discussed above, consider an example involving the prioritization of criteria used to buy a house for a family whose members cooperated to provide the judgments (Table 1).

Table 1
A family's house buying pairwise comparison matrix for the criteria

	Size	Trans.	Nbrhd.	Age	Yard	Modern	Cond.	Finance	w	v
Size	1	5	3	7	6	6	1/3	1/4	0.173	0.047
Trans.	1/5	1	1/3	5	3	3	1/5	1/7	0.054	0.117
Nbrhd.	1/3	3	1	6	3	4	6	1/5	0.188	0.052
Age	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8	0.018	0.349
Yard	1/6	1/3	1/3	3	1	1/2	1/5	1/6	0.031	0.190
Modern	1/6	1/3	1/4	4	2	1	1/5	1/6	0.036	0.166
Cond.	3	5	1/6	7	5	5	1	1/2	0.167	0.059
Finance	4	7	5	8	6	6	2	1	0.333	0.020

$\lambda_{\max} = 9.669$
Consistency ratio (C.R.) = 0.17

Table 2 gives the array of partial derivatives for the matrix of criteria in Table 1.

The (4,8) entry in Table 2 (in bold print) is largest in absolute value. Thus, the family could be asked to reconsider their judgment (4,8) of Age vs. Finance. One needs to know how much to change a judgment to improve consistency, and we show that next. One can then repeat this process with the goal of bringing the C.R. within the desired range. If the indicated judgments cannot be changed fully according to one's understanding, they can be changed partially. Failing the attainment of a consistency level with justifiable judgments, one needs to learn more before proceeding with the decision.

Two other methods, presented here in order of increasing observed efficiency in practice, are conceptually different. They are based on the fact that

$$n\lambda_{\max} - n = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} (\varepsilon_{ij} + \varepsilon_{ij}^{-1}).$$

This suggests that we examine the judgment for which ε_{ij} is farthest from one, that is, an entry a_{ij} for which $a_{ij}w_j/w_i$ is the largest, and see if this entry can reasonably be made smaller. We hope that such a change of a_{ij} also results in a new comparison matrix with that has a smaller Perron eigenvalue. To demonstrate how improving judgments works, take the house example matrix in Table 1. To identify an entry ripe for consideration, construct the matrix $\varepsilon_{ij} = a_{ij}w_j/w_i$ (Table 3). The largest value in Table 3 is 5.32156, which focuses attention on $a_{37} = 6$.

How does one determine the most consistent entry for the (3,7) position? Harker has shown that when we compute the new eigenvector w after changing the (3,7) entry, we want the new (3,7) entry to be w_3/w_7 and the new (7,3) to be w_7/w_3. On replacing a_{37} by w_3/w_7 and a_{73} by w_7/w_3 and multiplying by the vector w one obtains the same product as one would by replacing a_{37} and a_{73} by zeros and the two corresponding diagonal entries by two (see Table 4).

We take the Perron vector of the latter matrix to be our w and use the now-known values of w_3/w_7 and w_7/w_3 to replace a_{37} and a_{73} in the original matrix. The family is now invited to change their judgment towards this new value of a_{37} as much as they can. Here the value was

Table 2
Partial derivatives for the house example

	Size	Trans.	Nbrhd.	Age	Yard	Modern	Cond.	Finance
Size	–	0.001721	0.007814	−0.00041	0.00054	0.000906	−0.08415	−0.03911
Trans.	–	–	−0.00331	0.001291	0.002485	0.003249	−0.06021	−0.01336
Nbrhd.	–	–	–	−0.0091	−0.00236	−5.7E-05	0.008376	−0.07561
Age	–	–	–	–	−0.01913	−0.03372	0.007638	**0.094293**
Yard	–	–	–	–	–	−0.01366	−0.01409	0.041309
Modern	–	–	–	–	–	–	−0.02599	0.029355
Cond.	–	–	–	–	–	–	–	0.006487
Finance	–	–	–	–	–	–	–	–

Table 3
$\varepsilon_{ij} = a_{ij}w_j/w_i$

1.00000	1.55965	3.26120	0.70829	1.07648	1.25947	0.32138	0.48143
0.64117	1.00000	1.16165	1.62191	1.72551	2.01882	0.61818	0.88194
0.30664	0.86084	1.00000	0.55848	0.49513	0.77239	**5.32156**	0.35430
1.41185	0.61656	1.79056	1.00000	0.59104	0.51863	1.36123	2.37899
0.92895	0.57954	2.01967	1.69193	1.00000	0.58499	1.07478	1.78893
0.79399	0.49534	1.29467	1.92815	1.70942	1.00000	0.91862	1.52901
3.11156	1.61765	2.25498	0.73463	0.93042	1.08858	1.00000	0.99868
2.07712	1.13386	2.82246	0.42035	0.55899	0.65402	1.00133	1.00000

Table 4

	Size	Trans.	Nbrhd.	Age	Yard	Modern	Cond.	Finance	w
Size	1	1.7779	1.756208	0.774933	1.163989	1.418734	0.425449	0.494088	0.174
Trans.	0.562461	1	0.548777	1.556678	1.636746	1.994957	0.717895	0.794016	0.062
Nbrhd.	0.569408	1.822233	2	1.134652	0.994177	1.615679	0	0.675211	0.102
Age	1.290434	0.642394	0.881328	1	0.584131	0.533978	1.64704	2.23156	0.019
Yard	0.859115	0.610968	1.005857	1.711945	1	0.609428	1.315833	1.697915	0.034
Modern	0.704854	0.501264	0.618935	1.872735	1.640883	1	1.079564	1.39304	0.041
Cond.	2.35046	1.392962	0	0.60715	0.759975	0.9263	2	0.774223	0.223
Finance	2.02393	1.259421	1.481018	0.448117	0.588958	0.717855	0.291617	1	0.345

Table 5
Modified matrix in the a_{37} and a_{73} positions

	Size	Trans.	Nbrhd.	Age	Yard	Modern	Cond.	Finance	w	v
Size	1	5	3	7	6	6	1/3	1/4	0.175	0.042
Trans.	1/5	1	1/3	5	3	3	1/5	1/7	0.062	0.114
Nbrhd.	1/3	3	1	6	3	4	1/2	1/5	0.103	0.063
Age	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8	0.019	0.368
Yard	1/6	1/3	1/3	3	1	1/2	1/5	1/6	0.034	0.194
Modern	1/6	1/3	1/4	4	2	1	1/5	1/6	0.041	0.168
Cond.	3	5	2	7	5	5	1	1/2	0.221	0.030
Finance	4	7	5	8	6	6	2	1	0.345	0.021
$\lambda_{max} = 8.811$										
Consistency Ratio (C.R.) = 0.083										

$a_{37} = 0.102/0.223 = 1/2.18$, approximated by 1/2 from the AHP integer valued scale and we hypothetically changed it to 1/2 to illustrate the procedure (see Table 5). If the family does not wish to change the original value of a_{37}, one considers the second most inconsistent judgment and repeats the process. The procedure just described is used in the AHP software *Expert Choice*.

A refinement of this approach is due to W. Adams. One by one, each reciprocal pair a_{ij} and a_{ji} in the matrix is replaced by zero and the corresponding diagonal entries a_{ii} and a_{jj} are replaced by 2, the principal eigenvalue λ_{max} is then computed. The entry with the largest resulting λ_{max} is identified for change as described above. This method is in use in the Analytic Network Process (ANP) software program Superdecisions [5].

5. Conclusions

We have shown that if inconsistency is allowed in a positive reciprocal pairwise comparison matrix (which we have shown it must), the principal eigenvector is necessary for representing the priorities associated with that matrix, providing that the inconsistency is less than or equal to a desired value [6]. We also mentioned three ways and illustrated two of them, as to how to improve the consistency of judgments and transform an inconsistent matrix to a near consistent one.

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