This is one of the earlier versions of a paper that will be Chapter 27 in the *Handbook of Social Choice and Welfare, II,* ed. K. Arrow, A. Sen, and K. Suzumura, to be published by Elsevier. While it is my understanding that this version can be freely circulated at our AIM meeting, I ask that this be the limit. Outside of this meeting, please use the final, more accurate version that is expected to appear probably within the next couple of months.

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GEOMETRY OF VOTING[1]

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Abstract. It is shown how simple geometry can be used to analyze and discover new properties about pairwise and positional voting rules as well as for those rules (e.g., runoffs and Approval Voting) that rely on these methods. The description starts by providing a geometric way to depict profiles, which simplifies the computation of the election outcomes. This geometry is then used to motivate the development of a "profile coordinate system", which evolves into a tool to analyze voting rules. This tool, for instance, completely explains various long-standing "paradoxes" such as why a Condorcet winner need not be elected with certain voting rules. A different geometry is developed to indicate whether certain voting "oddities" can be dismissed or must be taken seriously, and to explain why other mysteries, such as strategic voting and the no-show

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paradox (where a voter is rewarded by not voting) arise. Still another use of geometry extends McGarvey's theorem about possible pairwise election rankings to identify the actual tallies that can arise (a result that is needed to analyze supermajority voting). Geometry is also developed to identify all possible positional and Approval Voting election outcomes that are admitted by a given profile; the converse becomes a geometric tool that can be used to discover new election relationships. Finally, it is shown how lessons learned in social choice, such as the seminal Arrow's and Sen's theorems and the expanding literature about the properties of positional rules, provide insights into difficulties that are experienced by other disciplines.

1 Introduction

A goal of the "geometry of voting" is to capture the sense that "a picture is worth a thousand words." Geometry after all, has long served as a powerful tool to provide a global perspective of whatever we are studying while exposing unexpected relationships. This is why we graph functions, plot data, study the Edgeworth box from economics, and use diagrams to enhance lectures. Similarly, a purpose of the geometry of voting is to create appropriate geometric tools to enable us to more easily capture global aspects of decision and voting rules while exposing new relationships. Since most, if not all practical voting rules combine pairwise or positional methods (e.g., runoffs, Approval Voting,cumulative voting, etc.), or use the rule directly, these are the emphasized methods.

A major reason why the area of social choice is so complex is that it is subject to the "curse of dimensionality;" e.g., this curse is what prevents us from using standard geometric approaches to address the challenges of this area. Already with the first interesting setting of three alternatives, the $3! = 6$ dimensions of profile space overwhelm any hope to use standard graphs to connect profiles with election outcomes. (Unless stated otherwise, a profile specifies how many voters have each preference ranking.) As standard "graphing" approaches fail, we need to create new geometric tools. This is the purpose of this chapter.

As illustrations, in Sect. 2 profiles are geometrically depicted in a manner to simplify the tallying process. Lessons learned from this geometric tallying approach are then used to create a "coordinate system" for the six-dimensional profile space; as it will be shown, this profile coordinate system allows us to better understand basic mysteries of social choice and voting theory. In Sect. 3, a geometric approach is created to more easily determine (e.g., rather than computing likelihoods) whether various paradoxical settings must be taken seriously, or can be dismissed as isolated anomalies. The section ends by describing the geometry of the three-candidate profile space in a manner that identifies which profiles give different outcomes for different voting rules.

A different theme, developed in Sect. 4, is motivated by those "nail-biting" close elections that involve three or more candidates. For many of us, encountering such an election creates an irresistible temptation to explore whether the outcome would have changed had a different election rule been used. (Had a different voting rule been used in the 2000 US presidential election, for instance, could Gore have beaten Bush?) Published results exploring these concerns typically consider only certain better known methods, which leads us to wonder what could happen had any of the infinite number of other rules been used. The geometric approach in Sect. 4 resolves this problem by showing how to depict *all possible positional and Approval Voting outcomes* for any specified profile. The provided references show how to discover all possible outcomes for any voting rule that offers a voter more than one way to tally his ballot; e.g., cumulative voting. The converse of these approaches creates a tool to identify new election relationships.

The theme of Sect. 5 is motivated by the wealth of surprising conclusions and information that have been discovered in social choice. What connects this information with concerns coming from other disciplines is that social choice emphasizes aggregation rules — rules of voting and group decision making. But aggregation methods are widely used in almost all disciplines (e.g., statistics creates ways to aggregate data), which makes it reasonable to wonder whether the hard-won results found in social choice might transfer to other disciplines. Can other areas benefit from, say, the insights of Arrow's and Sen's results, or from the substantial amount of new knowledge that has been discovered about voting rules? As shown, this is the case.

2 From simple geometry to profile coordinates

As a way to partly sidestep that dimensionality cure, I show how to list profiles in a manner that roughly mimics the structure of profile space. An advantage of this *"geometric profile representation"* is that it makes it much easier to tally positional and pairwise ballots. An added benefit of this tallying approach is that it identifies why, with the same profile, different rules can have conflicting election outcomes. This information, which is exploited to create a "coordinate system" for profiles, is used to answer several voting theory concerns.

2.1 Using geometry to represent profiles

A traditional way to describe a profile for the three alternatives A, B, C is to list how many of the voters' preferences are represented by each of the six rankings; e.g.,

Number	Ranking	Number	Ranking
7	$A \succ B \succ C$	12	$C \succ B \succ A$
15	$A \succ C \succ B$	4	$B \succ C \succ A$
2	$C \succ A \succ B$	12	$B \succ A \succ C$

(1)

But, let us be honest, tallying such a ballot can be be tedious; it requires sifting through the data to find how many voters rank each candidate in different ways. To avoid this annoyance, I developed the following geometric approach (Saari, 1994, 1995, 2001) that significantly simplifies the tallying process. (For applications to actual elections, see Nurmi (1999, 2000, 2002) and Tabarrok (2001).)

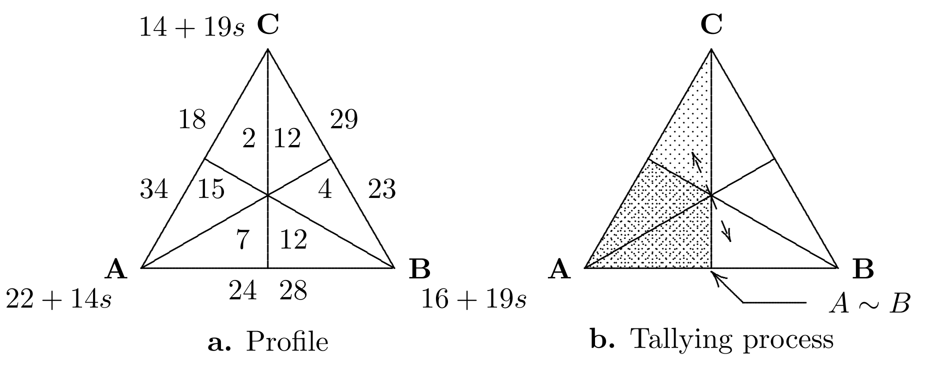


Fig. 1. Profile representation and tallies

Identify each candidate with a vertex of an equilateral triangle (Fig. 1a). A point in the triangle defines a ranking according to its distances to the three vertices where "closer is better." As the points on the vertical line of Fig. 1a are equal distance from the A and B vertices, for instance, they represent the $A \sim B$ tie or indifference. Similar lines define the $A \sim C$ and $B \sim C$ regions. This binary relationship divides the triangle into the thirteen Fig. 1a regions: the six small open triangles represent strict rankings and the seven remaining ranking regions, which include tied outcomes, are portions of the lines. In Fig. 1a, for instance, the number 15 is in the region that is closest to the A vertex, next closest to C, and farthest from B; this region corresponds to an $A \succ C \succ B$ ranking. This geometric positioning of ranking regions is similar to that of profile space in that adjacent "ranking regions" differ only by the ranking of an adjacent pair.

The *geometric profile representation* places the number of voters with a particular ranking in the associated ranking region; e.g., Fig. 1a represents the Eq. 1 profile. When representing a profile in a vector form, the convention is to start with the number of voters with the $A \succ B \succ C$ ranking and list the values in a clockwise manner ending with the $B \succ A \succ C$ value. Thus the numbers in Fig. 1b denote the "ranking types;" e.g., a type-two ranking (the Fig. 1b region with a 2) is $A \succ C \succ B$. With this notation, the Fig. 1a profile has the vector representation $(7, 15, 2, 12, 4, 12)$.

To find the $\{A, B\}$ majority vote tally, observe that those voters preferring A to B are listed to the left of the vertical $A \sim B$ line. Thus, A's tally in a $\{A, B\}$ election is the sum of values placed in the type $\{1, 2, 3\}$ regions; these terms would be in the Fig. 1b three shaded regions; B's tally would be the sum from the three unshaded regions (types $\{4, 5, 6\}$). With the Fig. 1a profile, the $B \succ A$ tally of $28 : 24$ is listed under the A–B leg. All other pairwise tallies are similarly computed and listed near the appropriate triangle edge. The outcome is the *cycle* $B \succ A, A \succ C, C \succ B$.

Candidate A's plurality tally is the number of voters who have her top-ranked, so this tally would be the Fig. 1b sum of the numbers in the two heavily shaded regions (types $\{1, 2\}$). Using this approach with Fig. 1a, the plurality ranking is $A \succ B \succ C$ with a $22 : 16 : 14$ tally: these are the $s = 0$ values for the expressions located next to the appropriate vertices.

A *positional election* assigns specified points to candidates based on how they are *positioned* on a ballot; for three candidates, such a rule is defined by the *voting vector* (w_1, w_2, w_3), $w_1 \geq w_2 \geq w_3$, $w_1 > w_3$, where w_j points are assigned to the j^{th} positioned candidate, To normalize these rules, let $w_3 = 0$ and divide the weights by w_1 to obtain $\mathbf{w}_s = (\frac{w_1}{w_1}, \frac{w_2}{w_1}, 0) = (1, s, 0)$ for $0 \leq s \leq 1$. For instance, a rule assigning 7, 5, and 0 points, respectively, to a voter's first, second, and third positioned candidate has the voting vector $(7, 5, 0)$ with the normalized $\mathbf{w}_{\frac{5}{7}} = (\frac{7}{7}, \frac{5}{7}, 0)$ form. (The $(7, 5, 0)$ tally is seven times the normalized $\mathbf{w}_{\frac{5}{7}}$ tally.) The plurality and antiplurality[2] rules are defined, respectively, by $(1, 0, 0)$ and $(1, 1, 0)$, while the Borda Count is defined by $(2, 1, 0)$ with the normalized $\mathbf{w}_{\frac{1}{2}} = (1, \frac{1}{2}, 0)$. (For n candidates, the Borda Count is defined by $(n-1, n-2, \ldots, 1, 0)$.)

With this normalization, A's $\mathbf{w}_s$-election tally adds s times the number of voters who have her second ranked to her plurality tally. In Fig. 1b, this is A's plurality vote plus s times the sum of type 3 and 6 voters, or

> [the sum of the numbers in the two heavily shaded regions] plus s times [the sum of numbers in the two adjacent regions indicated by the arrow].

Using Fig. 1a, this $[15 + 7] + s[2 + 12] = 22 + 14s$ tally is listed by the A vertex. The similarly computed $\mathbf{w}_s$-tallies for the two other candidates are posted by the appropriate vertices.

2.2 Different information; different voting outcomes

As shown next, these triangles can be used to identify the source of most "voting paradoxes": Namely, different rules use different information from a profile. If this comment is true, then we must anticipate different outcomes coming from different rules. Indeed, as the geometry of voting reveals, *all differences in election outcomes, all conflicts (including Arrow's and Sen's theorems) reflect differences in how voting rules use, or ignore, information from a profile.*

To develop intuition about this comment, let me suggest that the reader use this profile representation and the tallying approach to solve (before reading ahead) the following three problems:

[2]It is called "antiplurality" because it is equivalent to voting against one candidate.

1. While fixing the plurality tallies, modify the Fig. 1a profile so that the pairwise votes define the opposite cycle of $A \succ B, B \succ C, C \succ A$.
2. While fixing the plurality tallies, modify the Fig. 1a profile so that the paired comparisons define the $B \succ A \succ C$ outcome where B is the Condorcet winner. (A candidate is the Condorcet winner if she beats all other candidates in two-person majority votes.)
3. While fixing the plurality tallies, modify the Fig. 1a profile so that the antiplurality election ranking now is $C \succ B \succ A$.

To fix the Fig. 1a plurality tallies, the sum of the two numbers by a vertex must equal the sum of the Fig. 1a values; e.g., the integers adjacent to the A vertex must sum to 22.

The geometric tallying approach simplifies these challenges. To create a profile with the specified cycle, for instance, move voter preferences so that enough of them are in the $A \succ B$ vertical region (the light shaded region of Fig. 2a), the $B \succ C$ upward sloping shaded region, and the heavier shaded $C \succ A$ region; this defines the Fig. 2a shaded triangle. Thus, by concentrating the voter preferences (from each vertex's two ranking regions) in the regions with a vertex of the shaded triangle – types $\{1, 3, 5\}$ – we can create the desired cycle that has the strongest supporting tallies.

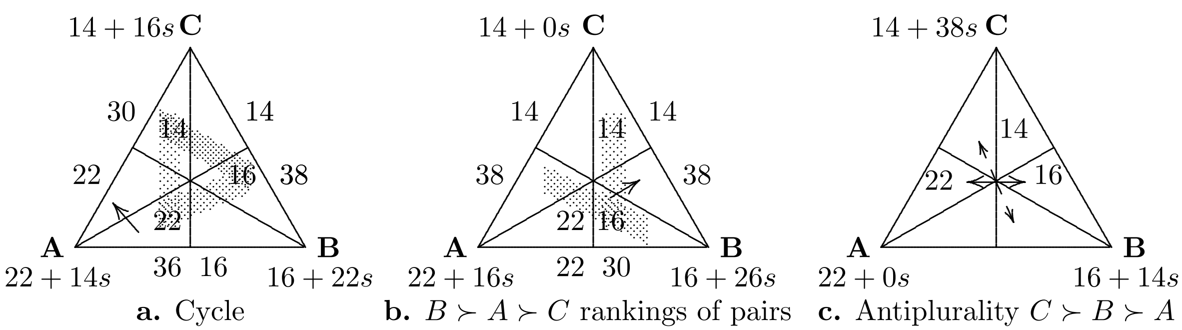


Fig. 2. Creating "paradoxes"

Not only do the Fig. 2a plurality votes remain the same as in Fig. 1a, but moving three of the 22 voters from the $A \succ B \succ C$ region to $A \succ C \succ B$ (indicated by the Fig. 2a arrow) creates the $(19, 3, 14, 0, 16, 0)$ profile where the tallies for *all positional rules* agree with their Fig. 1a values but the paired comparisons cycle in the opposite direction. Stated in terms of the consequences for voting rules, this construction indicates that there exists a partial disconnect between positional outcomes and the majority votes over pairs. The next exercise provides added support for this comment.

This second question is handled with a similar geometric approach. Namely, move enough voters to the two shaded Fig. 2b regions (where numbers remain in regions next to the same vertex to fix the Fig. 1a plurality outcomes) to attain the desired outcome. To create an example where the paired rankings define $B \succ C \succ A$, just slide the slanted shaded region above the $A \sim C$ indifference line; e.g., for an extreme choice, move the 16 voters from $B \succ A \succ C$ into the adjacent region, as indicated by the Fig. 2b arrow, to create the $(22, 0, 0, 14, 16, 0)$ profile.

These exercises capture the essence of my comment that differences in election outcomes reflect how different rules use different aspects of information from a profile. Each shaded region in Figs. 2a, b, for instance, describes the information needed to determine a specified paired comparison outcome, but each shaded region also contributes to (i.e., influences) the plurality outcome for *two of the candidates.* As two shaded regions can meet regions defining the plurality outcome for all three candidates (Fig. 2b), it now becomes easy to adjust a profile so that the plurality tallies remain fixed, but the majority vote outcomes can vary. In other words, anticipate results asserting that *the plurality and paired comparison rankings need not have anything in common.* This is made precise below. (While there are restrictions, the first analysis exploring this concern that I have seen is a 2005 preprint of (McDonald and Sieberg 2010).)

The third exercise identifies informational differences that create different positional outcomes. To reduce A's $\mathbf{w}_s$ tally while fixing the plurality outcome, decrease her second place votes — the values in the two Fig. 2c regions (types 3 and 6) with dashed arrows. Similarly, to increase C's antiplurality tally, move some of these voters into the regions (with solid arrows – types 2 and 5) providing second place votes for C. Guided by these arrows, the extreme Fig. 2c case has the $C \succ B \succ A$ antiplurality tally of 52:30:22; there are many more moderate examples.

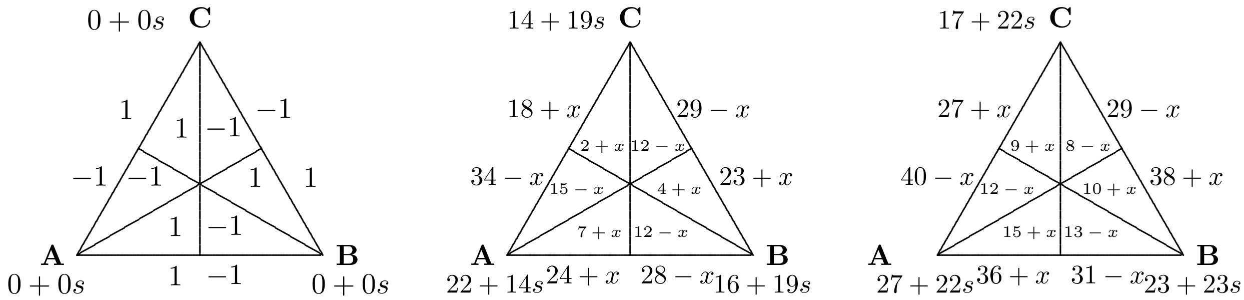
The informational difference between plurality and other positional methods, then, is captured by the "second place" arrows; two of the three choices are indicated in Fig. 2c. As each arrow meets the plurality regions for two candidates, expect assertions that there need not be any compatibility between the plurality and, say, the antiplurality rankings. This, too, is made precise below.

2.3 Profile coordinates

The geometric tallying method, then, identifies which portions of a profile are used by different rules. The natural next step is to exploit this information to create a general tool (that I call "profile coordinates" (Saari 1999, 2000, 200a, 2008)) that allow us to more easily analyze voting rules. Before doing so, notice that a coordinate system is of particular value if it simplifies the problem of interest by capturing relevant structures. Rather than helping, the normally useful x-y Cartesian system, for instance, can complicate the analysis of tracking a satellite; polar coordinates which specify the radius and angle defined by the object are more natural and valuable. In the same spirit, a way to significantly simplify the analysis of voting problems is to convert natural configurations of voter preferences into a profile coordinate system.

Start with what I call the "kernel profile;" these are the "neutral" profiles in that they merely yield complete ties for all positional rules and majority votes over pairs. A basis vector for this space is $\mathbf{K}^3 = (1, 1, 1, 1, 1, 1)$; this is where each ranking is supported by one voter. A general kernel profile, then, is given by $c\mathbf{K}^3$ for scalar $c > 0$. For three candidates, the kernel vectors define an one-dimensional subspace. But for $n > 3$ candidates, the kernel vectors reside in a $[n! - 2^{n-1}(n-2) - 1]$ dimensional space; e.g., for six candidate, this expression requires the kernel to consume over 80% of the 720 dimensions (591 of them) of profile space, and for ten candidates the kernel claims over 99.88% of the profile space dimensions! With the kernel voraciously devouring so many dimensions from the $n!$-dimensional profile space (Saari 2000, 2000a), we reach the ironic conclusion that a major source of the dimensionality curse plaguing social choice is due to profiles that, alone, do nothing!

The next natural profile direction (motivated by Fig. 2a) identifies those configurations of preferences that affect pairwise tallies, but nothing else. The extreme setting (Fig. 2a) assigns a voter to each ranking region with a vertex of the shaded triangle (types $\{1, 3, 5\}$); this defines $\mathbf{v} = (1, 0, 1, 0, 1, 0)$. To create a coordinate system with vectors orthogonal to $\mathbf{K}^3$ (to reduce technical problems), the sum of the coordinates must equal zero; I use $\mathbf{C}^3 = 2\mathbf{v} - \mathbf{K}^3 = (1, -1, 1, -1, 1, -1)$ (the Fig. 3a listing). Interpret the -1 terms in $\mathbf{C}^3$ as representing where a voter leaves this preference to adopt a preference with a $+1$; i.e., treat $\mathbf{C}^3$ as a vector that changes a given profile without changing the number of voters. The relevant properties of this *Condorcet profile* $\mathbf{C}^3$ are captured by the Fig. 3a tallies: $\mathbf{C}^3$ does not change any positional tally, but it generates a majority vote cycle. Even stronger, as asserted below, all possible inconsistencies and problems with three-candidate pairwise rankings and tallies are caused by $\mathbf{C}^3$; *nothing else is involved!*



a. $\mathbf{C}^3$ **b.** Changes in pairwise tallies **c.** Another example

Fig. 3. $\mathbf{C}^3$: Condorcet profile direction

Figure 3b illustrates this $\mathbf{C}^3$ affect; it lists all ways to fix *all of the Fig. 1a positional tallies* while changing the majority vote tallies. To do so, just add $x\mathbf{C}^3$ to the Fig. 1a profile where, to ensure non-negative Fig. 3b profile entries, x is an integer satisfying $-2 \le x \le 12$. The pairwise outcomes start with the $B \succ A$, $A \succ C$, $C \succ B$ cycle (for $-2 \le x \le 1$), progress to where A is the Condorcet (and Borda) winner with $A \succ B \succ C$ ($3 \le x \le 7$), and conclude with the $A \succ B$, $B \succ C$, $C \succ A$ cycle ($9 \le x \le 12$); with all 15 profiles, all positional tallies remain fixed! The Fig. 3c example is described below.

To find the basis for $n \ge 3$ candidates, start with a ranking of n-candidates, say $B \succ A \succ D \succ C \succ \ldots \succ Z$ and construct the "Condorcet n-tuple" by moving the top-ranked candidate to bottom place in the next ranking; e.g., the next two rankings are $A \succ D \succ C \succ \ldots \succ Z \succ B$ and $D \succ C \succ \ldots, \succ Z \succ B \succ A$. For each ranking in this n-tuple, assign -1 voters to the reversal of the ranking. I call this configuration of $2n$ preference rankings a *"Condorcet basis vector."*

An analysis of voting rules often requires determining whether a given profile has some portion in a Condorcet (or other) direction. To show how to do so, recall that if vector $\mathbf{b}$ has Euclidean length $||\mathbf{b}||$, then the amount of vector $\mathbf{a}$ in the direction $\frac{1}{||\mathbf{b}||}\mathbf{b}$ is given by the scalar product $< \mathbf{a}, \frac{1}{||\mathbf{b}||}\mathbf{b} >$. So, because $||\mathbf{C}^3|| = \sqrt{1^2 + (-1)^2 + \ldots + (-1)^2} = \sqrt{6}$, the amount of the Fig. 1a profile in the $\frac{1}{\sqrt{6}}\mathbf{C}^3$ direction is the fairly large value

$$< (7, 15, 2, 12, 4, 12), \frac{1}{\sqrt{6}}(1, -1, 1, -1, 1, -1) >= \frac{1}{\sqrt{6}}[7 - 15 + 2 - 12 + 4 - 12] = -\frac{26}{\sqrt{6}}.$$

This means that the Fig. 1a profile has a $-\frac{26}{\sqrt{6}}[\frac{1}{\sqrt{6}}\mathbf{C}^3] = -\frac{13}{3}\mathbf{C}^3$ component; the negative value corresponds to the sign of x in $x\mathbf{C}^3$; it means that the direction of the cyclic effect of the pairs is given by $B \succ A, A \succ C, C \succ B$. In comparison, as

$$< (3, 6, 2, 1, 2, 0), \frac{1}{\sqrt{6}}\mathbf{C}^3 >= \frac{1}{\sqrt{6}}[3 - 6 + 2 - 1 + 2 - 0] = 0, \tag{2}$$

profile $(3, 6, 2, 1, 2, 0)$ does not have any Condorcet components; it is orthogonal to this space.

2.3.1 Cycles and "Borda vs. Condorcet"

The Condorcet basis vectors can be used to answer several long standing concerns from social choice including the 230 year debate as to whether the Borda or the Condorcet winner more accurately reflects the views of the voters. An often repeated criticism (based on Condorcet's 1784 example) of the Borda Count is that it need not elect the Condorcet winner. The next theorem identifies both why this can happen and what is the source of all possible differences between the rankings of the two approaches. Another criticism of the Borda Count involves examples that can be created to illustrate inconsistencies in Borda rankings over different subsets of candidates; this theorem identifies the total cause of all of possible Borda difficulties.

Theorem 1 *(Saari 1999, 2000, 2000a, 2008) Assume there are $n \geq 3$ candidates.*

1. *There are $\frac{(n-1)!}{2}$ linearly independent Condorcet basis vectors; i.e., they form a $\frac{(n-1)!}{2}$ dimensional subspace of the $n!$ dimensional profile space. The n-candidate positional tally of a Condorcet basis vector is zero for each candidate, but the pairwise rankings include a cycle where each tally is $(n-2) : -(n-2)$.*

2. *A profile is orthogonal to all Condorcet basis vectors if and only if the pairwise rankings define a transitive ranking and for each X, Y, Z triplet, the tallies satisfy*

$$\tau(X, Y) + \tau(Y, Z) = \tau(X, Z) \tag{3}$$

   where $\tau(X, Y)$ is the difference between X's and Y's majority vote tallies in a $\{X, Y\}$ vote. (Thus, all possible pairwise cycles are completely due to Condorcet terms in a profile.)

3. *All possible differences between an n-candidate Borda ranking and the majority vote rankings of the pairs are due to Condorcet terms in a profile.*

4. *The n-candidate Borda tally for candidate X is the sum of the number of points she received over all of her majority vote pairwise comparisons. For any k satisfying $3 \leq k < n$, there is a constant depending on k so that candidate X's n-candidate Borda tally is this multiple times the sum of her Borda tallies over all subsets of k candidates.*

5. *All possible differences between a Borda ranking of a set of k alternatives, $k \geq 2$, and the Borda ranking of $n > k$ alternatives, are caused by Condorcet terms in a profile.*

As profile $(3, 6, 2, 1, 2, 0)$ is orthogonal to $\mathbf{C}^3$ (Eq. 2), it illustrates the surprising Eq. 3 equality; e.g., the outcome $A \succ B$ is by 11:3, $C \succ B$ by 9:5, and $A \succ C$ by 9:5, so $\tau(A, B) = 11 - 3 = 8$, $\tau(B, C) = 5 - 9 = -4$, and $\tau(A, C) = 9 - 5 = 4$, which satisfies $\tau(A, B) + \tau(B, C) = \tau(A, C)$. In contrast, the Fig. 1a profile (with $\mathbf{C}^3$ components) defines $\tau(A, B) = -4, \tau(B, C) = -6, \tau(A, C) = 16$, which fails to satisfy Eq. 3. By using $\tau(A, C) = -\tau(C, A)$, it is easy to prove for three candidates that the sign of $\tau(A, B) + \tau(B, C) + \tau(C, A)$ is the sign of the profile's multiple of $\mathbf{C}^3$.[3]

Parts 3 and 4 explain all possible differences between the Borda and majority vote rankings; namely, the Condorcet terms do not effect the Borda Count ranking but they do alter the pairwise tallies and can distort the pairwise rankings. This is nicely illustrated with Fig. 3c where $\mathbf{C}^3$ components are added to the base profile of $(15, 12, 9, 8, 10, 13)$. As ensured by part 4, each candidate's Borda tally is the sum of her tallies in her two pairwise votes; as this summation adds a $+x$ to a $-x$, the Borda tally cancels all Condorcet effects. Thus, over all 18 profiles represented in this figure,[4] the Borda ranking remains $A \succ B \succ C$ with a fixed 76:69:56 tally. In contrast, the pairwise rankings start with the cycle $A \succ C, C \succ B, B \succ A$ for the $-9 \leq x \leq -5$ values, advance to where the Condorcet winner in $B \succ A \succ C$ differs from the Borda winner for $x = -4, -3$, change the

[3] With the unanimity profile $(1, 0, 0, 0, 0, 0)$, $\tau(A, B) = \tau(A, C) = \tau(B, C) = 1$ so $\tau(A, B) + \tau(B, C) + \tau(C, A) = 1$. The guaranteed $\mathbf{C}^3$ component manifests itself in that the pairwise tallies of $\tau(A, B) = \tau(A, C) = 1$ fail to distinguish whether A is "more preferred" over B, or over C.

[4] To have non-negative entries in the profile, x must be an integer satisfying $-9 \leq x \leq 8$.

pairwise ranking to $A \succ B \succ C$ to agree with the Borda ranking for the nine $-2 \leq x \leq 6$ values, and conclude with the opposite cycle of $A \succ B, B \succ C, C \succ A$ for the remaining $x = 7, 8$ values. (It turns out that the Borda and Condorcet winners can differ with three candidates only if the difference between the Borda tallies of the first and second place candidates is smaller than the difference between the second and third place candidates (Saari 2001b); e.g., here it is $76 - 69 < 69 - 56$. Another result is that for any family, such as Figs. 3bc, there always are more choices where the Borda and transitive Condorcet rankings agree than disagree.)

Parts 4, 5 explain all possible paradoxical Borda Count outcomes. To expand on the statement, notice that all positional rules have a complete tie for any Condorcet four-tuple such as $\mathbf{C}^4 = \{A \succ B \succ C \succ D, B \succ C \succ D \succ A, C \succ D \succ A \succ B, D \succ A \succ B \succ C\}$. But dropping any one candidate, say D, from $\mathbf{C}^4$ defines the Condorcet triplet $A \succ B \succ C$, $B \succ C \succ A$, $C \succ A \succ B$, where all three candidate positional rules have a complete tie, *plus* the extra $A \succ B \succ C$ ranking. This extra ranking affects *all* three-candidate positional outcomes, but, surprisingly, the Condorcet directions are the *only* profile terms that can alter Borda outcomes for subsets of candidates.[5]

Over the four sets of three candidates, the extra rankings define a new kind of cycle,

$$A \succ B \succ C, \quad B \succ C \succ D, \quad C \succ D \succ A, \quad D \succ A \succ B \tag{4}$$

where each candidate is in first, second, and third place in different triplets. This listing makes it clear how to create all possible examples of profiles with, say, the Borda ranking is $A \succ D \succ C \succ B$, but by dropping bottom-ranked B, the new Borda ranking is the reversed $C \succ D \succ A$. Namely, start with a profile that has the desired Borda ranking of $A \succ D \succ C \succ B$, and then add a strong enough multiple of a $\mathbf{C}^4$ component to ensure that the extra $C \succ D \succ A$ terms dominate the three-candidate Borda tally. (The $\mathbf{C}^4$ term does not affect the four-candidate Borda tally.) This approach makes it easy to create examples with Borda Count inconsistencies over different sets of candidates, and (Thm. 1) it is the only way.

While the theorem makes it easy to create such examples, it also follows from Eq. 4 that if the Borda winner A does poorly in one subset of three candidates, she will do better in the other two subsets. This statement is further supported by part d, which requires A's low Borda tally in one subset to be offset by higher tallies in other subsets. (Part d does not hold for any other positional voting rule; instead, a wide array of outcomes could occur.)

Because *all possible differences* between the two rules are caused by Condorcet terms, it follows that to support the Condorcet approach over Borda, one must justify why a Condorcet n-tuple, such as $A \succ B \succ C$, $B \succ C \succ A$, $C \succ A \succ B$ (where each candidate is once in each position) should not be a complete tie. Until such a justification is developed, Condorcet's criticism is reversed; *a weakness of the Condorcet paired comparison approach is that it need not elect the Borda winner.*

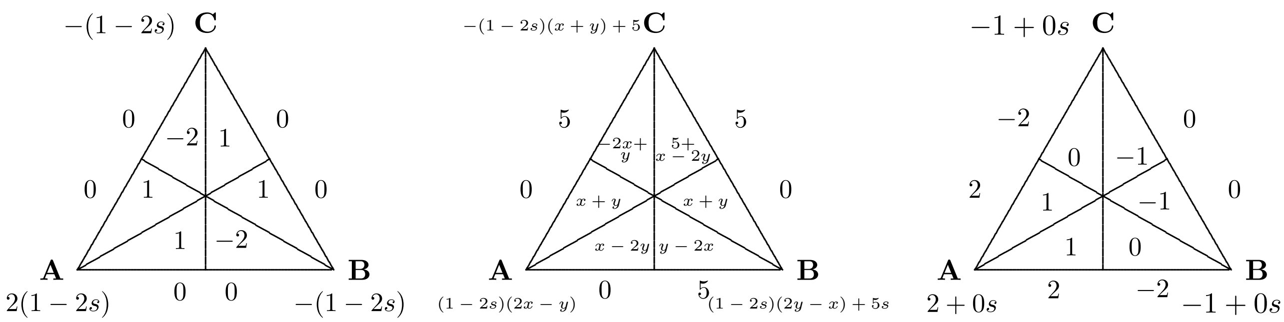
2.3.2 Positional differences

The next profile directions reflect the Fig. 2c symmetries. These coordinates have no influence on pairwise or Borda rankings, but they cause all possible differences that could ever occur among positional rules.

Of the several ways to define my *reversal* directions, I prefer to emphasize a particular candidate's plurality region; e.g., in Fig. 2c, both arrows are in the region emphasizing candidate B. The $\mathbf{R}_A$ choice in Fig. 4a, which I call "A-reversal," highlights candidate A; the negative values (where the coordinates sum to zero) make this choice orthogonal to both $\mathbf{K}^3$ and $\mathbf{C}^3$; they create

[5]Other positional rules are affected by other terms; e.g., the profile $A \succ B \succ C \succ D, D \succ C \succ B \succ A, B \succ A \succ D \succ C, C \succ D \succ A \succ B$ creates complete ties for all pairs and all four-candidate positional outcomes. When restricted to triplets, however, only the Borda Count has a complete tie.

the sense of voters changing preferences. The directions $\mathbf{R}_B$ and $\mathbf{R}_C$ are similarly defined; i.e., place a voter in each region where the indicated candidate is either top, or bottom ranked, and -2 voters in each region where the candidate is middle-ranked.



a. $\mathbf{R}_A$; Reversal direction **b.** Modifying unanimity **c.** $\mathbf{B}_A$; Basic direction

Fig. 4. Reversal profiles

The Fig. 4a tallies identify the properties of Reversal profile directions; namely, they have no impact whatsoever on pairwise and Borda rankings ($s = \frac{1}{2}$), but they influence all other positional rules. Notice how the outcomes for rules with $s < \frac{1}{2}$ have one kind of outcome, but those with $s > \frac{1}{2}$ have the opposite ranking. The basic properties of Reversal profiles follows:

Theorem 2 *(Saari 1999, 2000, 2000a, 2008) Assume the three candidates are $\{A, B, C\}$.*

1. *The reversal profiles satisfy*
$$\mathbf{R}_A + \mathbf{R}_B + \mathbf{R}_C = \mathbf{0}, \tag{5}$$
so they define a two dimensional subspace of the six-dimensional profile space.

2. *The $\mathbf{R}_X$ tally for candidate X is zero for the Borda Count and all majority votes over pairs; the $\mathbf{w}_s$ tally is $2(1-2s)$ for X and $-(1-2s)$ for the other two candidates. Thus, the $\mathbf{w}_s$ tally of a reversal profile has one ranking for $s < \frac{1}{2}$ and the opposite ranking for $s > \frac{1}{2}$.*

3. *Reversal components of a profile create differences between the tallies of candidates for different positional rules; all possible differences between the tallies of candidates for different rules are due to a profile's Reversal components.*

According to part 1, it suffices to select any two of these three profile directions to serve as a basis (but, it is not orthogonal); my choice usually is $\mathbf{R}_A$, $\mathbf{R}_B$. Part 3 has a surprising message; it means that the extensive literature describing differences among positional rules reduces to analyzing how positional rules behave over reversal components of a profile. These differences are specified in part 2.

This reversal effect is illustrated with Fig. 4b, which starts with a unanimity profile of five voters preferring $C \succ B \succ A$ and then adds $x\mathbf{R}_A + y\mathbf{R}_B$. Different from Figs. 3b, c, this time I did not include $\mathbf{K}^3$ components. So, after creating a desired example, add a $k\mathbf{R}^3$ vector where k is large enough to ensure that all components are non-negative.

As guaranteed by Thm. 2, part 2, all choices of x and y have the same pairwise tallies with $\tau(B, A) = \tau(C, B) = \tau(C, A) = 5$ and the same Borda Count ranking of $C \succ B \succ A$. The other positional rankings, however, can be converted into whatever is desired by assigning appropriate choices for the two variables. To have a conflicting Fig. 4b $A \succ B \succ C$ plurality ranking (i.e., $s = 0$), for instance, just solve the associated inequalities $2x - y > 2y - x > 5 - (x + y)$ (which reduce to $3y > 5, x > y$) with solutions $x = 3, y = 2$. By adding $4\mathbf{K}^3$, the resulting profile $(3, 9, 0, 8, 9, 0)$ has the desired properties.

The two remaining profile directions come from what I call the *Basic profiles*; the one for A is given in Fig. 4c. More generally, to create the direction for $\mathbf{B}_X$, assign one voter to each ranking where X is top-ranked and -1 to each ranking where X is bottom ranked. The properties of the Basic profile are reflected by the Fig. 4c tallies.

Theorem 3 *(Saari 1999, 2000, 2000a, 2008) Assume the three candidates are $\{A, B, C\}$.*

1. *The Basic profiles satisfy*
$$\mathbf{B}_A + \mathbf{B}_B + \mathbf{B}_C = \mathbf{0}, \tag{6}$$
so they define a two dimensional subspace of the six-dimensional profile space.

2. *For* all *positional rules, the $\mathbf{B}_X$ tally assigns two points to candidate X and -1 points for each of the other two candidates. In pairwise votes, X receives two points and the other candidate receives zero points. The pairwise vote between the two other candidates is a zero-zero tie.*

3. *Basic components of a profile have the same ranking for all positional rules and paired comparisons. Indeed, for this Basic component, the difference between tallies for any two candidates is precisely the same for all positional outcomes and $\frac{3}{4}$ the difference between majority vote tallies of these two candidates.*

4. *The common positional tally of a profile's Basic component equals the Borda Count tally of the profile minus the Borda Count's tally of the $k\mathbf{K}^3$ component.*

In other words, nothing goes wrong with a Basic profile! Not only do pairwise and *all* positional rules share the same ranking, but, for these rules, even differences in the tallies between any two candidates agree! The reason for this amazing property comes from the Fig. 2 shading approach, which identifies the different information from a profile used by different rules; the Basic profiles are orthogonal to these configurations of rankings that cause differences in outcomes.

The importance of profile coordinates for three alternatives is that they completely identify a profile's informational content in terms of the behavior of all positional and pairwise rules (along with voting methods that are built using these rules). Namely, everything that can happen with a positional method comes from the Basic and Reversal components of a profile; everything that can happen with the majority vote over pairs comes from the Basic and Condorcet components of a profile. Everything that can happen with the Borda Count, on the other hand, is based strictly on the Basic component of a profile. Because different rules use information from orthogonal subsets of information, different outcomes with different rules must be anticipated.

To provide extra content for this comment about the Borda Count, consider the profile $\mathbf{p} = 3\mathbf{B}_A + 2\mathbf{B}_B + 3\mathbf{R}_A + 9\mathbf{K}^3 = (6, 13, 10, 0, 11, 14)$. For all positional methods, the $3\mathbf{B}_A + 2\mathbf{B}_B$ tally is $(4, 1, -5)$. The normalized Borda outcome of $\mathbf{p}$ is $(31, 28, 22)$ while its tally of $9\mathbf{K}^3$ is $(27, 27, 27)$, and, indeed $(32, 28, 22) - (27, 27, 27) = (4, 1, -5)$. In other words, the Borda Count tally of a profile captures the common positional tally of the profile's Basic components; all possible differences in tallies reflect how different $\mathbf{w}_s$ rules handle reversal components.

2.4 Resolving classical problems and "paradoxes"

As a way to illustrate the power of this profile coordinate system, I now use it to explain what causes some of the better known classical three-candidate voting paradoxes.[6]

1. *"The Condorcet paradox: Given that the preference ordering of every voter ... is transitive, the (amalgamated) preference ordering of the majority of the voters ... may nevertheless be intransitive."*

   The answer is immediate; a necessary and sufficient condition for this to occur is if the profile has a sufficiently large $\mathbf{C}^3$ component. As it will be explained in Sect. 3.4.1, this $\mathbf{C}^3$ component has the effect of vitiating the crucial assumption that voters have transitive preferences!

2. *"The Condorcet Winner paradox: An alternative x is not elected despite the fact that x is preferred by a majority of the voters over each of the other competing alternatives."*

   Two explanations (with combinations) cover all possibilities. The first is captured by Fig. 3b, c and Thm. 1, which show that a profile's $\mathbf{C}^3$ components have absolutely no impact on positional outcomes, but they definitely influence pairwise outcomes. Thus an appropriately large multiple of $\mathbf{C}^3$ can change the Condorcet winner (as true with certain Fig. 3c values of x) without changing the outcome of any rule that is not affected by $\mathbf{C}^3$ terms (such as positional methods, or any method based on positional rules such as Approval or cumulative voting). The second effect reflects the fact that a profile's Reversal components have no impact on the pairwise rankings, but they can alter a non-Borda positional outcome to anything desired; e.g., the winner of such a positional method need not have anything to do with the paired comparison outcomes. (For more details, see Saari 2002.) Thus, if one accepts this paradox as a serious problem, all rules that give non-tied outcomes to $\mathbf{C}^3$ and/or to Reversal components should be avoided; this includes all rules based on paired comparisons and non-Borda positional outcomes.

3. *"The Condorcet Loser .. paradox: An alternative x is elected despite the fact that a majority of the voters prefer each of the remaining alternatives to x."*

   The complete answer is the Reversal component of a profile; this profile component, and only this kind of component, can force a non-Borda positional rule's outcome to be completely independent of the pairwise rankings. Rules constructed from non-Borda positional methods are subject to this problem if they do not include a pairwise vote.

4. *The Absolute Majority paradox: An alternative x may not be elected despite the fact that it is the only alternative ranked first by an absolute majority of the voters."*

   Such an alternative must be top-ranked with the plurality rule, but it need not be so ranked with any other positional rule. The explanation is a combination of the Reversal and $\mathbf{C}^3$ components. To see the idea, suppose 1000 voters prefer $A \succ B \succ C$ and 1000 prefer $B \succ C \succ A$; who do they prefer? Both A and B are in first place precisely 1000 times, but B is in second place while A is in last place the other 1000 votes, so it is easy to argue that B is the robust winner. Yet, by adding just one more voter in favor of A, she becomes the absolute majority winner and the plurality winner, but she is not the winner with most other positional rules. Again, it is easy to argue *for this kind of profile* that this behavior indicates

[6] I took this list directly from the charge for a August 2010 "Voting Power in Practice" workshop in Normandy, France, where a goal was "To try and formulate necessary and/or sufficient condition(s) for the occurrence of the main paradoxes [that they listed]." As shown here, this objective is readily accomplished with profile coordinates.

a weakness, rather than a strength, of the plurality vote. With a modification of the above profile decomposition (Saari 2002), it has been shown that this "paradox" occurs *only* with profiles of this kind; thus, it is arguable that the intended criticism of the paradox is poorly directed; it should be directed at the failings of the plurality vote. (Computing the scalar product of this profile with $\mathbf{C}^3$ and $\mathbf{R}_A$ shows that it has the strong components in these directions.)

5. *"The Absolute Loser paradox: An alternative x may be elected despite the fact that it is ranked last by a majority of the voters."*

   This property cannot occur with majority votes over pairs, so it is strictly a feature of how the Reversal profiles can force non-Borda positional methods to ignore information about pairs. This is the total explanation; as an illustration, the plurality vote where 202 prefer $A \succ C \succ B$, 201 prefer $B \succ C \succ A$, and 200 prefer $C \succ B \succ A$ has nearly two-thirds of the voters with A bottom ranked, but she is the plurality winner. The scalar product of this profile $(0, 202, 0, 200, 201, 0)$ with $\mathbf{R}_A$ demonstrates the profile's exceptionally strong Reversal component. Rules constructed from these positional methods, such as Approval and cumulative voting, are subject to this problem.

Other concerns are similarly addressed; e.g., there is a literature examining which voting rules will elect a Condorcet winner when one exists. The above profile decomposition provides a partial answer: The Condorcet winner is strictly determined by information from the Basic and Condorcet components, so, for a rule to elect the Condorcet winner, it must *not* be influenced by other types of profile information such as the Reversal components of a profile, but it *must be influenced* by Condorcet components. The statement about Reversal components means that all non-Borda Count positional rules, or methods based on these rules, cannot have this property. The second condition about Condorcet components means that the Borda Count cannot have this property, but methods that are based on the Borda Count over subsets of candidates might have this property. (This is because the Condorcet components affect Borda Count outcomes over subsets of candidates. This comment is reflected with the Nanson method, which is a Borda runoff, where at each stage all candidates not receiving at least the average number of votes are dropped. This leads to a Condorcet winner.)

The description of what happens for $n > 3$ alternatives is, as one must anticipate, more complicated. The intrepid reader is directed to (Saari 2000, 2000a).

3 Geometry of the likelihoods of voting problems

While Sect. 2 explains what goes wrong with standard voting rules, one might wonder whether these problems are isolated anomalies that can be safely ignored, or issues that must be seriously considered. In other words, how likely are the various paradoxes?

The real objective has nothing to do with finding precise likelihood values; after all, it is not clear where, if anywhere, such precise values have ever proved to be of any value. Adding doubt to the validity of these values is that often they are based on questionable probability distributions,[7] which means that the numerical values must not be taken seriously. But, while flawed, what makes these conclusions valued contributions to the social choice literature is that they provide insights

[7]An often used choice, for example, is "the impartial anonymous culture" (IAC) assumption popularized by W. Gehrlein, P. Fishburn, and others. While IAC asserts that all profiles are equally likely, I doubt whether anyone really believes that a profile with one voters preferring $A \succ B \succ C$ and 49 preferring $C \succ B \succ A$ is as likely as where the 50 voters are split, say, 24 preferring one choice and 26 the other.

about which voting behaviors are essentially isolated phenomenon and which ones reflect important, realistic difficulties. Because the real goal is to find this kind of qualitative information, it is natural to wonder whether answers can be found more easily by using the geometry of voting. This is the case; ways to do so are described after outlining a technical approach.

3.1 Central Limit Theorem

A way to appreciate what should be treated as an unexpected outcome (e.g., paradox) and what should not is to notice how easy it is to create Fig. 2b type examples with the same plurality tallies but with different pairwise and positional rankings. It is instructive, for instance, to use the Sect. 2 structures to find all profiles fixing the plurality $A \succ B \succ C$ tally of 17:16:15, but with different pairwise and/or antiplurality rankings.

Now, keeping the same 17:16:15 plurality tally, let me ask the reader to find all examples that exhibit complete agreement; i.e., the pairwise, plurality, and antiplurality rankings all share the same $A \succ B \succ C$ outcome. Try it, but you might not like the challenge problem, which is more difficult to solve because there are not many examples! As the numbers of possible examples serve as a surrogate for the likelihoods of various behaviors, this exercise suggests that consistency is more unlikely than inconsistency. This counter-intuitive observation forces us to wonder whether *the true paradoxical setting is to have complete consistency over all positional and pairwise rules, rather than inconsistencies.*

Such an assertion runs against accepted beliefs, but it is supported by analysis. By using methods of differential geometry and the central limit theorem combined with a geometric voting result described later (the "procedure line" in Sect. 4.3), Maria Tataru (a former graduate student of mine) and I developed an approach (Saari and Tataru 1999) to determine the likelihood that all positional outcomes agree.

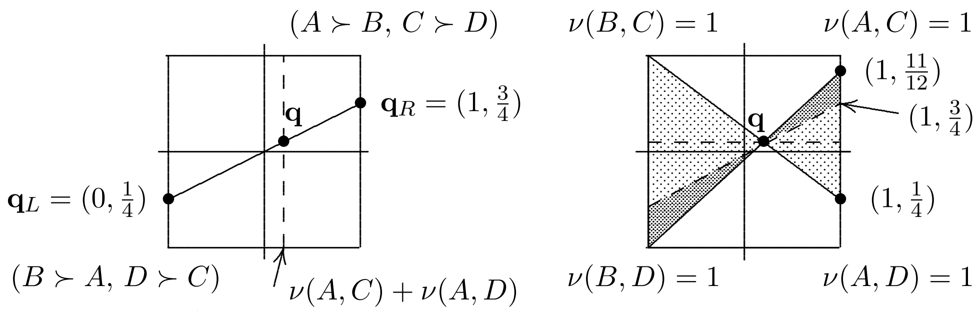
Theorem 4 *(Saari and Tataru, 1999) Assume there is a probably distribution for n voters over how they rank the three candidates where, in the limit, the distribution is IID with mean at complete indifference and with a positive standard distribution. In the limit and with probability* 0.69*, a profile will have different rankings of the candidates when tallied with certain different positional rules.*

In other words, with only the surprisingly small likelihood of 0.31 can we expect a profile to have common $\mathbf{w}_s$ rankings. Moreover (as suggested by Figs. 3b, c), this likelihood decreases once we include pairwise comparisons. (See (Saari and Tataru, 1999) for more general results.) This Saari-Tataru technique has subsequently been used by Merlin, Tataru, and Valognes (2000, 2001) to determine the likelihood the *winner* will change with the procedure and the likelihood of Condorcet profiles; other papers include Lepelley and Merlin (2001) and Tataru and Merlin (1997).

3.2 Geometry of paired comparisons

It is not uncommon to hear, after an election involving majority votes over pairs, how the outcome reflects the "will of the voters." But this need not be the case; e.g., Brams, Kilgour, and Zwicker (1998) describe a California "Yes-No" initiative election where not one of the millions of voters agreed with all of the actual outcomes. Partly motivated by their paper, Katri Sieberg (who did the statistical analysis for their paper) and I developed a geometric approach (Saari and Sieberg, 2001) to explain and provide geometric insights into these kinds of issues.

Start with a pairwise outcome where A beats B by receiving 60% of the vote. Geometrically represent this outcome on a line interval with endpoints A and B by placing a point 60% of the way from B to A. This point represents the profile and the outcome; with 100% certainty, 60% of these voters prefer A to B. Similarly with the two pairs $\{A, B\}$ and $\{C, D\}$, suppose A beats B and C beats D with, respectively, 60% and 55% of the vote. If the horizontal and vertical axes of a square depict, respectively, the A–B (A is on the right) and C–D (C is on the top) outcomes, then point $\mathbf{q}$ in the first quadrant of Fig. 5a represents this joint outcome.



a. A profile line **b.** Finding all supporting profiles

Fig. 5. Representing profiles

While with certainty 60% of the voters prefer A to B, and 55% prefer C to D, how should this outcome be interpreted? To capture the potential ambiguity of the interpretation, Sieberg created an example where 100 voters in a school district must decide whether to increase benefits for teachers (B is yes, A is no) and/or their salary (D is yes, C is no.) Does this joint outcome, where both propositions are defeated with at least a 55% vote, signal the voters' forceful rejection of the teachers? After all, 55% of voters might embrace both negative conclusions because

- 55 of them vote against both proposals ($A \succ B, C \succ D$),
- 5 vote against benefits ($A \succ B$) but accept a salary increase ($D \succ C$),
- while the remaining 40 voters support both proposals ($B \succ A, D \succ C$).

But this interpretation could be incorrect because the same negative election tallies occur where, rather than rejecting the teachers, 85% of the voters are trying to help them! Because of financial constraints, these voters choose to provide support in one or the other way. Namely,

- only 15 voters vote against the teachers on both issues ($A \succ B$, $C \succ D$)
- 45 of the 100 voters support a salary increase ($D \succ C$) over improving benefits ($A \succ B$),
- and 40 of the voters prefer improving benefits ($B \succ A$) over a salary increase ($C \succ D$).

Both issues lose by sizable margins, but because 85% of these voters wanted to help the teachers in one way or the other, they would be very disappointed with the combined decision.

The social choice issue is to determine whether this disappointment is an anomaly that can be safely ignored, or an issue that must be seriously addressed. To tackle these kinds of concerns we developed a geometric approach (Saari and Sieberg, 2001) to connect profiles with their Fig. 5a outcome. Let $\nu(A, C)$ be the fraction of all voters preferring $A \succ B$ and $C \succ D$; by using a similar notation with the other three choices, a profile becomes

$$\mathbf{p} = (\nu(A, C),\, \nu(A, D),\, \nu(B, C),\, \nu(B, D)). \tag{7}$$

Divide the voters into the "rightists" – the $A \succ B$ voters with preferences on the right edge, and the "leftists" – the $B \succ A$ voters with preferences on the left edge. (The horizontal axis is selected so that for $\mathbf{q} = (q_1, q_2)$, we have $q_1 \geq q_2$.) Further divide the rightists according to how they rank C and D; this defines the $\mathbf{q}_R$ point on the Fig. 5a right edge. A similar division of the leftist party defines $\mathbf{q}_L$ on the left edge. To interpret these points, as $\mathbf{q}_R$ is above the midpoint on the right edge, over half of the rightists support $C \succ D$; the lower position of $\mathbf{q}_L$ proves that most leftists prefer $D \succ C$. The general formulation for these points is

$$\mathbf{q}_L = (0, \frac{\nu(B,C)}{\nu(B,C)+\nu(B,D)}) = (0, \frac{\nu(B,C)}{1-(\nu(A,C)+\nu(A,D))}), \quad \mathbf{q}_R = (1, \frac{\nu(A,C)}{\nu(A,C)+\nu(A,D)}). \tag{8}$$

The straight line connecting $\mathbf{q}_L$ and $\mathbf{q}_R$, given by $(1-t)\mathbf{q}_L + t\mathbf{q}_R$, $0 \leq t \leq 1$, passes through $\mathbf{q}$ for $t = \nu(A,C) + \nu(A,D)$ (which is A's vote in the $\{A,B\}$ election); at this point we re-obtain the joint election outcome

$$\mathbf{q} = (\nu(A,C) + \nu(A,D), \nu(B,C) + \nu(A,C)).$$

In geometric terms, $\mathbf{q}$ is at the intersection of the Fig. 5a solid line connecting the endpoints and the dashed vertical line representing the $\{A,B\}$ pairwise tally. The important observation is that a profile can be equated with a line segment with

1. endpoints on the left and right edges of the square and
2. a distinguished point $\mathbf{q}$ on the line; $\mathbf{q}$ identifies the joint election outcome.

Call the line segment with its distinguished point a *profile line.*

Of particular importance is that the converse is true: a line segment with these properties defines an Eq. 7 profile where the segment becomes the profile line. The profile is found by carrying out the reverse computations. To illustrate, use the 60% vote of A over B along with the endpoints $\mathbf{q}_L = (0, \frac{1}{4})$ and $\mathbf{q}_R = (1, \frac{3}{4})$. As $\mathbf{q}_R$ requires $\frac{3}{4}$ of the rightist to prefer $C \succ D$, it follows that $\nu(A,C) = (0.6)(\frac{3}{4}) = 0.45$ while $\nu(A,D) = (0.6)(\frac{1}{4}) = 0.15$. Similarly, $\frac{1}{4}$ of the leftist prefer $C \succ D$, so $\nu(B,C) = (0.4)(\frac{1}{4}) = 0.1$ while $\nu(B,D) = (0.4)(\frac{3}{4}) = 0.3$. This particular *profile line*, then, represents the profile

$$(\nu(A,C),\ \nu(A,D),\ \nu(B,C),\ \nu(B,D)) = (0.45,\ 0.15,\ 0.10,\ 0.30).$$

To review; *all lines with endpoints on the side edges that pass through* $\mathbf{q}$ *represent profiles with outcome* $\mathbf{q}$. The set of all profiles with pairwise outcomes $\mathbf{q}$ are all lines in the shaded wedge of Fig. 5b; call it the *profile cone.* (To find the cone, set $\mathbf{q}_L$ at its extreme settings at the top and bottom vertices. The line from each of these vertices through $\mathbf{q}$ defines the cone's boundaries.) This profile cone identifies *all possible supporting profiles for the specified pairwise outcomes.*

3.3 Geometric likelihood estimates

The ability to geometrically identify all profile lines (i.e., all profiles) supporting a specified $\mathbf{q}$ outcome is what allows us to geometrically represent the likelihoods of different events. Motivated by Sieberg's example demonstrating the ambiguity of interpreting the Fig. 5a joint outcomes, it is natural to question how likely it is for certain percentages of the voters to accept both outcomes. A geometric answer is found by comparing the relative abundance of profile lines that satisfy a specified property relative to the full cone (Saari and Sieberg, 2001).

To examine the reasonable assumption that at least 55% of the Fig. 5 voters prefer both $A \succ B$ and $C \succ D$ outcomes, find all of the profiles lines that support this property. As only rightists support the $A \succ B$ conclusion, the analysis reduces to determining what fraction of them, given by

y, must also support $C \succ D$. This y value must satisfy $y(0.60) \geq 0.55$, or $y \geq \frac{11}{12}$; according to Fig. 5b, *precisely one profile satisfies this constraint.* With a smooth probability profile distribution, then, *a 55% level of support has essentially zero likelihood of occurring.*

This geometric approach makes it easy to determine whether 52%, or 50%, or any other percentage of the voters support both outcomes. To illustrate by examining the more modest condition that at least 45% of the voters support both outcomes, we must determine when $\nu(A, C) \geq 0.45$. To do so, find the fraction y of rightist with $C \succ D$ preferences, which is $y(0.60) \geq 0.45$, or $y \geq \frac{3}{4}$. In Fig. 5b, the relatively small size of this profile subset – the heavier shaded region in the profile cone – geometrically shows how unlikely it is to have even at least 45% of the voters supporting both outcomes![8]

The reader who prefers having actual probability values needs to specify a probability distribution for the profile lines. If one assumes, for instance, that each profile line is equally likely, then the likelihood (see Saari and Sieberg, 2001) is the length of the heavily shaded portion on the right edge divided by the length of the shaded portion on the right edge, or only $[\frac{11}{12} - \frac{3}{4}]/[\frac{11}{12} - \frac{1}{4}] = \frac{1}{4}$. Different distributions yield different outcomes; e.g., the more realistic binomial probability distribution drops the likelihood of 0.25 to a smaller value.

The geometry for more pairs, is higher dimensional. With three pairs, for instance, the square describing the outcomes for two pairs is replaced with a cube. But while the geometry is more complicated, the cone structure extends. For a sample of new results, it follows from the geometry of the square that with any two-pair outcome, there always exist voters who agree with the combined outcome.[9] For three or more pairs, however, the geometry proves it is possible for not a single voter to completely agree with the final outcome, which now a theoretical explanation for the (Brams, Kilgour, and Zwicker, 1998) empirical result. For details, more results and examples, and applications to "bundled voting," see (Saari and Sieberg, 2001).

3.4 Explaining and interpreting pairwise voting outcomes

As demonstrated in Sect. 3.2, as soon as voters vote over several pairs, it is likely that only a small fraction of the voters, or maybe even none, support all outcomes; i.e., any sense that the pairwise votes accurately reflect the views of most voters over all outcomes is doubtful. Why is this so, and how should such conclusions be interpreted? Explanations and interpretations are needed.

3.4.1 An explanation

Start with the problem of selecting a committee that must have a member from each of three groups. Each group has two candidates; from the *West*: (Deanna, Eric); from the *East*: (Florence, George); and from the *North*: (Helen, Joe). To keep everything simple, suppose there are only three voters where each is to vote for a candidate from each of the three districts. If Eric, George, and Joe each wins by 2:1, the social choice issue is to determine whether the voters are content with this outcome; does the conclusion reflect their aggregated views?

To analyze this problem, use reverse engineering by analyzing all possible supporting profiles. By ignoring the identity of each voter, there are five such profiles where, in the following list, the

[8] To determine what outcome has a 50% likelihood, find the midpoint of right edge in the cone, or $y = \frac{7}{12}$. As $\frac{7}{12}(0.6) = \frac{7}{20} = 0.35$, for a conclusion to be supported with a 50% likelihood, all we can say is that both outcomes are shared by at least 35% of the voters. Similar to the Thm. 4 conclusion, "consistency" need not be as likely as inconsistency

[9] To see why, let $\mathbf{q}$ be any point in the top quadrant of Fig. 5a satisfying the required $q_1 \geq q_2$. The worse situation is if $\mathbf{q}_L$ is located at the top, left vertex. But as the corresponding profile line passing through $\mathbf{q}$ still hits the right edge above the bottom vertex, there always must be a positive fraction of the rightists that accept both outcomes.

first letter of each candidate's name is used:

1. Voter 1: (E, G, J); Voter 2: (D, F, H); Voter 3: (E, G, J).
2. Voter 1: (D, G, J); Voter 2: (E, F, H); Voter 3: (E, G, J).
3. Voter 1: (E, F, J); Voter 2: (D, G, H); Voter 3: (E, G, J).
4. Voter 1: (E, G, H); Voter 2: (D, F, J); Voter 3: (E, G, J).
5. Voter 1: (D, G, J); Voter 2: (E, F, J); Voter 3: (E, G, H).

The (E, G, J) outcome appears to be reasonable for the first four profiles. This is because the directly opposing preferences of the first two voters create a tie that is broken by the remaining voter. But beyond appealing to the 2:1 tallies, justifying this conclusion for the last profile is problematic. A comforting observation, however, is that the outcome reflects the combined views of the voters for the vast majority of the scenarios — 80% of them.

But suppose the outlier, the last profile, is the actual one where each voter is disappointed with the outcome because each wanted to have at least one woman and one man on the committee: this wish reflects how they voted. On the other hand, it is reasonable to question whether this example constitutes a realistic criticism of the voting rule; after all, nothing is built into the rule to reflect this "mixed gender" requirement — or any other newly imposed stipulation. But, as described next, this mixed gender condition identifies the source of all pairwise voting paradoxes.

To illustrate what happens, convert this committee example into an equivalent, but more familiar setting just by changing names. Replace D with $B \succ A$ and E with $A \succ B$, F with $C \succ B$ and G with $B \succ C$, H with $A \succ C$ and J with $C \succ A$. With this name change, *the "mixed gender" condition is equivalent to social choice requirement that voters have transitive preferences;* a voter's "same gender" rankings of all men, or all women, translate, respectively, to the *cyclic* preferences $A \succ B$, $B \succ C$, $C \succ A$ or $B \succ A$, $A \succ C$, $C \succ B$.

The first four committee profiles correspond, then, to where at least one voter has cyclic preferences. Only the last one, the outlier, represents voters with transitive preferences: The preferences of voters 3, 1, and 2 correspond, respectively, to

$$A \succ B \succ C, \quad B \succ C \succ A, \quad C \succ A \succ B. \tag{9}$$

Recall how this Eq. 9 "Condorcet triplet" played a central role in Sect. 2.3.1.

As the only difference between the committee election and pairwise voting is a name-change, the analysis of both examples must be identical. Namely,

1. The majority vote over pairs loses information by severing all information that connects the pairs. Beyond the mixed gender requirement, the vote emasculates the individual rationality assumption by dropping all information about the transitivity of individual rankings.

2. Rather than responding to a specified profile, the majority vote reacts to the set of all profiles that support the specified tallies; it selects an outcome that best reflects "the largest portion" of these profiles.

These comments explain the majority vote cycle associated with the Eq. 9 Condorcet triplet; the cycle arises because it best reflects the properties of most (here, 80%) of the supporting profiles, rather than the specified one. The fact that this cycle reflects an appropriate outcome for non-admissible profiles (as they have cyclic voters) is immaterial – not to us, but to the mechanics of the voting rule. The rule strips away and ignores the connecting information and requirements that we accept as valuable when determining pairwise outcomes. (More is in Saari 2006, 2008.)

3.4.2 Other consequences

The above explanation is compatible with earlier interpretations of paired comparison results that were advanced in (Saari and Sieberg, 2001, Saari, 2004). A convenient one comes from the geometry of the Fig. 5b profile cone where $\nu(A, C)$ *has the largest component value* for most of the profile lines supporting $\mathbf{q}$. This suggests that the true role of $\mathbf{q}$ is *not* to identify what outcomes most voters in a specified profile prefer (which is false), but rather to identify which profile entry has the largest value for the largest number of supporting profiles in the profile cone. (For one pair, this statement supports common expectations; for several pairs, it conflicts with popular interpretations.) Thus, consistent with the Sect. 3.4.1 comments, rather than reflecting a specified profile, the outcome is appropriate for the largest portion of supporting profiles. "Paradoxes," then, are generated by the outliers in the set of supporting profiles.

Theorem 5 *(Saari and Sieberg, 2001) Let the probability distribution over profiles be either the uniform or the binomial distribution with $p = \frac{1}{2}$. For two or more pairs, the combined pairwise outcome agrees with the component that has the largest value with the largest number of profiles.*

To convert this theorem into a working tool, notice that two pairs define a profile with *four components,* so, according to Thm. 5, all we can say in general is that the dominating component value for the largest number of profiles is greater than $\frac{1}{4}$; not $\frac{1}{2}$. By using geometry (or the analytic approach as suggested by footnote 8), an even smaller value is needed to assert with 50% likelihood that the fraction of the voters support all outcomes. With N pairs, we only know that the dominating component is larger than $(\frac{1}{2})^N$; so, with $N = 3$ or 4 pairs, we only know, respectively, that the largest number of profile lines are where at least $\frac{1}{8}$ or $\frac{1}{16}$ of the voters approve all outcomes. With $N = 20$ "Yes-No" issues on the ballot, all we can expect is that largest number of profile lines have the combined outcome supported by $\frac{1}{1{,}048{,}576}$ of the voters.

So (without added information), there is no reason to expect the joint outcomes over several pairs to enjoy much support. A lesson of particular importance coming from the profile cone geometry is that pairwise majority vote outcomes need not reflect the properties of an actual profile; they reflect a statistical sense of the set of all possible profiles that yield the same result.

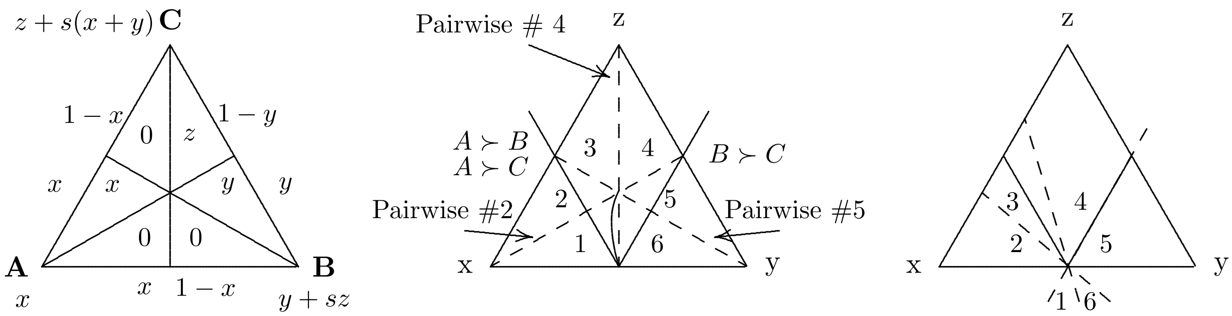
3.5 Geometry of all three-candidate pairwise and positional outcomes

A geometric representation of profile space showing all possible interactions among all three-candidate majority vote and positional outcomes would be very useful. If possible, we would like to have this geometry identify all inconsistencies among the outcomes for different rules, specify all supporting profiles, and indicate their likelihoods. A method that does some of this (using profile coordinates) is in (Saari 1999). The approach outlined next, which comes from (Saari and Valognes, 1998, 1999), captures these objectives by describing the geometry of profile space.

For reasons explained later, start with any profile involving only three types; the Fig. 6a choices are voters of types two, four, and five (Fig. 1b). If x, y, z represent the fraction of voters of the indicated types, then the positional and pairwise tallies are as specified in Fig. 6a.

3.5.1 Geometrically comparing positional and pairwise outcome

Reflecting my preferred and persistent use of equilateral triangles, notice how this set of points $\{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \leq x, y, z,\, x + y + z = 1\}$ defines a simplex (equilateral triangle in Fig. 6b) that passes through the three points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$; these points capture, respectively, where $x = 1, y = 1$, and $z = 1$. To find all profiles supporting specific election outcomes, on this triangle find the boundaries (i.e., ties) of the majority vote outcomes over pairs. According to Fig. 6a, the $A \sim B$ outcome is given by the line $x = \frac{1}{2}$; this is the solid downward slanting line in Fig. 6b.



a. Class of profiles **b.** Pairwise and $s = 0$ **c.** Pairwise and $s = \frac{1}{2}$

Fig. 6. Three candidate geometry

As the notation in Fig. 6b indicates, all profiles in the triangle to the left of this line represent $A \succ B$; the profiles to the right represent $B \succ A$. The same line captures the $A \sim C$ boundary, so all that remains is to find the $B \sim C$ boundary, which is given by the line $y = \frac{1}{2}$ and the upward slanted line in Fig. 6b. This profile space, then, is divided into three open regions; the small equilateral triangle on the left corresponds to type two profiles defining $A \succ C \succ B$, the small equilateral triangle on the right identifies all profiles defining the opposite type 5 rankings, and the remaining quadrilateral are all profiles defining a type 4 pairwise ranking.

While the same approach identifies which profiles define what positional outcomes, it is easier to handle the rules separately and then describe how they are connected. The $A \sim B$ plurality ($s = 0$) boundary of profiles is given (Fig. 6a tallies) by the line $x = y$—the vertical dashed line in Fig. 6b. Similarly, the $A \sim C$ and $B \succ C$ boundaries are given, respectively, by the $x = z$ and $y = z$ dashed lines. (The curved line is explained later.)

Results identifying differences in election outcomes are immediate; e.g., Fig. 6b proves for profiles of the Fig. 6a type that any of the thirteen possible plurality rankings can be accompanied by a type four ($C \succ B \succ A$) pairwise ranking. The geometry even shows how to find supporting profiles; e.g., to have a type one plurality outcome with the reversed type four pairwise ranking, just select a point (x, y, z) in the indicated region; e.g., any point satisfying $z < y < x < \frac{1}{2}$ suffices, such as $x = \frac{5}{12}$, $y = \frac{4}{12}$, $z = \frac{3}{12}$. The integer form of this profile is $(0, 5, 0, 4, 3, 0)$.

Figure 6c compares the pairwise and Borda outcomes. To compute the Borda ($s = \frac{1}{2}$) $A \sim B$, $B \sim C$, $A \sim C$ boundaries, it follows from the Fig. 6a tallies that we should graph, respectively, the Fig. 6c lines $x = y + \frac{x}{2}$, $y + \frac{x}{2} = z + \frac{1}{2}(x + y)$, $x = z + \frac{1}{2}(x + y)$. (The graphing is simple; two points determine each line and they can be found by setting one of the variables equal to zero.) Notice how the figures capture significant differences between the plurality and Borda ranking with respect to the pairwise outcomes; e.g., all profiles leading to a type five pairwise outcome define the same Borda ranking, but this is not so with the plurality rule. Profiles defining a type four pairwise outcome require a type three or four Borda outcome, but all possible plurality rankings can emerge. Indeed, with Fig. 6a profiles, it is impossible to have a Borda type one or six ranking.

The remaining extreme setting is the antiplurality vote ($s = 1$); here the $A \sim B$ boundary coincides with the $A \sim B$ pairwise boundary, the $A \sim C$ boundary is the vertex $(1, 0, 0)$ (or $x = 1$), and the $B \succ C$ boundary is the right triangle edge. In other words, with these profiles, the only strict antiplurality outcomes are of type three (the triangle near the x vertex) and type four. Comparing the dashed lines of Figs. 6b, c, it follows that there exist plenty of profiles providing a type four Borda ranking, and anything desired (i.e., any of the thirteen possibilities) for the plurality ranking.

To relate the outcomes for the various positional rules, plot how the completely tied $\mathbf{w}_s$ election point moves as s varies. As this point is the intersection of the $A \sim B$ and $B \sim C$ boundary surfaces, it follows from the associated algebraic equations that these tied points satisfy

$$(x, y, z) = (\frac{1+s}{3}, \frac{1-s+s^2}{3(1-s)}, \frac{1-2s}{3(1-s)}), \quad 0 \leq s \leq 1. \tag{10}$$

The $0 \leq s \leq \frac{1}{2}$ part of this curve is plotted in Fig. 6b; the rest of the curve is out of the triangle. In other words, as s increases, the $\mathbf{w}_s$ point of complete indifference moves down this curve from the plurality point carrying a modified version of the boundary lines for the $\mathbf{w}_s$ ranking regions.

3.5.2 Surprising conclusions

This geometry makes it surprisingly easy to pick out fascinating conclusions. For instance, notice how the type-four region for any $\mathbf{w}_s, 0 < s \leq \frac{1}{2}$ contains portions of *all plurality regions.* In fact because of the tilting of the $A \sim B$ indifference line, the geometry makes it simple to show that if $0 \leq s_1 < s_2 \leq \frac{1}{2}$, then *any* $\mathbf{w}_{s_1}$ ranking can be accompanied with a type four $\mathbf{w}_{s_2}$ ranking. For another property, select any point $\mathbf{p}$ in the region between the curved line and the $A \sim B$ plurality boundary. The plurality outcome winner is A. But as s increases, regions 6, 5, and 4 pass over $\mathbf{p}$; namely, for any such profile $\mathbf{p}$, *each candidate becomes the winner with some* $\mathbf{w}_s$*.* The geometry also shows that this is the only region in this setting where, as s changes, all three candidates can win. On the other side of the curved line, however, regions 5 and 6 cannot cross a point, but regions 2 and 3 can. So, in an appropriately selected portion, it follows for any such profile that both A and C can be a $\mathbf{w}_s$ winner, but B never can.

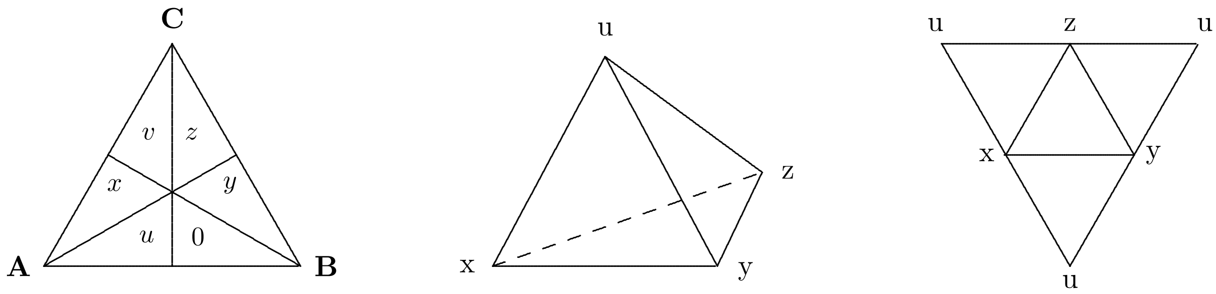
Finding the likelihoods of various events also becomes immediate. For the practical purpose of determining whether various inconsistencies are serious, or can be ignored, the answer is geometrically represented by the size of the appropriate regions relative to the area of the full triangle. Letting n_2, n_5, n_4 represent the number of voters of the various types, where $n = n_2 + n_4 + n_5$, we have that $x = \frac{n_2}{n}$, $y = \frac{n_5}{n}$, and $z = \frac{n_4}{n}$. As the value of n becomes much larger, it is not difficult to prove that the ratio of the areas of various regions tends, in the limit, to agree with the ratio of the number of points of these types in different regions. Namely, the geometry makes it easy to compute likelihoods; e.g., to find how likely it is for a Fig 6.a profile to have all candidates as winners by varying $\mathbf{w}_s$, compute the ratio of the region between the Fig. 6b curved line and vertical axis relative to the area of the triangle. Handling more realistic probability distributions involves integration over regions. Examples for all of this are in (Saari and Valognes, 1998, 1999).

3.5.3 The full profile space for three-candidates

The huge obstacle imposed by a three-candidate profile space is its large $3! = 6$ dimension. One dimension can be dropped by replacing the number of voters having each ranking with the fraction of all voters that have this ranking. To drop another dimension, notice that we can assign zero voters to some one ranking. To analyze $(4, 6, 3, 9, 7, 2)$, for instance, subtract $2\mathbf{K}^3$ to obtain $(2, 4, 1, 7, 5, 0)$. (This reduction primarily reduces the number of majority vote cycles.) By changing names, if necessary, the choice of the ranking with zero voters is arbitrary; in Fig. 7a it is type 6. Thus, relative to the $k\mathbf{K}^3$ reduction and the six possible name changes, Fig. 7a reflects a general profile. This set

$$\mathcal{P} = \{(x, y, z, u, v) \in \mathbb{R}^5 \,|\, x, y, z, u, v \geq 0, x + y + z + u + v = 1\} \tag{11}$$

defines an equilateral tetrahedron in a four-dimensional space. But, it remains challenging to envision four-dimensional objects. Another step is necessary.



a. All profiles **b.** Tetrahedron for four types **c.** Opened tetrahedron

Fig. 7. Finding the geometry of profile space

The approach is to determine what happens on $\mathcal{P}$'s "faces" where a variable equals zero. If, say, $v = 0$, the remaining variables, $\{(x, y, z, u) \in \mathbb{R}^4 \,|\, x, y, z, u \geq 0,\, x + y + z + u = 1\}$ create an equilateral tetrahedron as depicted in Fig. 7b. To understand the geometry of voting rules on this tetrahedron (i.e., with these four voter types), open it up. That is, select a vertex, say u, and cut down the three connecting edges. Folding the faces out generates a Fig. 7c type figure. In each of these four triangles, carry out an analysis similar to that used in Fig. 6 to determine what profiles cause what kinds of behaviors with different voting rules.[10] (This analysis, but where a right triangle is used rather than equilateral triangles, is developed in (Saari and Valognes, 1998, 1999).) Folding the edges up creates an excellent picture of the relationships among the outcomes for the rules.

Similar to how Fig. 7b is converted into Fig. 7c, "open" the Eq. 11 four-dimensional equilateral tetrahedron into five three-dimensional tetrahedrons; each is defined by setting one of the five variables equal to zero. Just as the opened three-dimensional tetrahedron attaches three equilateral triangles to edges of a central equilateral triangle, the four-dimensional object attaches four equilateral tetrahedrons each to a face of a central equilateral tetrahedron. The analysis of behavior on each tetrahedron is known, so examining connecting boundaries of tetrahedrons provides a picture of $\mathcal{P}$. This structure allows surprising results to be "seen."

4 Other geometric voting results

The reasonably sized literatures that discuss either paired comparisons or strategic voting make it worth indicating how geometry allows us to better "see" what is occurring in either case. Some of the geometry is outlined; details are left to references. A third concern reflects what many of us do after a close election; we check to see whether there would have been a different outcome had a different election rule been used. But the complexity of the analysis often means that we can check only what might have occurred with a couple of standard voting rules. The last part of this section shows how to use geometry to capture what would have happened with a wide selection of rules.

[10] Readers interested in extending to four candidates the Sect. 2 geometric approach to represent three-candidate profiles and to compute the tallies should use this unfolding of the tetrahedron. Details are in Saari 2000.

4.1 Paired comparisons; extending McGarey's theorem

A valued paired comparison result is McGarvey's theorem (1953) asserting that all possible pairwise rankings are possible. Namely, for each of the $\binom{n}{2}$ pairs of n alternatives, select one of the three possible rankings: there exists a profile yielding all of these majority vote rankings. The next step is to move beyond knowing what combination of rankings can arise to specify all possible tallies. This is of interest, for instance, when analyzing supermajority voting where McGarvey's result does not apply. The approach to do so (Saari 1994, 1995) mimics aspects of the construction in Sects. 3.2, 3.3[11] where an interval is identified with each pair of alternatives and a point x of the distance from X toward Y represents where Y receives the fraction x of the $\{X,Y\}$ vote. For three alternatives, the three pairs define a cube. (Thus n alternatives define a $\binom{n}{2}$ dimensional cube.) A convenient choice selects the vertices for each line to be at the values of ±1 so that 0 represents a tie and a negative or positive value represents a victory for the appropriate candidate.

Not all points in the cube represent pairwise tallies; e.g., it is impossible to use transitive preferences to attain the point representing unanimous support for $A \succ B$, $B \succ C$, and $C \succ A$. To find which tallies *are* admissible for $n = 3$, notice that only two of the eight vertices represent unanimity cyclic outcomes and they are diametrically located. (For n alternatives, there are $2^{\binom{n}{2}} - n!$ such vertices; e.g., for $n = 5$, the ten-dimensional cube has $2^{10} - 5! = 1024 - 120 = 904$ non-transitive and 120 transitive vertices.) Draw hyperplanes through vertices with transitive rankings where only non-transitive vertices are on one side; throw this side away. With three candidates, then, two oppositely located pyramids, each with a cyclic vertex, are dismissed (Saari 1994, 1995). All remaining points can be attained with pairwise outcomes; e.g., with k voters, these are all points in the truncated cube that have a common denominator k.

By using $-1 \le x_{X,Y} \le 1$ for each pair of candidates where, say $x_{A,B} = x_{B,C} = x_{C,A} = 1$ means that the first listed candidate in each pair wins unanimously, it follows that all rational points in

$$\{(x_{A,B}, x_{B,C}, x_{C,A}) \mid -1 \le x_{X,Y} \le 1, \quad -1 \le x_{A,B} + x_{B,C} + x_{C,A} \le 1\} \tag{12}$$

are paired comparison election tallies for some profile; no other points have this property. As an illustration, suppose we wish to determine all possible $\{B,C\}$ outcomes that can occur if A wins both of her elections with 90% of the vote (so $x_{A,Y} = -1 + .9(1-(-1)) = .8$). The answer from Eq. 12 is that $-1 \le 0.8 + x_{B,C} - 0.8 \le 1$, or $-1 \le x_{B,C} \le 1$, which means that there are no restrictions whatsoever on the $x_{B,C}$ tally.

This argument can be used to prove McGarvey's result. With the alternatives $\{A_1, \dots, A_n\}$, the center point of each face of the $\binom{n}{2}$ dimensional cube is the pairwise tally for some profile; e.g., a way to realize the center point of the $A_1 \succ A_2$ face, given by point $(1, 0, \dots, 0)$ (so, A_1 is the unanimous winner over A_2 and all other paired comparisons end in a tie), is to consider all rankings where A_1 is ranked immediately above A_2; the desired outcome arises by assigning one voter to each of these rankings.[12] As such, the convex hull defined by these midpoints generates a subset of the tallies that are achievable with profiles. This convex set includes a sizable neighborhood about the midpoint of the cube (given by $(0, \dots, 0)$), so it intersects all ranking regions. In turn, this means that any listing of pairwise rankings can be achieved. (But to have ties, the number of voters (which is given by a common denominator of a point) must have an even value.)

To connect this geometry with the profile coordinates, create what I call the *transitivity plane*; it is the set of pairwise tallies in the cube that satisfy Eq. 3 for each triplet, (The plane's normal

[11] The approaches differ in that the Sects. 3.2, 3.3 development identifies all associated profiles.

[12] Clearly, A_1 beats A_2 with a unanimous vote. For all other pairs, treat the $A_1 \succ A_2$ ranking as a unit denoted by Z, which reduces the number of variables to $n-1$. Because each ranking of these variables is accompanied by its reversal, the paired comparison ranking for all other pairs is a complete tie.

directions are defined by the Condorcet basis directions described in Sec. 2.3.) This plane provides an interesting, geometric way to represent the tallies for different voting rules (Saari 2000). As an example, for any pairwise tally $\mathbf{q}$ in the cube, the associated Borda Count outcome (and normalized tally) is the closest point (with the usual Euclidean distance) in the transitivity plane to $\mathbf{q}$. Associated with the Borda rule is the Copeland rule; this is where a winner of a pairwise outcome is assigned one point, the loser zero, and with a tie both receive $\frac{1}{2}$ points. (To use the cube with $[-1, 1]$ edges, change the Copeland tallies to 1, -1, and 0.) Find the vertex of the original uncut cube that is closest to $\mathbf{q}$, and then find the point in the transitivity plane that is closest to this vertex; this is the Copeland outcome. The Kemeny outcome (another well-studied voting rule) is the closest region to $\mathbf{q}$ with a transitive ranking (where "closest" now is the l_1 distance, or the sum of the magnitudes of the coordinates), and so forth (Saari, 2000). (For definitions of these voting rules, see the descriptions that Brams and Fishburn give in Chap. 4 of this series.)

This geometry makes it possible to create a fairly complete description of all Copeland and Kemeny outcomes and paradoxes that can occur by dropping candidates, or comparing them with other procedures (Saari and Merlin, 1996, 2000). Merlin and I proved, for instance, that the Kemeny method always ranks the Borda winner above the Borda loser, and the Borda method always ranks the Kemeny winner above the Kemeny loser. (Our paper (Saari and Merlin, 2000) extends many of the nice results about Kemeny's method that were discovered by Le Breton and Truchon (1997) and by Young (1978, 1988).)

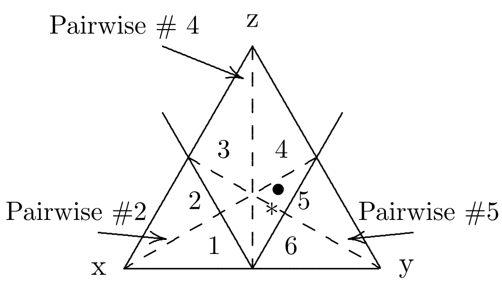
As true throughout social choice, expect to discover delightful surprises. As an illustration, while the Kemeny method handles cyclic rankings by finding the "closest" transitive ranking, the Dodgson method seeks a "winner" rather than a ranking, so it selects the "closest" ranking that has a Condorcet winner. The similarity between these rules makes it is reasonable to expect that the Dodgson winner is Kemeny top-ranked. This, however, need not be the case: by using my geometry of pairwise tallies, Ratliff (2001) proved that the Dodgson winner can be ranked *anywhere within a Kemeny ranking*, even last! He even proved (2002) that no relationship need exist between the Dodgson winner and the Borda ranking. When Ratliff generalized Dodgson's approach to create a method to select a committee (a committee of size k is found from $\mathbf{q}$ by finding the nearest ranking where each of the k candidates is preferred to all of the remaining $n - k$ candidates), he discovered (2003) all sorts of surprising conclusions! The "best" committee of five selected by his method, for instance, need not include anyone from the "best" committee of two, nor even the Dodgson winner. No wonder the area of social choice is so intriguing! (These mysteries, of course, are caused by a profile's Condorcet terms, which mix up behavior among different subsets of candidates. See Sect. 2.3.)

4.2 Geometry of strategic behavior

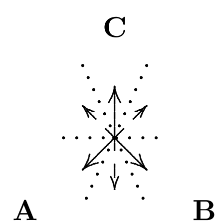
For other appealing issues, recall the Brams and Fishburn (1983) "no-show" paradox where a voter obtains a personally better outcome by not voting, and the Gibbard (1973) - Satterthwaite (1975) conclusion asserting that with three or more candidates, settings exist where voters can achieve personally better outcomes by voting strategically. These kind of results involve changes in profiles. Namely, if F is the voting rule and $\mathbf{p}$ the original profile, then the goal these kinds of problems (i.e., strategic voting, voters not voting, voters changing their views, or whatever else a theoretician may wish to examine) is to compare the $F(\mathbf{p} + \mathbf{v})$ outcome with that of $F(\mathbf{p})$, where $\mathbf{v}$ represents the change in the profile. For the outcomes to differ, $F(\mathbf{p} + \mathbf{v})$ and $F(\mathbf{p})$ must be in different ranking regions. If the change involves a limited number of voters (so $||\mathbf{v}||$ is "small"), both outcomes must be near a boundary.[13] A simple, non-mathematical way to address these kinds of issues is to examine outcomes near ties and determine who benefits by breaking them (Saari 2003).

In Fig. 8a, for instance, the bullet represents a $B \succ C \succ A$ plurality election outcome with a close vote between the top two candidates. Of the three Fig. 6a voter types, voters of types two and four may be disappointed in the outcome. Can either of these types be strategic to achieve a personally better conclusion? The geometry of what can occur is in Fig. 8b.

If a type four voter votes sincerely, the outcome helps C; this is the upward arrow pointing toward C in Fig. 8b. To be strategic, the voter does not vote for C, which is the Fig. 8b downward dashed arrow, and votes either for A or B, one of the two downward sloping solid arrows pointing toward the appropriate candidate's name. The combination of these actions changes the outcome in one of the two downward directions indicated by the dotted line. As both move the outcome more solidly into the type five region, this voter has no strategic opportunities. The type two voter, on the other hand, does have a strategic option. To see this, because a sincere vote is represented by the solid arrow pointing to the left, not voting moves the outcome in the opposite direction given by the dashed arrow pointing upwards to the right. By voting for C, the solid arrow pointing upwards, the combination moves the outcome, the bullet, in the direction of the upward dotted line, which can cross the boundary and change the winner.



a. Plurality and runoffs



b. Changing the outcome

Fig. 8. Strategic voting; unexpected behavior

Figure 8b captures the essence of the seminal Gibbard – Satterthwaite Theorem. Namely, whatever the voting rule, as long as each candidate can win with some profile, there exist boundaries of ties, or transitions, between what profiles elect one or another candidate. For a profile sufficiently near such a boundary, the question is whether there is a tactical way to cross it. To see what happens, first notice that the plurality vote offers only three possible ways to vote, but the combinations of not voting in one way (i.e., sincerely) and voting in another manner creates six directions (the dotted lines). Because the arrows are not all on one side of any line passing through the center point of Fig. 8b, it follows that at least one of these arrows (i.e., at least one strategy) will cross the boundary no matter how it is positioned. To extend this insight into a general result, conditions are imposed on the voting rule to ensure that at a point, the voters' possible votes are not all on one side of a line. This condition can be accomplished in several ways, such as requiring that votes for a candidate will help her, or even that such votes will hurt her

To "see" the geometry of the no-show paradox, where the act of not voting creates a personally better outcome, suppose the outcome for a plurality runoff is given by the asterisk in Fig. 8a. As

[13] For the mathematically more inclined reader, the comparison of $F(\mathbf{p}+\mathbf{v})$ and $F(\mathbf{p})$ suggests a Taylor series approximation with the directional derivative product $(\nabla F(\mathbf{p}), \mathbf{v})$. This observation motivated the geometric approach in (Saari 1995, 2001) where $\nabla F(\mathbf{p})$ is replaced by the normal to the profile boundary of tied election outcomes.

the plurality outcome is $B \succ A \succ C$, candidates A and B advance to the runoff where, as the figure shows, the pairwise outcome is of type four, so B wins. As this winner is a type two voter's bottom choice, it is easy to imagine scenarios where he refuses to vote. But by not voting, the outcome moves in the direction of the Fig. 8b dashed arrow pointing upwards and to the right. This action, then, could move the asterisk, the plurality outcome, into the type five region where B and C are advanced to the runoff. Here C, our negligent voter's second ranked candidate, wins: By not voting, the voter achieved a personally better outcome. Beyond the geometry, expect the no-show paradox, or violations of monotonicity (so more support can hurt a candidate), etc., to occur with any rule, such as a runoff, involving two or more votes (Saari 1995). The problem is that if the first vote determines who gets advanced to the second vote, all sorts of problems can arise.

One might wonder whether it is possible for a voter to elect his top choice by forgetting to vote. This is what Nurmi (2001) calls the *strong no-show paradox.* As it turns out, the geometry of Fig. 8b prevents this strong no-show paradox from occurring with the plurality vote. Other $\mathbf{w}_s$ methods, however, change the directions represented in Fig. 8b, so the conclusion changes. The following result, described in terms of "positive involvement," is proved in (Saari 1994, 1995). Extensions to $n > 3$ candidates are immediate.

Theorem 6 *All three-candidate $\mathbf{w}_s$-runoff methods admit the "no-show" paradox. With the exception of the plurality vote ($s = 0$), all other $\mathbf{w}_s$-runoff procedures allow the strong no-show paradox.*

The geometry of strategic behavior leads to other conclusions. For instance, because all procedures can be manipulated, the next question is to determine *which voting rules are least likely to allow a successful manipulation by a small fraction of the voters.* I answered this question for positional methods with any number of candidates in (Saari 1990); the conclusion for three alternatives is reproduced in (Saari 1995). While precise definitions are left to the references, think of the "level of susceptibility" as the number of profiles where a small number of voters can successfully manipulate the outcome; i.e., a positional procedure that permits fewer successful strategic opportunities is less susceptible.

Theorem 7 *(Saari, 1990, 1995) The positional method $\mathbf{w}_s$ that is least susceptible to a small, successful manipulation is the Borda Count. As the value of $|s - \frac{1}{2}|$ increases, so does the level of susceptibility of the positional method.*

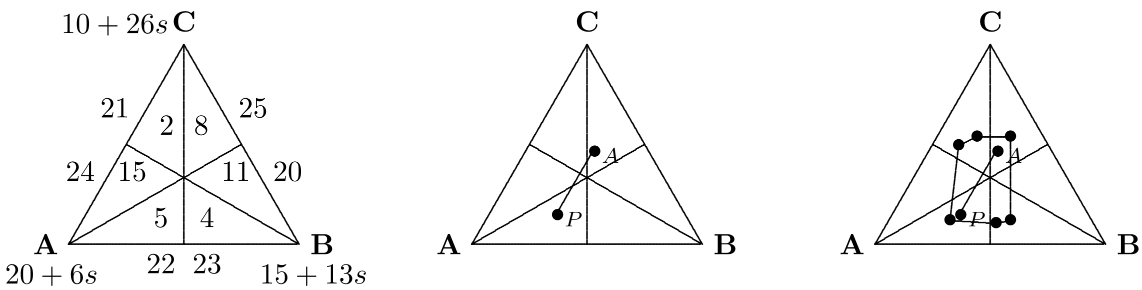
The most manipulable methods include the plurality vote, which is manifested by the "Don't waste your vote!" cries heard during three-candidate plurality elections. A reason the Borda Count is least susceptible is that strategic voting involves two components, opportunity and approach. With the assumption of a limited number of strategic voters, "opportunity" requires the sincere tally to be nearly tied for the two top-ranked candidates. Thus, a major part of the analysis involves finding the relative sizes of such boundaries in profile space. The conclusion reflects those problems of finding the rectangle of area one with the minimum perimeter, or the ellipse of area one with the smallest circumference. In both cases, the answer is the most symmetric figure – a square and a circle. Similarly in voting, the answer is the $\mathbf{w}_s$ method exhibiting the most symmetry between assigned w_j points: the Borda Count. A major reason for this conclusion, then, is that the Borda Count minimizes the opportunities for a small number of voters to successfully engage in strategic behavior.

4.3 Could my candidate have won with a different voting rule?

At least for me, a close "nail-biting" election creates an irresistible temptation to explore whether a different voting rule would have changed the outcome — particularly if "my" candidate lost. Could this be done more easily with the geometry of voting? It can, and the converse of the approach becomes a tool that allows us to better understand the theoretical structure of voting rules.

The clue suggesting how to find the outcomes for *all* positional methods is in Figs. 1a and 9a: the tallies define a linear equation. In Fig. 9a, the plurality votes, or "first place tallies," are **FPT**=$(20, 15, 10)$. The "second place tallies" specify how often each candidate is ranked in second place; e.g., for A it is the sum of voters with types 3 and 6 preferences. With Fig. 9b, we have **SPT**=$(6, 13, 26)$.[14] The general positional outcome has the form of a straight line

$$\mathbf{w}_s \textbf{ vote tally} = \mathbf{FPT} + s\,\mathbf{SPT}, \quad 0 \le s \le 1. \tag{13}$$



a. New Profile **b.** Procedure line **c.** AV Hull

Fig. 9. Finding all possible outcomes

Thus in $\mathbb{R}^3$(where the x, y, and z axes are identified, respectively, with the tallies of A, B, and C), the Fig. 9a and Eq. 13 tallies define a line connecting the the plurality ($s = 0$) and antiplurality ($s = 1$) endpoints. In the space of normalized tallies (i.e., where $x+y+z = 1$, which is, again, an equilateral triangle), just plot the normalized plurality and antiplurality tallies (depicted, respectively, in Fig. 9b with $\bullet_P$ and $\bullet_A$).[15] The connecting line (which I call a "procedure line") identifies all all possible $\mathbf{w}_s$ tallies. (See Fig. 9b.) As the line crosses seven regions, the Fig. 9a profile defines seven different election rankings ranging from $A \succ B \succ C$ to the reversed $C \succ B \succ A$.

Use elementary algebra to determine which rules define which outcomes; e.g., in Fig. 9b, the first tie is $B \sim C$, which means that $15 + 13s = 10 + 26s$, or $s = \frac{5}{13}$. Thus the $\mathbf{w}_s$ outcome for this profile is $A \succ B \succ C$ for $0 \le s < \frac{5}{13}$ and $A \succ B \sim C$ for $s = \frac{5}{13}$.

As an application of procedure lines, recall that W. Clinton won the 1992 US Presidential election with less than a majority vote. It is reasonable to wonder whether one of his opponents, Bush or Perot, could have won with a different election method. But by using the procedure line, Tabarrok (2001) showed that Clinton's support was more solid than previously believed: Clinton would have been victorious with *any* positional method. (On the other hand, by using the technique from (Saari 1995) that is described in the next subsection, Tabarrok showed with Approval Voting (AV) that any candidate—Bush, Clinton, or Perot—could have won depending on which voters voted for one or two candidates.) Tabarrok then showed how to use the procedure line to make it into a useful tool to analyze whether a candidate is the voters' solid choice. Elsewhere, Tabarrok and Spector (1999) used the higher dimensional version of this approach (Saari, 1992, 2001) to

[14]For more candidates, compute the "third place tallies," etc.

[15]Namely, there are 45 voters, so plot $(\frac{20}{45}, \frac{15}{45}, \frac{10}{45})$ for the plurality vote and $(\frac{26}{90}, \frac{28}{90}, \frac{37}{90})$ for the normalized antiplurality tally.

analyze the important 1860 US presidential election that precipitated the Civil War. Also see Nurmi 2002.

The 1992 election provides an interesting setting to discuss AV outcomes. Start with the reality that it is strategically irrational for an AV voter to vote for his second place candidate if this candidate could beat his top choice. A voter with Clinton $\succ$ Bush $\succ$ Perot preferences, for instance, would *not* vote for Clinton and Bush as it would cancel his impact on the {Clinton, Bush} outcome. Thus it is reasonable to believe that only those AV voters with Perot second ranked would vote for two candidates; if enough had done so, we would have had President Perot. In other words, Approval Voting has troubling properties.

4.3.1 AV and hulls for multiple voting rules

To find all Approval Voting outcomes (where voters can vote for one or two candidates) for the Fig. 9a profile, notice that A's tally is 20 should she receive only first place votes, and 26 should she receive the votes of everyone who ranked her in first or second place. Thus, A has seven possible tallies. In general, the number of different AV tallies for a candidate is one more than her SPT value; the total number of different AV tallies is the product of these values. The Fig. 9a profile, then, admits $7 \times 14 \times 27 = 2646$ different AV tallies.

Plotting all 2646 values is not realistic. Fortunately, however, they can be found in terms of the eight extreme AV tallies where candidates receive either all possible first and second place votes, or only first place votes. These tallies are $(20, 15, 10)$, $(26, 15, 10)$, $(20, 28, 10)$, $(20, 15, 36)$, $(26, 28, 36)$, $(20, 28, 36)$, $(26, 15, 36)$, $(26, 28, 10)$ with the associated normalized tallies $(\frac{20}{45}, \frac{15}{45}, \frac{10}{45})$, $(\frac{26}{51}, \frac{15}{51}, \frac{10}{51})$, $(\frac{20}{58}, \frac{28}{58}, \frac{10}{58})$, $(\frac{20}{71}, \frac{15}{71}, \frac{36}{71})$, $(\frac{26}{90}, \frac{28}{90}, \frac{37}{90})$, $(\frac{20}{84}, \frac{28}{84}, \frac{36}{84})$, $(\frac{26}{77}, \frac{15}{77}, \frac{36}{77})$, $(\frac{26}{64}, \frac{28}{64}, \frac{10}{64})$. To obtain what I call the *AV hull*, plot these eight normalized points, and connect them to find the associated convex hull. (See Fig. 9c where the lines defining the convex region is the AV hull.) The 2644 AV normalized values are close to being equally spaced within this region.

Notice that the plurality and antiplurality tallies always must be among the eight AV hull vertices. This requires any $\mathbf{w}_s$ outcome for a profile to also be an AV outcome; e.g., if a profile has an excellent, or a questionable $\mathbf{w}_s$ outcome, this same ranking occurs with AV. AV, however, admits many other outcomes; e.g., with the Fig. 9a profile, it follows from Fig. 9c that *any of the thirteen ways* to rank three candidates is an admissible AV ranking for this profile. The AV hull includes the procedure line and usually much more, so with any natural probability or statistical assumption on voter preferences, it is unlikely for the AV outcome to agree with a positional tally.

The following illustrates another kind of result that follows from the geometry: While the outcome is described for three alternative, it holds for any number of them.

Theorem 8 *If* $\mathbf{p}$ *is a profile where some* $\mathbf{w}_s$ *method allows a voter to be strategic; then* $\mathbf{p}$ *also offers AV strategic opportunities. On the other hand, there are sets of profiles permitting AV strategic settings for some voter, but no strategic opportunities for any positional method.*

The reason AV is more prone to strategic action follows directly from the geometry. The procedure line is a subset of the AV hull, so if a positional outcome is near a tie allowing a strategic opportunity, then so are AV outcomes. The AV hull, however, admits other strategic settings near a tie where no positional method does. (See Saari, 1995, 2001).)

The properties and outcomes associated with other "multiple voting rules" are found in a similar manner (Saari 2010). A multiple rule is where a voter ranks the candidates and then selects a voting rule to tally his ballot. The following makes this more precise.

Definition 1 *An n-candidate scoring vector is given by* $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ *where the* $s_j \in \mathbb{R}$ *has* *a specified value. A multiple rule is defined by a specified set of scoring vectors $\mathcal{S}_{rule}$. In voting, a voter ranks the candidates and then selects $\mathbf{s} \in \mathcal{S}_{rule}$ to tally the ballot.*

An example of a multiple rule is where a voter strategically truncates his ballot to avoid assigning points to lower ranked candidates. If the specified positional rule is $(7, 4, 0)$, for instance, and if the one candidate listed on a truncated ballot is assigned seven points, then $\mathcal{S}_{truncate} = \{(7, 4, 0), (7, 0, 0)\}$. A standard strategic plurality vote is to vote for your second ranked choice, so $\mathcal{S}_{plurality} = \{(1, 0, 0),\ (0, 1, 0)\}$. To provide examples of other multiple rules,

Definition 2 *"Approval Voting" (AV) is where a voter votes "yes" or "no" for each candidate; in tallying the ballot, a candidate receiving a "yes" vote receives one point. For three candidates, then, $\mathcal{S}_{AV} = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.*

"Cumulative voting" (CV) is where, with n candidates, each voter has $n - 1$ points; the voter assigns integer values to the candidates in any desired manner as long as the sum does not exceed the specified value. For three candidates, $\mathcal{S}_{CV} = \{(2, 0, 0), (1, 1, 0), (1, 0, 0)\}$.

"Divide the points voting" (DPV) is where each voter is given a fixed number of points. The voter assigns non-negative values to the candidates, whether integer valued or not, in any desired manner as long as the sum equals the specified value; say 2. For three candidates, $\mathcal{S}_{DPV} = \{(w_1, w_2, 0) \,|\, w_1 + w_2 = 2\}$.

"Range Voting" (RV) is where each voter assigns to each candidate a number of points coming from a specified range, say, any integer between 0 and 100. Thus, for three candidates, $\mathcal{S}_{RV} = \{(w_1, w_2, w_3) \,|\, 0 \leq w_1 \leq 100,\ 0 \leq w_2 \leq w_1,\ 0 \leq w_3 \leq w_2\,\}$.

In all cases, each candidate is ranked according to the sum of assigned points.

For each multiple rule, a "tally hull" can be constructed in a manner similar to the approach used to construct the AV hull. Of importance for what is described in the next section, the hulls for multiple rules using positional methods can be constructed just by knowing the FPT and SPT values. See (Saari 2010) for details.

Definition 3 *For a voting rule and a profile $\mathbf{p}$, the voting rule's "tally hull" is the convex hull of all outcomes after each tally is normalized to represent the fraction of the total vote assigned to each candidate. It is denoted by $\mathcal{TH}_{voting\ rule}(\mathbf{p})$*

This tool makes it possible to discover difference among multiple rules; the approach is to compare the structure of their hulls. For instance, is it possible to have a Borda outcome that could never occur with Cumulative voting? Theorem 9, which proves that this is impossible, identifies what can occur with positional and the four defined multiple rules.

Theorem 9 *(Saari 2010) For any given profile $\mathbf{p}$ and $n \geq 3$ candidates, the following inequality holds:*

$$\mathcal{TH}_{positional\ rule}(\mathbf{p}) \subset \mathcal{TH}_{AV}(\mathbf{p}) \subset \mathcal{TH}_{CV}(\mathbf{p}) = \mathcal{TH}_{DPV}(\mathbf{p}) \subset \mathcal{TH}_{RV}(\mathbf{p}). \tag{14}$$

To illustrate Thm. 9, notice how Fig. 9c displays the ample opportunities for AV strategic opportunities; they are where the AV hull crosses tied vote boundaries. According to Thm. 9 these strategic opportunities only increase with Cumulative and Range voting.

A worrisome property of multiple rules is how they spawn a large set of admissible outcomes rather than one conclusion. To avoid the trouble of having to compute and determine the appropriate tally hull, it would be useful if we had simple "rules of thumb" that would identify all outcomes that can occur with each multiple rules. Such an approach follows from the geometry of the tally hulls; I illustrate the ideas with AV primarily because the arguments are easier.

It is reasonable to expect that the AV bottom ranked candidates are those who are not highly ranked by most voters. The next definition pushes this notion to an extreme by excluding only those candidates who are bottom ranked by too many voters.

Definition 4 *Candidate X is said to be "in the mix" with candidate Y if the number of voters who do not have X bottom ranked is more than the number of voters who have Y top-ranked.*

To illustrate, suppose A and B receive, respectively, 45% and 40% of the first-place rankings, while C is bottom ranked by 51% of the voters. As 49% of the voters do not have C bottom ranked, she is in the mix with both A and B. As A is not bottom ranked by at least 45% of the voters and as C is top ranked by 15%, A is in the mix with everyone; as B is not bottom ranked by at least 51% of the voters (who have C bottom ranked), she too is in the mix. The next theorem provides a quick way to use this tool to find all possible admissible AV rankings.

Theorem 10 *(Saari 2010) For a given three-candidate profile, if X is in the mix with Y, there exists a sincere AV election outcome where X beats Y. If each candidate is in the mix with any other candidate, then all possible ways to rank the candidates is an admissible AV election outcome. However, if X is not in the mix with Y, then no AV outcome can rank X above Y.*

As the above example has all candidates in the mix with all other candidates, any of the thirteen ways to rank three candidates is an admissible AV outcome. For rules-of-thumb addressing the CV outcomes, etc., see (Saari 2010).

4.3.2 Creating examples

Rather than using a profile to find the procedure line, the converse approach would be to start with a line segment and then determine whether it is the procedure line for some profile. If it is possible to do this, then properties of voting rules can be found just by considering all admissible ways to position lines in the space of normalized election tallies (Saari 1995). Similarly, rather than searching for a profile to create a desired kind of tally hull for a specified multiple rule, an easier way to find the rule's properties would be if we could start with what appears to be a hull and then determine if it is supported by some profile. All of this can be done for the procedure line and several multiple rules, such as AV, CV, and DPV. Details are in (Saari 2010), but the main steps are outlined next.

The first step is to understand how a change in a profile can change the shape and size of the various hulls. A clue comes from the assertion that the hulls described in Thm. 9 rely on FPT and SPT information; i.e., all vertices of a hull are determined from the plurality and antiplurality tallies. This dependency on SPT accurately suggests that a profile's reversal components (Thm. 2) play major roles in changing the size and shape of the hull. An interesting feature is that these hulls are centered about a particular positional voting rule's outcome. The Borda outcome, for example, always is located at the center of the AV tally hull; e.g., by connecting the opposite vertices of the AV hull, the four lines cross at the Borda outcome (Saari 2010). (With the Fig. 9a profile, the Borda outcome is $A \sim C \succ B$, or where the procedure line crosses the $A \sim C$ line in Fig. 9b. It is not difficult to see which pairs of vertices of the AV hull connect and pass through the Borda outcome.) Thus, a first step in designing a hypothetical hull is to select appropriate potential choices for the plurality, antiplurality tallies.

The problem reduces to determining which choices of potential plurality and antiplurality tallies can be supported by profiles. When a supporting profile does exist, a candidate's antiplurality tally, A_X, is bounded below by her plurality tally Pl_X, or $Pl_X \leq A_X$. As each voter has twice as many antiplurality votes, when normalizing with n candidates, we have $\frac{1}{2}\frac{Pl_X}{n} \leq \frac{A_X}{2n}$. As a candidate can receive at most half of the antiplurality votes, her normalized $s = 1$ tally is bounded above by $\frac{1}{2}$. That is,

$$\frac{1}{2}\frac{Pl_X}{n} \leq \frac{A_X}{2n} \leq \frac{1}{2}. \tag{15}$$

Surprisingly, Eq. 15 suffices to ensure that selecting points are supported by a profile; namely, if any two rational points in the simplex, $\mathbf{p}_j = (p^j_A, p^j_B, p^j_C)$, $j = 1, 2$, satisfy $\frac{1}{2}p^1_X \leq p^2_X \leq \frac{1}{2}$ (i.e., Eq. 15), then $\mathbf{p}_1$ and $\mathbf{p}_2$ serve as the normalized plurality and antiplurality tallies for some profiles! The class of supporting profiles differ in two ways; the first (reflecting the $\mathbf{K}^3$ profile component) is a scaling factor that allows a supporting profile to be multiplied by any positive integer to create another supporting profile. The second way profiles can differ is by adding multiples of $\mathbf{C}^3$.

To illustrate, because $\mathbf{p}_1 = (\frac{4}{13}, \frac{3}{13}, \frac{6}{13})$ and $\mathbf{p}_2 = (\frac{8}{26}, \frac{8}{26}, \frac{5}{13})$ satisfy Eq. 15, they are normalized plurality and antiplurality tallies for a class of profiles. To find these profiles, first find integer tallies for these points; to find a plurality tally, multiply $\mathbf{p}_1$ by its common denominator of 13 to have $(4, 3, 6)$. The associated antiplurality tally is found by multiplying $\mathbf{p}_2$ by twice this value, or 26, to obtain the antiplurality tally of $(8, 8, 10)$. This means that $SPT = (8, 8, 10) - (4, 3, 6) = (4, 5, 4)$, while the plurality tallies provide the FPT information.

To find one supporting profile, assign zero voters to some ranking. To do so, notice that the smallest FTP and STP value is where B has only 3 first place votes. So, assign zero voters to one ranking where B is top-ranked, say $B \succ C \succ A$ and three to the other ranking $B \succ A \succ C$. This choice of a ranking with zero voters and C's STP value of 4 means that four voters have the $A \succ C \succ B$ ranking. Now that three entries are known, the rest of the profile of $(0, 4, 1, 5, 0, 3)$ is easily found. Thus, for any positive integer t, $(0, 4t, t, 5t, 0, 3t)$ is another supporting profile.

For each t, the class of supporting profiles can be further enlarged by adding appropriate $x\mathbf{C}^3$ terms. To illustrate with $t = 1$, we find the following 13-voter profiles:

Number	**Ranking**	**Number**	**Ranking**
x	$A \succ B \succ C$	$5 - x$	$C \succ B \succ A$
$4 - x$	$A \succ C \succ B$	x	$B \succ C \succ A$
$1 + x$	$C \succ A \succ B$	$3 - x$	$B \succ A \succ C$

where the $0 \leq x \leq 3$ admissible x values define the different $\mathbf{C}^3$ multiples.

For any selected rational points that satisfy Eq. 15, the set of supporting profiles can be found in this same simple manner.

5 Exporting lessons learned from social choice

Voting rules are aggregation methods: the rule aggregates the voters' preferences into a societal ranking or outcome. But aggregation procedures are central to almost all disciplines; in astronomy, for instance, each particle's angular rotation is aggregated into the system's angular momentum, in statistics, data are aggregated to provide information about the "whole," in economics, each individual's willingness to buy and sell at a given price is aggregated into a price mechanism that, hopefully, finds the appropriate price where supply equals demand. Indeed, much of what we analyze in the social and behavioral sciences involves aggregation.

The theme to be explored in this section is whether the various insights about aggregation rules that have been developed in social choice can provide guidance about what to expect with the aggregation rules from other disciplines. This can be done; an immediate example is (Saari 2005), which shows how to use information about voting rules to significantly extend Luce's insightful axioms (Luce 1959) for human discision making. While several other illustrating examples could be provided, the ones used in this section suffice to make my point. 1) I show how the profile decomposition of Sect. 2 provides insights into the structure of nonparametric statistics, 2) how Arrow's Theorem speaks to concerns as disperse as alcoholism and nanotechnology, 3) how Sen's Theorem provides insights into those troubling periods of transition within a society, and 4) how Chichilnisky's topological dictator can be used to better understand concerns from psychology.

5.1 From voting to nonparametric statistics

When asking about which nonparametric test should be used in a particular setting, a common response is that the choice is subjective. This comment suggests that the structure of the various statistical tests is unknown. Here, social choice has much to offer because of the strong relationship between these tests and positional voting rules.

To explain, suppose the goal is to test the level of some ingredient (where more is better) in three brands of coffee; the data is in the first table of Eq. 16. An approach to handle this non-parametric analysis is to replace the actual data values with rankings where more is better; this leads to the second table's ranked data set.

$$\begin{array}{c|c|c} A & B & C \\ \hline 3.2 & 2.9 & 3.3 \\ 2.7 & 3.4 & 3.0 \end{array} \qquad \begin{array}{c|c|c} A & B & C \\ \hline 4 & 2 & 5 \\ 1 & 6 & 3 \end{array} \tag{16}$$

Next, combine the ranking values in all possible ways, which defines the eight triplets $(1,6,3)$, $(4,6,3)$, $(1,2,3)$, $(1,6,5)$, $(4,2,5)$, $(1,2,5)$, $(4,6,5)$, $(4,2,3)$. One way in which conclusions about this data are obtained is to analyze the triplets; e.g., one test counts how often an alternative is top ranked in a triplet. Notice; the way in which these triplets are used means that they can be converted into a profile; e.g., none of these triplets have $A \succ B \succ C$, but one of them has $A \succ C \succ B$. The associated profile, then is $(0,1,1,2,3,1)$. Thus, the application of this approach to the Eq. 16 data set reduces to the social choice problem of computing the plurality outcome for the profile; the outcome is $B \succ C \succ A$.

I asked my former graduate student Deanna Haunsperger to more carefully examine these connections. As part of her PhD thesis (1992), she proved that a large class of nonparametric statistical rules can be analyzed precisely in this manner; that is, first convert the data into profiles, and then use a positional voting rule to rank the alternatives. The ranking associated with the widely used Kruskal-Wallis rule, for instance, is determined in this manner by using the Borda Count. (The Kruskal-Wallis ranking of the Eq. 16 data is $B \sim C \succ A$.) Haunsperger then used my results (Saari 1989) about the dictionaries of positional voting rules[16] to characterize all possible paradoxical behavior that could ever occur with any specified non-parametric rules.

Closely related to Haunsperger's article is a paper by Laruelle and Merlin (2002) where they mapped cooperative games into profiles. Laruelle and Merlin then used a different dictionary result of mine (where I found all possible positional outcomes that could be associated with a profile (Saari 1992)) to demonstrate the wild behavior of power indices. Independently, Sieberg and I (2001a) obtained similar conclusions, but we did so by developing a direct geometric approach.[17]

From Haunsperger's results, we now know that should different nonparametric rules be applied to the same data set, they can define different rankings. The natural next question is to understand

[16]These dictionaries list all possible rankings that ever could occur when specified positional rules are used with profiles; i.e., these results extend McGarvey's paired comparison conclusion to all positional rules

[17]Although (Saari and Sieberg 2001a) was published first, I expect that Laruelle and Merlin (2002) obtained their main results before we did.

what aspects of data affect different methods; i.e., it would useful to have a data decomposition similar to the Sect. 2 profile decomposition. This was done in (Bargagliotti and Saari, 2009) where, surprisingly, everything is similar to the Sect. 2 description. Subsequent work in this direction is by Bargagliotti and Orrison, and by Crisman, .

5.2 "Divide and Conquer;" a generalized Arrow's Theorem

A way to appreciate why Arrow's seminal theorem (1952) has implications in a variety of areas that were previously unsuspected is to offer a different interpretation. Recall, his assumptions are:

1. Voters have complete and transitive preferences; there are no restrictions.

2. The societal outcome is complete and transitive.

3. (Pareto) If everyone ranks some pair in the same manner, then that common ranking is the society ranking of the pair.

4. (IIA) The outcome of any pair depends only on the relative rankings of that pair. That is, if $\mathbf{p}_1$ and $\mathbf{p}_2$ are two profiles where each voter has the same relative ranking of a specified pair, then the pair's societal ranking is the same for both profiles.

The conclusion, of course, is that with three or more alternatives, the only rule satisfying these conditions is a dictator; i.e., there is a voter so that, for all profiles, the societal outcome always agrees with that voter's ranking.

First, notice how the conditions that are imposed on the rule mandate that the societal ranking is constructed by determining appropriate outcomes for each pair. In a real sense, these conditions require the rule to adopt a "divide-and-conquer" approach where the structure of the whole — the societal outcome — is determined by dividing the problem into its parts — the pairs.

Now, suppose for $n \geq 3$ alternatives that an appropriate societal ranking always exists. After all, I suspect that the supporters of different voting rules might argue that their approach always delivers the "correct" outcome. In this setting, Arrow's result asserts that

> *if an appropriate complete and transitive societal ranking always exists, there are settings where this ranking cannot be obtained in terms of finding appropriate outcomes for each of the pairs;*

that is, there always exist settings where the divide-and-conquer approach fails. The same statement holds in trying to construct a "whole" from the parts.

This "divide-and-conquer" methodology is commonly used in many disciplines: this is true in engineering, the nano-sciences, medicine, economics, alcoholic addiction, organizational design, and just about everything. With the above interpretation of Arrow's result, one must wonder whether a similar message extends to other disciplines; i.e., are there settings where the whole cannot be obtained from the parts? It does (Saari 2010a); the proof of this assertion is based on replacing transitivity definition of rankings with conditions that encompass more general settings.

This objective is accomplished by examining the structure of transitivity, and extracting appropriate properties. Transitivity, for instance, divides the ranking into pairs, so assume that the divide-and-conquer approach defines disjoint sets of objects $\mathcal{C}_j$, $j = 1, \ldots, k$. To illustrate, for an industry that is producing a product, divide the process into the components of design $\mathcal{D}$, manufacturing $\mathcal{M}$, and sales $\mathcal{S}$. $\mathcal{D}$, for example, could consist of the infinite number of possible designs that may be based on results coming from partial differential equations, etc.

In transitivity, completeness means that any admissible combination of pairs must include a ranking for each pair. For a general setting, assert that a combination $(c_1, \ldots, c_k)$, $c_j \in \mathcal{C}_j$, is complete only if it has an entry from each component part. To illustrate with the industry problem, for completeness we need a triplet (d, m, s) where $d \in \mathcal{D}$ is a suggested design, $m \in \mathcal{M}$ is a proposed manufacturing approach, and $s \in \mathcal{S}$ is the marketing plan.

Transitivity asserts that there is a right way, and a wrong way, to assemble the pairs. But if the pairs define a nontransitive outcome, the set of rankings can be converted into a transitive one with an appropriate change in some entries. (For instance, the pairs $A \succ B, B \succ C, C \succ A$ can be converted into a transitive ranking by reversing any one of these pairwise rankings.) A similar condition is imposed on divide-and-conquer approaches. For instance, it may be that (d, m, s) is incompatible with a specified feasibility condition because the design, d, cannot be manufactured. But, this triplet can be altered by selecting a different design d^* so that (d^*, m, s) is compatible.

The component sets $\mathcal{C}_j$ could be anything; a continuum, solutions of partial differential equations, plans for design and marketing, and so forth. The restrictions arise by how "compatibility" is defined; these conditions are determined by the field of interest; they list when a combination is, or is not acceptable. After imposing these conditions, the rest of the structure of Arrow's theorem extends. The conclusion is that there must be settings where the divide-and-conquer approach fails.

As the true goal is to find positive conclusions, a first step requires finding a positive version of Arrow's result (see Chap. 2, (Saari 2008)). The approach to do so is based on Sect. 3.4.1, which essentially shows that the IIA condition nullifies the assumption that voters have transitive preferences. (A geometric proof is in the Appendix of (Saari 2001a).) Namely, similar to the way the pairwise vote ignores the "mixed gender" condition of Sect. 3.4.1, the IIA condition strips away the connecting information that the voters connect the pairs in a transitive manner. Consequently, to achieve a positive condition, IIA needs to be modified to allow using the crucial assumption that voters have transitive preferences.[18] A similar story applies to the generalized version of Arrow's result as applied to divide-and-conquer methods, but general results have yet to be found.

5.3 Dysfunctional societies; lessons from Sen's Theorem

The Sect. 3.4.1 argument also explains the source of Sen's important theorem (1970). It is interesting how, by doing so, Sen's result gains wider interest.

Sen's only requirements for a rule (which leads to the impossibility of a paretian liberal) are that it satisfies Arrow's Pareto condition and the minimal liberal condition (where at least two agents are assigned at least one pair; each "decisive agent" determines the societal outcome of the assigned pairs). This emphasis on pairs, then, means that the rule must dismiss the information that voters have transitive preferences. (So, rather than the usual assertion about a conflict between Pareto and Minimal Liberalism, the result reflects the emphasis on building a result by analyzing pairs.) As such, it now becomes easy to create all possible Sen cycles (Saari 2008).

Namely, select a desired societal cyclic ranking, and assign it to all voters; e.g., select the "two-cycle" choice of $A \succ B, B \succ C, C \succ A, C \succ D, D \succ B$. Temporarily assign these rankings to all agents. Next, assign at least two decisive agents to each cycle where each is decisive over at least one of the pairs. As an illustration, the $\{B, C\}$ pair is in both cycles, so let Ann be decisive over $\{B, C\}$. Let Barb be decisive over $\{A, B\}$ and $\{D, B\}$ – a pair from each of the two cycles.

If an agent is decisive over a pair, the way all other voters rank this pair is immaterial. So, where necessary, reverse a pair's ranking for the non-decisive agent to create a transitive preference;

[18] For instance, if IIA is modified so that the rule can use each voter's relative ranking of a pair, and the number of alternatives that separate the two alternatives, Arrow's dictator conclusion is replaced by the Borda Count.

this always can be done! To illustrate, as Anne is decisive over $\{B, C\}$, reversing Barb's ranking of this pair converts her original cyclic rankings into the transitive preferences of $C \succ A \succ D \succ B$. Similarly, Barb is decisive over two pairs; by reversing Ann's rankings of each of these pairs converts her originally assigned cyclic rankings into her new transitive preference ranking of $B \succ C \succ D \succ A$. This assignment of decisive voters and transitive preferences generates the desired two cycles.

Notice how Ann's choice of $B \succ C$ over her assigned pair creates a "strong negative externality" for Barb in that not only does Barb disagree with this societal ranking, but she strongly disagrees as reflected by the fact that another alternative separates her $C \succ B$ ranking. Similarly, Ann suffers a strong negative externality over each of Barb's choices. This is a general conclusion.

Theorem 11 *(Saari 2008) In any Sen cycle, all agents suffer a strong negative externality.*

As developed in (Saari 2001a), (Saari and Petron 2006), and (Li and Saari, 2008), and summarized in (Saari 2008), this strong negative externality feature can be used to temper the Sen cycles, and it also introduces new issues. The fact each person suffers a strong negative externality, for instance, introduces opportunities to create a "tit-for-tat" structure from the Prisoners' Dilemma to reduce the effects of the Sen cycles.

The fact *everyone* must suffer a strong negative externality suggests that a Sen cycle also models dysfunctional societal settings. It can, for instance, capture the occasionally disruptive transition stage that can occur when a previously disenfranchised group finally gains power. As an example, consider smoking in restaurants with the three options of A = "Smoking is acceptable", B = "Smoking is not acceptable", and C= "Complain to the maitre de." Non-smokers preferences are $B \succ C \succ A$ while a smoker has the preferences $A \succ B \succ C$. Prior to no-smoking requirements, the smoker was a decisive agent over $\{A, B\}$, so he smoked and created a strong negative externality for the non-smoker. While the non-smoker has the right to take action over $\{A, C\}$, the problem is that she is not decisive so the outcome remains $C \succ A$, which defines the transitive $A \succ B \succ C$. But during the transition stage to non-smoking, the non-smoker does become decisive, and a Sen cycle emerges. Once the new social "non-smoking" social norm becomes solidified, the smoker no longer is decisive, so the transitive societal outcome of $B \succ C \succ A$ now holds. Namely, Sen's cycle in this example captures the transition behavior between the two social norms.

5.4 Applying Chichilnisky's topological dictator to psychology

Another delightful result is Chichilnisky's (1982) topological dictator conclusion. A simple case (that differs from Chichilnisky's intent) is where two agents are trying to decide where to picnic on the beach of an island. In this setting, each person's preferred choice can be described as a point on a circle S^1. As the circle also is the space of outcomes, the decision rule F can be described as

$$F : S^1 \times S^1 \to S^1. \tag{17}$$

Natural properties to impose on F are 1) continuity (we do not want trivial changes in an agent's choice to create a jump in the outcome), 2) unanimity (if both people want to picnic at the same spot, that should be the outcome), and 3) Pareto; i.e., if there is an arc of shortest distance between the two agents' preferred location, then the outcome should be on that arc (e.g., if one agent wants to be on the East side while the other on the North, then the outcome should not be in the southwest side). Chichilnisky's surprising conclusion is that any such F must be "homotopic to a dictator." Stating this conclusion in a more reader friendly manner, it means that one agent makes the decision while the other, up to certain bounds, can modify the outcome.

The connection between this result and problems in psychology is that many psychological processes involves two or more "parts," such as two eyes, or the two sides of a brain. Also, stimuli can be higher dimensional, such as the color circle, or faces expressing emotions. As such, Eq. 17 can model the psychological process using essentially the same requirements on F. In this way the mathematics from social choice can explain why one side of the brain can dominate in some tasks. This direction, along with others, are being developed with my colleague Louis Narens.

As indicated above (and developed in Saari 2008), what causes Arrow's and Sen's results is their emphasis on constructing the whole from pairs; this means that the intended transitivity information about the domain, the space of individual preferences, is not being used. A similar explanation holds for Chichilniskys result. As described in (Kronewetter and Saari 2007) and developed more extensively in (Saari 2008), the culprit is the continuity of F. To explain, continuity is a technical condition ensuring that what occurs at a point is similar to what occurs nearby. As such, continuity alone forces F to treat the domain as a line, or a "contractable space." But if the space does not have this feature (and a circle does not), expect negative results to occur. Thus, to develop a positive result, information about the true structure of the domain must be included.

6 Summary

A goal of this chapter is to make the geometry of voting more intuitive by emphasizing approaches that, after described, should be transparent and applicable elsewhere, and to indicate a variety of new results and techniques. My expectation is that the descriptions will suggest to the reader new ways to analyze other choice rules and problems.

To suggest what else is possible, by combining geometry with other techniques such as concepts from dynamical "chaos" (Saari, 1995a), it becomes possible to list all possible paradoxical positional election outcomes that can occur with any number of candidates, any combination of positional procedures (for the different subsets of candidates) and any profile Saari (1989, 1995a). In other words, we now know all possible election paradoxes that could ever occur with any positional and pairwise methods, and, by extension, for all voting rules that use them. These conclusions identify a surprisingly large numbers of paradoxes.

Conversely, suppose a voting rule never admits certain kinds of election rankings. These missing listings constitute properties of the rule; e.g., the Borda Count never ranks a Condorcet loser over a Condorcet winner. It is easy to convert missing listings into an "axiomatic characterization." Because the Borda Count admits, by far, the smallest number and kinds of lists of election rankings over the different subsets of candidates (Saari, 1989, 1990a), it enjoys the most "properties," which makes it possible to generate all sorts of new "axiomatic characterizations" for the Borda Count (Saari, 1990a, 1995a).

This comment, suggesting how to find new "axiomatic" representations leads to my concluding point. While the geometry of voting has provided many new and different insights into several choice theory issues, this approach remains at an early stage of development. This approach can and should be extended in other directions.

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