Motivation

Recall the Fourier transform of functions on \((\mathbb{R},+)\): if \(f : \mathbb{R} \to \mathbb{R}\), \(\int_{\mathbb{R}} f^2 < \infty\), then the Fourier transform of \(f\) is the function \(\hat{f} : \mathbb{R} \to \mathbb{R}\) defined by
\[
\hat{f}(h) := \int_{\mathbb{R}} \exp(-ihx) f(x) \, dx
\]

We want to define a similar transformation on (compact) groups. In this tutorial we study the Fourier transform on \(S_n\), the symmetric group on \(n\) elements.

There are three aspects of Fourier transform:

- **Algebraic**: in a sense, the Fourier transform preserves some important algebraic structures of the group. For instance, if we act on the group \((\mathbb{R},+)\) by a left translation: \(f'(x) = f(x-t)\), then this corresponds to a natural action on the Fourier transform of \(f\): \(\hat{f}'(h) = \exp(-ih) \hat{f}(h)\). Or if we have convolution: \(\hat{f} \ast \hat{g}(h) = \hat{f}(h) \hat{g}(h)\).
- **Analytic**: terms in the Fourier transform gives smoothness information on the function. This is important in signal processing.
- **Algorithm**: the efficiency of the Fast Fourier transform (FFT) makes it popular in practice.

Fourier Transform on \(S_n\)

**Definition 1.** A representation of a group \(G\) on a vector space \(V\) is a group homomorphism \(\phi : G \to GL(V, \mathbb{F})\), where \(GL(V, \mathbb{F})\) is the general linear group of a vector space \(V\) over the field \(\mathbb{F}\).

When \(V\) is of dimension \(d < \infty\) (which it is in our case), then we can identify \(GL(V, \mathbb{F})\) with \(GL_d(\mathbb{F})\), which is the space of invertible \(d \times d\) matrices with entries in \(\mathbb{F}\).

**Example** Let \(G = S_n\). Then \(\rho : S_n \to GL_d(\mathbb{F})\) is a representation of \(S_n\) if and only if \(\rho\) is a homomorphism:
\[
\rho(\sigma_1 \sigma_2) = \rho(\sigma_1) \rho(\sigma_2) \quad \text{for} \quad \sigma_1, \sigma_2 \in S_n.
\]

**Example** The exponential function \(x \mapsto \exp(-ihx)\) is a representation of \((\mathbb{R},+)\) on \(GL_1(\mathbb{C})\).

This is the key in the usual Fourier transform. Note that \(h\) serves as an indexing over all possible representations of the group \((\mathbb{R},+)\). Therefore, generalizing this idea, we define the Fourier transform for functions \(f : S_n \to \mathbb{C}\) as:
\[
\hat{f}(\lambda) = \sum_{\sigma \in S_n} f(\sigma) \rho_\lambda(\sigma)
\]
where \(\lambda\) (for the moment) serves as an ‘indexing’ parameter.

**Definition 2.** An irreducible representation of a group is a group representation that has no nontrivial invariant subspaces. Otherwise it is called reducible.

On a compact group \(G\), reducible representations over \(\mathbb{C}\) can be written as direct sum of irreducible representations. Hence we are interested in irreducible representations for \(S_n\). What are the possible representations on \(S_n\)?
Young diagram and representations of $S_n$

**Young diagram.** Let $\{\lambda_i : i = 1 \ldots k\}$ be the cardinality of a partition of $n$ objects into $k$ boxes. In other words, $\lambda_i \in \mathbb{N}$, $\sum_{i=1}^{k} \lambda_i = n$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$. Arranging the boxes in a stack, the diagram obtained is called the Young diagram. An example of a Young diagram for the partition $5, 4, 1$ on 10 objects is included below. The boxes are filled with numbers from 1 to $n$, and the resulting table with entries is called a Young tableau. In a standard Young tableau, the entries increase from left to right, top to bottom. The dimension of a (standard) Young diagram is the number of distinct ways the boxes can be filled.

**Young tableaux and representations of $S_n$.** There is a one-to-one correspondence between Young diagrams and irreducible representations of the symmetric group $S_n$ over $\mathbb{C}$. Let $\lambda$ refers to a Young diagram. Therefore we can write

$$\hat{f}(\lambda) = \sum_{\sigma \in S_n} f(\sigma) \rho_\lambda(\sigma)$$

where $\rho_\lambda$ denotes an irreducible representation of $S_n$ that correspond to $\lambda$.

**Given a Young diagram $\lambda$, how can we construct $\rho_\lambda$?** In this tutorial we give the formula and an example on $S_3$. We do not prove the construction. Interested readers can refer to *Group representations in probability and statistics* (Diaconis), or *the symmetric group: representations, combinatorial algorithms and symmetric functions* (Sagan).

Let $d$ be the dimension of $\lambda$. Then $\rho_\lambda$ maps $S_n$ to $GL_d(\mathbb{C})$, therefore we can index the entries of the matrix $\rho_\lambda(\sigma)$ by distinct Young tableaux $\tau, \tau'$ of $\lambda$. Furthermore, any $\sigma \in S_n$ can be written as products of adjacent transpositions, which are of the form $(i, i+1)$. Therefore, it is sufficient to define $[\rho_\lambda(i, i+1)]_{\tau, \tau'}$. The Young’s orthogonal representation is:

$$[\rho_\lambda(i, i+1)]_{\tau, \tau'} = \begin{cases} 
\frac{d^{-1}(i, i+1)}{1 - d^2(i, i+1)} & \text{if } \tau = \tau' \\
0 & \text{if } \tau' = (i, i+1)(\tau)
\end{cases}$$

where:

- $d_\tau$ is the number of steps it take to move $i$ to $i+1$ where north and east movements (up and right) are taken as positive, and south and west movements (down and left) are taken as negative.
- $(i, i+1)(\tau)$ refers to a filling of $\lambda$ obtained from $\tau$ by applying the transposition $(i, i+1)$ (swapping $i$ an $i+1$).

Note that this results in a sparse, symmetric matrix.

Presenter: R. I. Kondor  
Page 2
Example on $S_3$. There are 3 Young diagrams on $n = 3$, and these are listed as unfilled boxes in the diagram below. Denote them $\lambda_1, \lambda_2, \lambda_3$ respectively. Note that $\rho_{\lambda_1}, \rho_{\lambda_3}$ are of dimension 1, and $\rho_{\lambda_2}$ is of dimension 2. Let $\tau$ and $\tau'$ denote these two Young tableaux respectively. Then $\rho_{\lambda_2} : S_3 \to G_2(\mathbb{C})$, and

$$
\rho_{\lambda_2}((1, 2)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho_{\lambda_2}(2, 3) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}
$$

Algebraic properties of the Fourier transform on $S_n$

The Fourier transform on $S_n$ defined in equation 1 satisfies the following properties:

- It is an invertible, norm-preserving transformation, where the norm of $f : S_n \to \mathbb{C}$ is defined by

  $$
  \|f\|^2 = \sum_{\sigma \in S_n} |f(\sigma)|^2
  $$

  and the norm of $\hat{f}$ is defined by

  $$
  \|\hat{f}\| = \frac{1}{n!} \sum_{\lambda} d_{\lambda} \|\hat{f}_\lambda\|_F^2
  $$

  where $d_{\lambda}$ is the dimensionality of $\rho_{\lambda}$, and $\|\hat{f}_\lambda\|_F$ denotes the Frobenius norm of the matrix $\hat{f}_\lambda$.

- The inversion formula is

  $$
  f(\sigma) = \frac{1}{n!} \sum_{\lambda} d_{\lambda} tr(\hat{f}(\lambda)(\rho_{\lambda}(\sigma))^{-1})
  $$

- Translation theorem: fix $\tau \in S_n$. If $f^\tau(\sigma) = f(\tau^{-1}\sigma)$, then

  $$
  \hat{f}^\tau(\lambda) = \rho_{\lambda}(\tau) \hat{f}(\lambda)
  $$

- Convolution: let $(f * g)(\sigma) := \sum_{\tau} f(\sigma\tau^{-1})g(\tau)$. Then

  $$
  \hat{f} * \hat{g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)
  $$

  This is where we get computational gain.
Analytic viewpoint and connections to ranking

Let $\sigma \in S_n$ denotes the ranking in which candidate $i$ is ranked in position $\sigma(i)$. Define $f : S_n \rightarrow \mathbb{R}$, $f(\sigma) =$ number of people voted for this ranking. Then the Fourier transform coefficients $\hat{f}(\lambda)$ gives ‘smoothness’ information of $f$. For example, the first term $\hat{f}((n)) = \sum_\sigma f(\sigma)$ gives the mean of the function. The first and second term $\hat{f}((n-1,1)) = \sum_{\sigma: \sigma(i)=j} f(\sigma)$ gives the number of votes for ranking $i$ in position $j$ (first order statistics). Inclusion of higher terms allow one to obtain higher order statistics.

Applications and references

On kernel computation:
R. Kondor and M. Barbosa: Ranking with kernels in Fourier space (COLT 2010):
http://www.its.caltech.edu/~risi/papers/KondorBarbosaCOLT10.pdf

Multi-object tracking:
R. Kondor, A. Howard and T. Jebara: Multi-object tracking with representations of the symmetric group (AISTATS 2007)
http://www.its.caltech.edu/~risi/papers/KondorHowardJebaraAISTATS07.pdf


Classical reference on the subject: